Transformation Groups, Vol. 1, Nos.1 & 2 , 1996, pp. 1–33

TILTING MODULES FOR CLASSICAL GROUPS AND HOWE DUALITY IN POSITIVE CHARACTERISTIC

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Abstract. We use the theory of tilting modules for algebraic groups to propose a characteristic free approach to "Howe duality" in the exterior algebra.

To any series of classical groups (general linear, symplectic, orthogonal, or spinor) over an algebraically closed field $\mathbf{k}$, we set in correspondence another series of classical groups (usually the same one). Denote by $G_1(m)$ the group of rank m from the first series and by $G_2(n)$ the group of rank n from the second series. For any pair $(G_1(m), G_2(n))$ we construct the $G_1(m) \times G_2(n)$-module $\mathbf{M}(m,n)$. The construction of $\mathbf{M}(m,n)$ is independent of characteristic; for $\operatorname{char} \mathbf{k} = 0$, the actions of $G_1(m)$ and $G_2(n)$ on $\mathbf{M}(m,n)$ form a reductive dual pair in the sense of Howe.

We prove that $\mathbf{M}(m,n)$ is a tilting $G_1(m)$- and $G_2(n)$-module and that $\operatorname{End}_{G_1(m)} \mathbf{M}(m,n)$ is generated by $G_2(n)$ and vice versa. The existence of such a module provides much information about the relations between the category $\mathcal{K}_1(m,n)$ of rational $G_1(m)$-modules with highest weights bounded in a certain sense by n and the category $\mathcal{K}_2(m,n)$ of rational $G_2(n)$-modules with highest weights bounded in the same sense by m. More specifically, we prove that there is a bijection of the set of dominant weights of $G_1(m)$-modules from $\mathcal{K}_1(m,n)$ to the set of dominant weights of $G_2(m)$-modules from $\mathcal{K}_2(m,n)$ such that Ext groups for induced $G_1(m)$-modules from $\mathcal{K}_1(m,n)$ are isomorphic to Ext groups for corresponding Weyl modules over $G_2(n)$. Moreover, the derived categories $D^b\mathcal{K}_1(m,n)$ and $D^b\mathcal{K}_2(m,n)$ appear to be equivalent.

We also use our study of the modules $\mathbf{M}(m,n)$ to find generators and relations for the algebra of all G-invariants in $\Lambda^\bullet(\mathbf{V}^{n_1} \oplus (\mathbf{V}^*)^{n_2})$, where $G = GL_m, Sp_{2m}, O_m$ and $\mathbf{V}$ is the natural G-module.

* Research was supported in part by Grant M7N000/M7N300 from the International Science Foundation and Russian Government and by INTAS Grant 94-4720.

** Research was supported in part by Grant M8H000/M8H300 from the International Science Foundation and Russian Government and by INTAS Grant 94-4720.

Received September 24, 1995, revised version received November 29, 1995. Accepted February 4, 1996.

Introduction

In this paper we study modules for reductive algebraic groups from the classical series (general linear, symplectic, orthogonal, and spinor). For example, let us describe our results in the case of the symplectic series.

Suppose $\Bbbk$ is an algebraically closed field of arbitrary characteristic. Denote by $G_1 = G_1(m)$ the symplectic group Sp_{2m} over the field $\Bbbk$ and by $\mathbf{V}$ the natural $2m$-dimensional G_1-module.

Let $\mathbf{M} = \Lambda^\bullet \mathbf{V}^n$ and let Υ_1 be the set of all dominant weights of the group G_1 such that the corresponding Young diagram lies in the rectangle $m \times n$. Note that Υ_1 is the set of all dominant weights of the G_1-module $\mathbf{M}$. Moreover, for each $\lambda \in \Upsilon_1$ there is an G_1-extremal λ-weight vector in $\mathbf{M}$. Let $\mathcal{K}_1$ be the category of all rational G_1-modules such that all their simple subquotients are of the form $\mathbf{L}(\lambda)$ with $\lambda \in \Upsilon_1$.

Suppose $\operatorname{char} \Bbbk = 0$. Then it is well known that the algebra $\operatorname{End}_{G_1} \mathbf{M}$ is generated by a distinguished Lie subalgebra that is isomorphic to the Lie algebra of the group $G_2 = G_2(n) = Sp_{2n}$. It is a simple special case of Howe's theory of "reductive dual pairs" (see [9]).

As a G_2-module, $\mathbf{M} \simeq \Lambda^\bullet \mathbf{W}^m$, where $\mathbf{W}$ is the natural $2n$-dimensional G_2-module. Let us define the set Υ_2 of dominant weights for the group G_2 and the category $\mathcal{K}_2$ of G_2-modules in the same way as Υ_1 and $\mathcal{K}_1$ (with m instead of n). Since in characteristic 0 any G_i-module ($i = 1, 2$) is semisimple, it follows that $\mathbf{M}$ is a $G_1 \times G_2$-module with simple spectrum and the functor $F : \mathbf{X} \mapsto \operatorname{Hom}(\mathbf{M}, \mathbf{X})$ yields an equivalence of the category $\mathcal{K}_1$ and the category $\mathcal{K}_2$.

We show that in the case of arbitrary characteristic the group G_2 also acts on $\mathbf{M}$ by G_1-automorphisms, and there is a G_2-isomorphism $\mathbf{M} \simeq \Lambda^\bullet \mathbf{W}^m$. Since the category of G_i-modules is not semisimple for $\operatorname{char} \Bbbk \neq 0$, the situation becomes more complicated. To study it we use the theory of tilting modules for quasi-hereditary algebras, see [4, 12, 7].

The category $\mathcal{K}_i$ is naturally equivalent to the category of modules over the generalized Schur algebra $S_i = S(G_i, \Upsilon_i)$ (see [6]), which is quasi-hereditary. Any G_i-module from $\mathcal{K}_i$ can be viewed as S_i-module and vice versa.

We prove that $\mathbf{M}$ is a tilting G_i-module, that is, $\mathbf{M}$ has both Weyl and good filtrations. Moreover, we show that $\mathbf{M}$ is a full tilting S_i-module. Using this, we prove that the ring of G_1-endomorphisms of $\mathbf{M}$ is generated by the image of G_2 and vice versa. More precisely, we show that $\operatorname{End}_{S_1} \mathbf{M} \simeq S_2$ and $\operatorname{End}_{S_2} \mathbf{M} \simeq S_1$. We consider this as a generalization of Howe's theory of reductive dual pairs to the case of arbitrary characteristic.

The theory of tilting modules provides us with more information than just the fact that S_1 and S_2 are mutual commutants. For example, we prove that the functor F takes any induced G_1-module $\nabla_1(\lambda)$ with $\lambda \in \Upsilon_1$ to Weyl

G_2-module $\Delta_2(\lambda^\dagger)$, where $\lambda \to \lambda^\dagger$ is an order-reversing bijection of Υ_1 onto Υ_2. Moreover, the functor F yields an equivalence of the category $\mathcal{F}(\nabla_1)$ of S_1-modules admitting good filtration and the category $\mathcal{F}(\Delta_2)$ of S_2-modules admitting Weyl filtration. Besides, for any two modules $\mathbf{X}, \mathbf{Y} \in \mathcal{F}(\nabla_1)$ and any $k \geqslant 0$ we have

$$\mathrm{Ext}^k_{G_2}(F\mathbf{X}, F\mathbf{Y}) = \mathrm{Ext}^k_{G_1}(\mathbf{X}, \mathbf{Y}).$$

In particular,

$$\mathrm{Ext}^k_{G_2}(\Delta_2(\lambda^\dagger), \Delta_2(\mu^\dagger)) = \mathrm{Ext}^k_{G_1}(\nabla_1(\lambda), \nabla_1(\mu))$$

for any $k \geqslant 0$ and $\lambda, \mu \in \Upsilon_1$.

In the language of derived categories the corresponding result sounds even better: the functor $RF : D^b\mathcal{K}_1 \to D^b\mathcal{K}_2$ yields an equivalence of triangulated categories.

In the study of dual pairs [9] Howe used the well-known description of the ring of vector invariants for the classical group G_1 in terms of generators and relations (see [14]). He considered (for $\operatorname{char} \Bbbk = 0$) three alternatives: the invariants in the symmetric algebra, in the tensor algebra, and in the exterior algebra of the sum of several copies of the natural G_1-module. For $\operatorname{char} \Bbbk > 0$, the invariants in the symmetric algebra were studied by De Concini and Procesi in [5]. The description of invariants in the tensor algebra follows easily from the description of invariants in the symmetric algebra.

We produce a characteristic free description of invariants in the remaining case of the exterior algebra. The exterior algebra $\Lambda^\bullet\mathbf{V}^n$ is just our module $\mathbf{M}$, and the subalgebra of G_1-invariants is the space of all G_1-extremal 0-weight vectors. It follows from our results that this space is isomorphic to the Weyl module for the group G_2 with highest weight $0^\dagger$. Using this, we show that

(1) The subalgebra of G_1-invariants is generated by the divided powers $\psi_{rs}^{(k)}$ of the classical basic invariants ψ_{rs}.
(2) All the relations for these generators follow from the standard relations for divided powers and the relations of the form $\mathfrak{s}(\psi_{rs}) = 0$, where $\mathfrak{s}(\psi_{rs})$ is a skew-symmetric tensor of rank more than $2m$ expressed in terms of the generators ψ_{rs}.

We get similar results for all other classical series as well. In the case of the general linear series our results (except for the description of invariants) were proved by Donkin in [7] (or follow easily from the facts proved there). Our proof is based on a different approach and applies to all series of classical groups.

The paper is organized as follows. In Section 1 we recall some definitions and facts about algebraic groups, quasi-hereditary algebras, and tilting modules. In Section 2 we state our main result, i.e., Theorem 2.1, and derive some of its consequences. Sections 3 and 4 contain the proof of Theorem 2.1. In Section 5 we use Theorem 2.1 to find generators and relations for the subring of G-invariants in the exterior algebra.

1. Preliminaries

Throughout this paper the ground field $\Bbbk$ is algebraically closed.

Let G be a connected reductive linear algebraic group over $\Bbbk$, let T be a maximal torus of G, and let $B^+ \supset T$ and $B^- \supset T$ be two opposite Borel subgroups.

Let $X = X(T)$ be the weight lattice, R the root system, R^+ the system of positive roots which makes B^+ the positive Borel subgroup and B^- the negative Borel subgroup, $R^- = -R^+$, and let Π be the set of simple roots. The weight lattice X is partially ordered: for $\lambda, \mu \in X$ we write $\lambda \leqslant \mu$ iff $\mu - \lambda$ is a sum of simple roots. We define the T-module L_λ as $\Bbbk$ with the action of T via $\lambda \in X$.

Let $W = N_G(T)/C_G(T)$ be the Weyl group, let $\langle\ ,\ \rangle$ be a nonsingular, symmetric, positive definite W-invariant form on $\mathbb{R} \otimes_{\mathbb{Z}} X$, and let $\|\lambda\| = \sqrt{\langle \lambda, \lambda \rangle}$ for $\lambda \in \mathbb{R} \otimes_{\mathbb{Z}} X$. Let $X^+ = \{\lambda \in X \mid \langle \alpha, \lambda \rangle \geqslant 0 \text{ for all } \alpha \in \Pi\}$ be the set of dominant weights. If $\lambda \in X^+$ and $\mu > \lambda$, then $\|\mu\| > \|\lambda\|$.

Suppose $\mathbf{V}$ is a rational G-module. By $\mathbf{V}^\lambda$ we denote the λ-weight space of $\mathbf{V}$. Let U^+ and U^- be the unipotent radicals of the groups B^+ and B^-, respectively. If $\mathbf{v} \in \mathbf{V}^\lambda$ and $u \in U^+$, then $u\mathbf{v} = \mathbf{v} + \mathbf{w}$, where $\mathbf{w} \in \bigoplus_{\mu > \lambda} \mathbf{V}^\mu$ (if $u \in U^-$, then $\mathbf{w} \in \bigoplus_{\mu < \lambda} \mathbf{V}^\mu$).

An element $\mathbf{v}$ of a rational G-module $\mathbf{V}$ is called an extremal vector if $u\mathbf{v} = \mathbf{v}$ for all $u \in U^+$. The subspace of all extremal vectors in $\mathbf{V}$ is equal to the sum of its intersections with weight spaces. Any nonzero extremal vector of weight λ in $\mathbf{V}$ generates a submodule with unique highest weight λ. Conversely, any highest weight vector is extremal.

Let $\chi(\lambda)$ be the Weyl character corresponding to a dominant weight $\lambda \in X^+$. The following two indecomposable G-modules have the same character $\chi(\lambda)$:

(1) The Weyl module $\Delta(\lambda)$ with highest weight λ. It is a universal G-module generated by a vector of highest weight λ. By the word "universal" we mean that for any G-module $\mathbf{V}$ that is generated by a vector of highest weight λ, there is an epimorphism $\Delta(\lambda) \to \mathbf{V}$ such that the highest weight vector of $\Delta(\lambda)$ maps to the highest weight vector of $\mathbf{V}$.

(2) The induced module $\nabla(\lambda) = \operatorname{Ind}_{B^-}^{G} L_\lambda$, where L_λ is considered as a B^--module with trivial action of U^-. The socle $\mathbf{L}(\lambda)$ of $\nabla(\lambda)$ is a

unique (up to isomorphism) simple G-module with highest weight λ. The head of $\Delta(\lambda)$ is isomorphic to $\mathbf{L}(\lambda)$.

If $\mu, \lambda \in X^+$ and $\mu \not> \lambda$, then

$$\mathrm{Ext}^1_G(\nabla(\mu), \nabla(\lambda)) = \mathrm{Ext}^1_G(\Delta(\lambda), \Delta(\mu)) = 0.$$

An ascending filtration $\mathbf{0} = \mathbf{M}_0 \subset \mathbf{M}_1 \subset \cdots \subset \mathbf{M}$ of a G-module $\mathbf{M}$ is called a ∇-filtration (Δ-filtration) if each successive quotient is isomorphic to $\nabla(\lambda)$ (respectively $\Delta(\lambda)$) for some $\lambda \in X^+$. A G-module $\mathbf{M}$ is called a tilting module if it has both ∇-filtration and Δ-filtration. We denote the class of all tilting modules by $\mathcal{T}_G$. Any direct summand of a tilting module is also a tilting module (see [12, Theorem 2]).

Theorem 1.1. (Ringel, [12]; Donkin, [7, (1.1)]) *For each $\lambda \in X^+$ there is an indecomposable G-module $\mathbf{T}(\lambda) \in \mathcal{T}_G$ which has unique highest weight λ. Furthermore, λ occurs with multiplicity* 1 *as a weight of* $\mathbf{T}(\lambda)$. *The modules $\mathbf{T}(\lambda)$ form a complete set of nonequivalent indecomposable modules in $\mathcal{T}_G$.*

Now we recall some definitions from [6, 7].

Let Υ be a finite subset of X^+ which is saturated, i.e., whenever $\lambda \in \Upsilon$ and μ is an element of X^+ satisfying $\mu \leqslant \lambda$, we have $\mu \in \Upsilon$. We say that a G-module $\mathbf{M}$ belongs to Υ if the highest weights of all composition factors of $\mathbf{M}$ belong to Υ. Among all G-submodules belonging to Υ of an arbitrary G-module $\mathbf{M}$, there is a unique maximal one which we denote by $O_\Upsilon(\mathbf{M})$. In particular, regarding $\Bbbk[G]$ as a left G-module via $(x \cdot f)(z) = f(zx)$, for $x, z \in G$, $f \in \Bbbk[G]$, we get a submodule $O_\Upsilon(\Bbbk[G])$. In fact, $O_\Upsilon(\Bbbk[G])$ is a subcoalgebra of $\Bbbk[G]$. The dual algebra $S = S(G, \Upsilon) = O_\Upsilon(\Bbbk[G])^*$ is called a generalized Schur algebra. We have

$$\dim S(G, \Upsilon) = \sum_{\lambda \in \Upsilon} (\dim \Delta(\lambda))^2.$$

Let $\mathbf{M}$ be a rational G-module belonging to Υ. Thus $\mathbf{M}$ is a right $\Bbbk[G]$-comodule with structure map $\tau_{\mathbf{M}} : \mathbf{M} \to \mathbf{M} \otimes \Bbbk[G]$, say. Then $\tau_{\mathbf{M}}$ is a G-module map, where G acts on $\mathbf{M} \otimes \Bbbk[G]$ with trivial action on the left factor. Applying the functor O_Υ, we get the restriction $\tau'_{\mathbf{M}} : O_\Upsilon(\mathbf{M}) \to \mathbf{M} \otimes O_\Upsilon(\Bbbk[G])$. But $O_\Upsilon(\mathbf{M}) = \mathbf{M}$, therefore $\tau'_{\mathbf{M}}$ gives $\mathbf{M}$ the structure of a right $O_\Upsilon(\Bbbk[G])$-comodule and hence a left S-module. Conversely, starting with a left S-module $\mathbf{M}$, and reversing the procedure, we obtain on $\mathbf{M}$ a structure of rational G-module belonging to Υ. In this way, the category of G-modules belonging to Υ is equivalent to the category of S-modules. Further on we study only finite-dimensional rational G-modules and do not distinguish between a G-module belonging to Υ and the corresponding S-module.

Proposition 1.2. (Donkin, [6, (2.2d)]) *For all left S-modules $\mathbf{X}$ and $\mathbf{Y}$ and $k \geqslant 0$ we have* $\operatorname{Ext}^k_S(\mathbf{X},\mathbf{Y}) = \operatorname{Ext}^k_G(\mathbf{X},\mathbf{Y})$.

It is clear from the definitions that the set $\{\mathbf{L}(\lambda) \mid \lambda \in \Upsilon\}$ is a full set of simple S-modules. Let $\Upsilon = \{\lambda(1),\dots,\lambda(r)\}$, where $i < j$ whenever $\lambda(i) < \lambda(j)$. The algebra S is quasi-hereditary with respect to the ordering $\mathbf{L}(\lambda(1)),\dots,\mathbf{L}(\lambda(r))$ of simple S-modules (see [6, 7]). The modules $\Delta(\lambda(i))$ form the set of standard S-modules with simple head; the modules $\nabla(\lambda(i))$ form the set of costandard S-modules with simple socle. By $\mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\nabla)$) denote the class of finite-dimensional S-modules admitting a Δ-filtration (resp. ∇-filtration). For $\mathbf{M} \in \mathcal{F}(\Delta)$ (resp. $\mathbf{N} \in \mathcal{F}(\nabla)$) we denote by $(\mathbf{M} : \Delta(\lambda))$ (resp. $(\mathbf{N} : \nabla(\lambda))$) the multiplicity of $\Delta(\lambda)$ in a Δ-filtration of $\mathbf{M}$ (resp. $\nabla(\lambda)$ in a ∇-filtration of $\mathbf{N}$). An S-module $\mathbf{M}$ is called a tilting module if it admits a ∇-filtration and admits a Δ-filtration. Such an $\mathbf{M}$ is a sum of $\mathbf{T}(\lambda(i))$'s and we call $\mathbf{M}$ a full tilting module if $\mathbf{T}(\lambda(i))$ occurs as a direct summand for each $i = 1,\dots,r$. We denote the class of all tilting S-modules by $\mathcal{T}_S$. Clearly, S-module $\mathbf{M} \in \mathcal{T}_S$ iff $\mathbf{M} \in \mathcal{T}_G$ as a G-module.

Let $\mathbf{M} \in \mathcal{T}_S$ be a full tilting module and let $S^\dagger = \operatorname{End}_S \mathbf{M}$. We write endomorphisms to the right of $\mathbf{M}$; thus $\mathbf{M}$ is a *right* $S^\dagger$-module. There is a functor $F : \mathbf{X} \mapsto \operatorname{Hom}_S(\mathbf{M},\mathbf{X})$ from the category of left S-modules to the category of left $S^\dagger$-modules.

Theorem 1.3. (Ringel, [12, Theorem 6]) *Let $\Delta^\dagger_i = F\nabla(\lambda(i))$. Then for each $i = 1,\dots,r$ the $S^\dagger$-module $\Delta^\dagger_i$ has simple head, $\mathbf{L}^\dagger_i$ say, $\{\mathbf{L}^\dagger_i \mid i = 1,\dots,r\}$ is a full set of simple $S^\dagger$-modules, and $S^\dagger$ is a quasi-hereditary algebra with respect to the ordering $\mathbf{L}^\dagger_r,\dots,\mathbf{L}^\dagger_1$ of simple $S^\dagger$-modules. The modules $\Delta^\dagger_i$ form the set of standard $S^\dagger$-modules with simple head. Besides, the functor F yields an equivalence between the category $\mathcal{F}(\nabla)$ of S-modules admitting a ∇-filtration and the category $\mathcal{F}(\Delta^\dagger)$ of $S^\dagger$-modules admitting a $\Delta^\dagger$-filtration.*

By [12, Theorem 4 and Corollary 3] we have $\operatorname{Ext}^k_S(\mathbf{X},\mathbf{Y}) = 0$ whenever $\mathbf{X} \in \mathcal{F}(\Delta)$, $\mathbf{Y} \in \mathcal{F}(\nabla)$, and $k > 0$. Hence $\operatorname{Ext}^k_S(\mathbf{M},\mathbf{M}) = 0$ for $k > 0$. Besides, the module $\mathbf{M}$ has finite projective dimension, and the ring S has a finite resolution $\mathbf{0} \to S \to \mathbf{T}_0 \to \mathbf{T}_1 \to \cdots \to \mathbf{T}_m \to \mathbf{0}$, where $\mathbf{T}_i \in \mathcal{T}_S$ (see [12, Theorem 5]). Thus by [3, (2.1)] we have

Proposition 1.4. *The functor $RF : D^b(S\text{-}mod) \to D^b(S^\dagger\text{-}mod)$ yields an equivalence of triangulated categories.*

We also need the following two propositions:

Proposition 1.5. (Donkin, [7, (3.2)]) *For any* $\mathbf{X}, \mathbf{Y} \in \mathcal{F}(\nabla)$ *and* $k \geqslant 0$ *we have* $\operatorname{Ext}^k_{S^\dagger}(F\mathbf{X}, F\mathbf{Y}) = \operatorname{Ext}^k_S(\mathbf{X}, \mathbf{Y})$. *In particular,*

$$\operatorname{Ext}^k_{S^\dagger}(\Delta^\dagger_j, \Delta^\dagger_i) = \operatorname{Ext}^k_S(\nabla(\lambda(j)), \nabla(\lambda(i)))$$

for any $k \geqslant 0$ *and* $j, i \in [1, r]$.

Denote costandard $S^\dagger$-modules with simple socle by $\nabla^\dagger_i$.

Proposition 1.6. (Donkin, [7, (3.1)]) *For* $1 \leqslant i, j \leqslant r$ *we have*

$$(\mathbf{T}(\lambda(i)) : \nabla(\lambda(j))) = [\nabla^\dagger_j : \mathbf{L}^\dagger_i],$$

where $[\nabla^\dagger_j : \mathbf{L}^\dagger_i]$ *is the composition multiplicity.*

Let $\sigma : G \to G$ $(g \mapsto g^\sigma)$ be an involutive anti-automorphism such that $\sigma|_T = \mathrm{id}_T$. For example, if $G = GL_m$, then we take g^σ to be the transpose of the matrix $g \in G$. Clearly, σ maps U^+ to U^- and U^- to U^+. Besides, such an anti-automorphism induces an anti-automorphism of any generalized Schur algebra $S = S(G, \Upsilon)$.

In the sequel we fix such an anti-automorphism σ for any group G under consideration. Thus we may view any left G-module (or S-module) $\mathbf{V}$ as a right G-module (S-module) via the anti-automorphism σ: for any $\mathbf{v} \in \mathbf{V}$ and $g \in G$ we put $\mathbf{v}g = g^\sigma \mathbf{v}$. Further on we shall not distinguish between left and right G-modules (S-modules).

Let $\mathbf{V}$ be a G-module. We define the transpose $\mathbf{V}^\sigma$ of the module $\mathbf{V}$ in the following way: as a vector space $\mathbf{V}^\sigma = \operatorname{Hom}(\mathbf{V}, \Bbbk)$, and the action of G on $\mathbf{V}^\sigma$ is given by $(gf)(\mathbf{v}) = f(g^\sigma \mathbf{v})$ for all $\mathbf{v} \in \mathbf{V}$, $f \in \operatorname{Hom}(\mathbf{V}, \Bbbk)$, $g \in G$. The functor "Transpose" is contravariant and exact. Besides, it has the following properties:

- $(\mathbf{V}^\sigma)^\sigma \simeq \mathbf{V}$;
- $\operatorname{ch}_G \mathbf{V}^\sigma = \operatorname{ch}_G \mathbf{V}$;
- $\operatorname{Ext}^k_G(\mathbf{V}, \mathbf{W}) \simeq \operatorname{Ext}^k_G(\mathbf{W}^\sigma, \mathbf{V}^\sigma)$ for $k \geqslant 0$;
- $(\nabla(\lambda))^\sigma \simeq \Delta(\lambda)$, $(\Delta(\lambda))^\sigma \simeq \nabla(\lambda)$;
- If $\mathbf{V}$ admits a ∇-filtration, then $\mathbf{V}^\sigma$ admits a Δ-filtration, and vice versa.

Let $\mathbf{M}$ be a G-module and let $(\ ,\ )$ be a bilinear form on $\mathbf{M}$. This form is called contravariant if $(g\mathbf{v}_1, \mathbf{v}_2) = (\mathbf{v}_1, g^\sigma \mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{M}$, $g \in G$. Since σ is identical on the torus T, we see that different weight spaces of $\mathbf{M}$ are orthogonal w. r. t. contravariant form. It is clear that if $\mathbf{V}$ is a G-submodule of $\mathbf{M}$ and the form $(\ ,\ )$ is contravariant and nonsingular, then $\mathbf{M}/\mathbf{V}^\perp \simeq \mathbf{V}^\sigma$. In particular, $\mathbf{M} \simeq \mathbf{M}^\sigma$ for any G-module $\mathbf{M}$ with nonsingular contravariant form.

In [10] Mathieu proved

Theorem 1.7. *The G-module $\nabla(\lambda) \otimes \nabla(\mu)$ admits a ∇-filtration.* $\square$

With the help of the functor "Transpose" we get

Corollary 1.8. *The G-module $\Delta(\lambda) \otimes \Delta(\mu)$ admits a Δ-filtration.* $\square$

Corollary 1.9. *Suppose* $\mathbf{v}$ *is highest weight vector in* $\Delta(\lambda) \otimes \Delta(\mu)$ *(i.e.,* $\mathbf{v}$ *is the tensor product of highest weight vectors of the two factors); then it generates a submodule that is isomorphic to the Weyl module:* $\langle G\mathbf{v}\rangle \simeq \Delta(\lambda+\mu)$. $\square$

Combining Theorem 1.7 and Corollary 1.8 (see also [7]), we get

Theorem 1.10. *If* $\mathbf{M}, \mathbf{N} \in \mathcal{T}_G$, *then* $\mathbf{M} \otimes \mathbf{N} \in \mathcal{T}_G$. $\square$

Now let us say a few words about classical groups. In the sequel we use the following notations.

The symplectic group Sp_{2m} consists of all matrices preserving the bilinear form

$$\langle {\sum_{i=-m}^{m}}' x^i \mathbf{e}_i, {\sum_{i=-m}^{m}}' y^i \mathbf{e}_i \rangle = \sum_{i=1}^{m} (x^i y^{-i} - x^{-i} y^i)$$

on the vector space $\mathbb{k}^{2m}$ with the standard basis $(\mathbf{e}_1, \ldots, \mathbf{e}_m, \mathbf{e}_{-m}, \ldots, \mathbf{e}_{-1})$. Here $\sum'$ means that the summation is taken over $i \neq 0$.

The orthogonal group O_{2m} consists of all matrices preserving the quadratic form

$$\Phi({\sum_{i=-m}^{m}}' x^i \mathbf{e}_i) = \sum_{i=1}^{m} x^i x^{-i}$$

on the vector space $\mathbb{k}^{2m}$ with the standard basis $(\mathbf{e}_1, \ldots, \mathbf{e}_m, \mathbf{e}_{-m}, \ldots, \mathbf{e}_{-1})$. The orthogonal group O_{2m+1} consists of all matrices preserving the quadratic form

$$\Phi(\sum_{i=-m}^{m} x^i \mathbf{e}_i) = \sum_{i=1}^{m} x^i x^{-i} - (x^0)^2$$

on the vector space $\mathbb{k}^{2m+1}$ with the standard basis $(\mathbf{e}_1, \ldots, \mathbf{e}_m, \mathbf{e}_0, \mathbf{e}_{-m}, \ldots, \mathbf{e}_{-1})$.

We consider orthogonal groups only for $\operatorname{char} \mathbb{k} \neq 2$.

For the groups GL_m, Sp_{2m}, SO_{2m}, and SO_{2m+1} we take the group of diagonal matrices from G for T and the groups of upper and lower triangular matrices from G for B^+ and B^-, respectively. The group $X = X(T)$ has the natural basis $(\varepsilon^1, \ldots, \varepsilon^m)$, where the character ε^i takes each matrix from T to its i-th diagonal entry. By $\lambda_1, \ldots, \lambda_m$ we denote the coordinates of the weight $\lambda \in X$ in the basis (ε_i).

If $G = GL_m$, Sp_{2m}, and O_{2m}, then we take matrix transposition for the anti-automorphism σ. Thus the bilinear form given by the unity matrix in the standard basis of the natural G-module is contravariant. If $G = O_{2m+1}$, then we put $g^\sigma = Q^{-1}g^tQ$, where

$$Q = \mathrm{diag}(\underbrace{1,\dots,1}_{m}, 2, \underbrace{1,\dots,1}_{m}).$$

The standard contravariant bilinear form on the natural G-module is given by the matrix Q.

We say that a weight λ of the group GL_m is *a polynomial weight* if all $\lambda_i \geqslant 0$. The Young diagram $\mathfrak{Y}(\lambda)$ associated to a polynomial dominant weight λ consists of $\lambda_1 + \dots + \lambda_m$ squares arranged in consecutive rows so that i-th row has λ_i squares. The rows are counted from top to bottom and arranged so that they all start from the same left extremity. In the same way we associate a Young diagram to any dominant weight of the groups Sp_{2m} and SO_{2m+1}. In all these three cases the dominant weight λ is uniquely determined by the Young diagram $\mathfrak{Y}(\lambda)$.

For any dominant weight λ of the group SO_{2m}, we have $\lambda_1 \geqslant \dots \geqslant \lambda_{m-1} \geqslant \lambda_m \geqslant -\lambda_{m-1}$. Let $\mathfrak{Y}(\lambda)$ be the Young diagram with rows $\lambda_1, \dots, \lambda_{m-1}, |\lambda_m|$. The dominant weight λ is uniquely determined by the Young diagram $\mathfrak{Y}(\lambda)$ and $\mathfrak{S}(\lambda) = \mathrm{sign}(\lambda_m)$.

The groups O_{2m} and O_{2m+1} are not connected. Nonetheless the theory of tilting modules and generalized Schur algebras may be applied to them.

Let G be one of these groups and let $\check{G}$ be its maximal connected subgroup. Thus $\check{G} = SO_{2m}$ or SO_{2m+1}. Let us describe what we mean by dominant weights, Weyl modules, Weyl characters, etc. for the group G. As a rule, we denote the corresponding objects for the group $\check{G}$ by the same letter with $\check{}$ over it: for example, by $\check{T}$ we denote the maximal torus of $\check{G}$, etc.

The Weyl group of the group G is defined in the same way as for connected groups: $W = N_G(\check{T})/C_G(\check{T})$. Its action on $\mathbb{R} \otimes_{\mathbf{Z}} \check{X}$ preserves the same nonsingular, symmetric, positive definite bilinear form $\langle\ ,\ \rangle$.

Let $B^\pm = N_G(\check{B}^\pm)$ and let $T = N_G(\check{T}) \cap B^+$. The group T is generated by $\check{T}$ and the element $s \in G \setminus \check{G}$, where

$$s(\mathbf{e}_i) = \begin{cases} \mathbf{e}_{-i}, & \text{if } G = O_{2m} \text{ and } i = \pm m; \\ -\mathbf{e}_i, & \text{if } G = O_{2m+1} \text{ and } i = 0; \\ \mathbf{e}_i, & \text{otherwise.} \end{cases}$$

Denote by $\bar{s}$ the image of s in the Weyl group W. Clearly, if $G = O_{2m+1}$, then $\bar{s}$ is the unity element, and if $G = O_{2m}$, then $\bar{s}(\check{\lambda}) = \lambda_1\varepsilon^1 + \dots + \lambda_{m-1}\varepsilon^{m-1} - \lambda_m\varepsilon^m$ for any $\check{\lambda} = \lambda_1\varepsilon^1 + \dots + \lambda_m\varepsilon^m \in \check{X}$.

Any G-module $\mathbf{M}$ can be decomposed in a direct sum of simple T-modules. The isotypical components of $\mathbf{M}$ as T-module play the role of weight spaces for the G-module $\mathbf{M}$.

Let X be the set of all pairs $(\mathfrak{O}, \mathfrak{S})$ such that $\mathfrak{O}$ is an $\bar{s}$-orbit in $\check{X}$ and

$$\mathfrak{S} \in \begin{cases} \{0\}, & \text{if } |\mathfrak{O}| = 2; \\ \{-1, +1\}, & \text{if } |\mathfrak{O}| = 1. \end{cases}$$

Suppose $\lambda \in X$; then by $\mathfrak{S}(\lambda)$ we mean the second element of the pair $(\mathfrak{O}, \mathfrak{S}) = \lambda$. If $|\mathfrak{O}| = 1$, then we denote by $\check{\lambda}$ the unique element of $\mathfrak{O}$; if $|\mathfrak{O}| = 2$, then we denote by $\check{\lambda}$ the element of $\mathfrak{O}$ with the last coordinate positive. In any case we denote by $\lambda_1, \dots, \lambda_m$ the coordinates of $\check{\lambda}$ in the standard basis of $\check{X}$.

Let

$$L_\lambda = \begin{cases} \check{L}_{\check{\lambda}}, & \text{if } \mathfrak{S}(\lambda) = \pm 1; \\ \check{L}_{\check{\lambda}} \oplus \check{L}_{\bar{s}\check{\lambda}}, & \text{if } \mathfrak{S}(\lambda) = 0. \end{cases}$$

We define the action of s on L_λ as follows: if $\mathfrak{S}(\lambda) = \pm 1$, then s acts via multiplication by $\mathfrak{S}(\lambda)$; if $\mathfrak{S}(\lambda) = 0$, then s acts via interchanging the direct summands $\check{L}_{\check{\lambda}}$ and $\check{L}_{\bar{s}\check{\lambda}}$. Thus L_λ becomes a T-module. It is clear that $\{L_\lambda \mid \lambda \in X\}$ is a full set of simple T-modules. We call the elements of X *the weights* of the group G.

We denote by $\mathcal{G}$ the free $\mathbb{Z}$-module with the basis $(e^\lambda)_{\lambda \in X}$. For any finite-dimensional G-module $\mathbf{M}$, we define its formal character

$$\operatorname{ch} \mathbf{M} = \sum_{\lambda \in X} [\mathbf{M} : L_\lambda] e^\lambda \in \mathcal{G}.$$

We call a weight $\lambda \in X$ *dominant*, if the weight $\check{\lambda}$ of the group $\check{G}$ is dominant. The set of dominant weights is denoted by X^+. For a dominant weight $\lambda \in X^+$ we denote by $\mathfrak{Y}(\lambda)$ the Young diagram, corresponding to $\check{\lambda}$. Clearly, the dominant weight λ is uniquely determined by $\mathfrak{Y}(\lambda)$ and $\mathfrak{S}(\lambda)$.

Let us introduce the following partial order in the set X: put $\lambda > \mu$ iff $\check{\lambda} > \check{\mu}$ or $\check{\lambda} > \bar{s}\check{\mu}$. The simple G-modules are parametrized by their highest weights, which belong to X^+. Denote the simple G-module with highest weight λ by $\mathbf{L}(\lambda)$. We define the Weyl character $\chi(\lambda)$ as the formal character of $\mathbf{L}(\lambda)$ in characteristic zero.

The Weyl module $\Delta(\lambda)$ and the induced module $\nabla(\lambda)$ are defined in the same way as for connected groups. We have $\operatorname{ch} \Delta(\lambda) = \operatorname{ch} \nabla(\lambda) = \chi(\lambda)$. It is readily seen that with these conventions the theory of tilting modules and generalized Schur algebras for the group G is the same as for connected groups.

We shall also study spinor groups. Let us recall their definition. Assume that $\operatorname{char} \mathbb{k} \neq 2$. Let $\mathbf{V} = \mathbb{k}^{2m}$, and let $C(\Phi)$ be the Clifford algebra of the quadratic form Φ preserved by O_{2m}. We regard $\mathbf{V}$ as a subspace of $C(\Phi)$. Let us put $\mathbf{V}_- = \langle \mathbf{e}_{-m}, \dots, \mathbf{e}_{-1} \rangle$ and $\mathbf{U} = \Lambda^\bullet \mathbf{V}_-$. The basis of $\mathbf{U}$ consisting of elements $\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}$ with $i_1 < \cdots < i_k$ will be referred to as the standard basis of $\mathbf{U}$. It is well known that $\mathbf{U}$ is a unique (up to isomorphism) simple $C(\Phi)$-module.

Denote by Pin_{2m} the subgroup of the group of all invertible elements of $C(\Phi)$ generated by the elements $\mathbf{v} \in \mathbf{V} \subset C(\Phi)$ such that $\mathbf{v}^2 = \mathbf{1}$. It is a two-fold cover of the orthogonal group O_{2m}. Abusing terminology, we shall call the natural O_{2m}-module V also *the natural* Pin_{2m}*-module.* The vector space $\mathbf{U}$, regarded as a Pin_{2m}-module, is called the *spinor* Pin_{2m}*-module.*

The preimage of SO_{2m} under the natural projection $Pin_{2m} \to O_{2m}$ is denoted by $Spin_{2m}$. Note that $Spin_{2m} = Pin_{2m} \cap C_+(\Phi)$, where $C_+(\Phi)$ is the even part of the Clifford algebra $C(\Phi)$. Let us call $Spin_{2m}$ the *special spinor group.*

The anti-automorphism σ can be naturally extended from the orthogonal group to the corresponding spinor group We consider the anti-automorphism of the Clifford algebra that takes $\mathbf{e}_i$ to $\mathbf{e}_{-i}$ for $i = \pm 1, \dots, \pm m$. The restriction of this anti-automorphism to the spinor group (special or general) is the desired anti-automorphism σ. The bilinear form given by the unity matrix in the standard basis of $\mathbf{U}$ is contravariant.

Now let $\mathbf{V} = \Bbbk^{2m+1}$ and let Φ be the quadratic form on $\mathbf{V}$ that is preserved by the group O_{2m+1}. The vector space $\mathbf{U} = \Lambda^\bullet \mathbf{V}_-$ is a unique (up to isomorphism) simple $C_+(\Phi)$-module. The spinor group $Spin_{2m+1}$ is a subset of $C_+(\Phi)$ defined in the same way as $Spin_{2m}$. It is a two-fold cover of the special orthogonal group SO_{2m+1}. The anti-automorphism σ can be naturally extended from SO_{2m+1} to $Spin_{2m+1}$ in the same way as in the even case. The standard contravariant bilinear form on $\mathbf{U}$ is again given by the unity matrix in the standard basis.

Any weight λ of the special spinor group $Spin_{2m}$ or $Spin_{2m+1}$ can be expressed as $\lambda = \lambda_1\varepsilon^1 + \cdots + \lambda_m\varepsilon^m$, where either $\lambda_i \in \mathbb{Z}$ for all $i \in [1,m]$, or $\lambda_i \in \mathbb{Z} + \frac{1}{2}$ for all $i \in [1,m]$. Here by ε^i we denote the weights that correspond to the elements of the standard basis for the weight lattice of the corresponding special orthogonal group.

For a dominant weight λ of the group $Spin_{2m}$ or $Spin_{2m+1}$, we define $\mathfrak{Y}(\lambda)$ as the Young diagram with rows $[\lambda_1], \dots, [\lambda_{m-1}], [|\lambda_m|]$, where brackets denote the integral part. We also put $\mathfrak{S}(\lambda) = \operatorname{sign} \lambda_m$. The dominant weights for the group Pin_{2m} are defined in the same way as for the group O_{2m}.

Let G be one of the groups GL_m, Sp_{2m}, O_{2m}, SO_{2m}, O_{2m+1}, SO_{2m+1}, $Spin_{2m}$, Pin_{2m}, and $Spin_{2m+1}$ (for orthogonal and spinor groups we assume that $\operatorname{char} \Bbbk \neq 2$).

We shall consider the following category $\mathcal{C}$ of G-modules. If G is a symplectic or orthogonal group, then $\mathcal{C}$ is the category of all finite-dimensional rational modules. If G is a general linear group, then $\mathcal{C}$ is the category of all finite-dimensional polynomial modules (that is, the modules such that all their weights are polynomial). If G is a spinor group, then the category of all finite-dimensional rational modules is the direct sum of two subcategories: the first one consists of the modules with integral weights (this subcategory is equivalent to the category of all finite-dimensional rational modules over the corresponding special orthogonal group) and the second one consists of modules with the coordinates λ_i of all the weights in $\mathbb{Z} + \frac{1}{2}$. We take the latter subcategory for $\mathcal{C}$.

Let $\mathfrak{X}^+$ be the set of dominant weights of G-modules from $\mathcal{C}$. To any dominant weight λ we have assigned the Young diagram $\mathfrak{Y}(\lambda)$. If $G = SO_{2m}$, $Spin_{2m}$, O_{2m}, O_{2m+1}, then the dominant weight $\lambda \in \mathfrak{X}^+$ is uniquely determined by $\mathfrak{Y}(\lambda)$ and $\mathfrak{S}(\lambda)$; if G is any other group from our list, then the dominant weight $\lambda \in \mathfrak{X}^+$ is uniquely determined by $\mathfrak{Y}(\lambda)$ alone.

There is a standard partial order in the set of Young diagrams: $\mathfrak{Y} \geqslant \mathfrak{Y}'$ iff $\sum_{i=1}^{j} k_i \geqslant \sum_{i=1}^{j} k_i'$ for all j, where k_i (resp. k_i') is the length of the i-th row of the diagram $\mathfrak{Y}$ (resp. $\mathfrak{Y}'$). We introduce the induced partial order in the set $\mathfrak{X}^+$, i.e., we write $\lambda > \mu$ iff $\mathfrak{Y}(\lambda) > \mathfrak{Y}(\mu)$. This partial order is stronger than the restriction of the standard partial order in the weight lattice X to $\mathfrak{X}^+$.

2. The main theorem

Let m and n be nonnegative integers and let (G_1, G_2) be one of the following pairs of classical groups: (GL_m, GL_n), (Sp_{2m}, Sp_{2n}), (O_{2m}, SO_{2n}), $(O_{2m+1}, Spin_{2n})$, (Pin_{2m}, SO_{2n+1}), and $(Spin_{2m+1}, Spin_{2n+1})$. For orthogonal and spinor groups we assume that $\operatorname{char} \Bbbk \neq 2$.

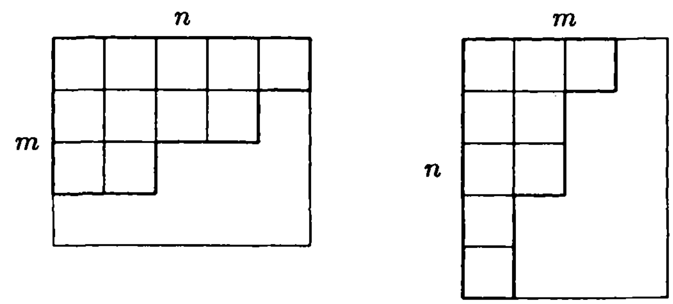
We mark the sets X, $\mathfrak{X}^+$, category $\mathcal{C}$, elements $\{\varepsilon^i\}$, etc. corresponding to the group G_1 (resp. G_2) by the subscript 1 (resp. 2).

Note that m is the rank of the group G_1 and n is the rank of the group G_2. When we want to emphasize it, we shall write $G_1(m)$ instead of G_1, $G_2(n)$ instead of G_2, $\mathfrak{X}_1^+(m)$ instead of $\mathfrak{X}_1^+$, $\mathcal{C}_1(m)$ instead of $\mathcal{C}_1$, and so on.

Let $\mathcal{K}_1(m,n)$ be the category consisting of $G_1(m)$-modules from $\mathcal{C}_1(m)$ with the weights such that all their coordinates are less than $n+1$. We have $\mathcal{K}_1(m,n) \subset \mathcal{K}_1(m,n+1)$ for $n = 0, 1, \dots$, and $\mathcal{C}_1(m) = \bigcup_{n=0}^{\infty} \mathcal{K}_1(m,n)$. When m and n are fixed, we write $\mathcal{K}_1$ instead of $\mathcal{K}_1(m,n)$.

Denote the set of dominant weights of modules from $\mathcal{K}_1$ by $\Upsilon_1 = \Upsilon_1(m,n)$. Clearly, the weight $\lambda \in \mathfrak{X}_1^+$ belongs to Υ_1 iff the corresponding Young diagram $\mathfrak{Y}(\lambda)$ lies in the rectangle $m \times n$. The set Υ_1 is partially ordered as a subset of the partially ordered set $\mathfrak{X}_1^+$. Recall that this order is stronger than the standard order in the weight lattice X_1.

It is readily seen that the set Υ_1 is saturated, and thus the category $\mathcal{K}_1$ is equivalent to the category of finite-dimensional modules over the generalized Schur algebra $S_1 = S_1(m,n) = S(G_1, \Upsilon_1)$.



$\mathfrak{Y}(\lambda)$ $\mathfrak{Y}(\lambda^\dagger)$

FIGURE 1

We define the subcategory $\mathcal{K}_2 = \mathcal{K}_2(m,n) \subset \mathcal{C}_2(m)$ and the partially ordered subset $\Upsilon_2 = \Upsilon_2(m,n) \subset \mathfrak{X}_2^+$ in the same way as $\mathcal{K}_1$ and Υ_1 (with m and n interchanged). The weight $\lambda \in \mathfrak{X}_2^+$ belongs to Υ_2 iff the corresponding Young diagram $\mathfrak{Y}(\lambda)$ lies in the rectangle $n \times m$. The set Υ_2 is saturated and the category $\mathcal{K}_2$ is equivalent to the category of finite-dimensional modules over the generalized Schur algebra $S_2 = S_2(m,n) = S(G_2, \Upsilon_2)$.

For $\lambda \in \Upsilon_1$ we define $\lambda^\dagger \in \Upsilon_2$ by the following conditions:

(1) $\mathfrak{Y}(\lambda^\dagger)$ is the transpose of the complement of $\mathfrak{Y}(\lambda)$ in the rectangle $m \times n$ (see fig. 1).
(2) if $G_1 = O_{2m}, O_{2m+1}$, then $\mathfrak{S}(\lambda^\dagger) = \mathfrak{S}(\lambda)$.

It is clear that the correspondence $\lambda \mapsto \lambda^\dagger$ is an order-reversing bijection $\Upsilon_1 \to \Upsilon_2$.

For $i = 1, 2$ by $\Delta_i(\lambda)$ (resp. $\nabla_i(\lambda)$) we denote the Weyl (resp. induced) G_i-module with highest weight $\lambda \in \mathfrak{X}_i^+$. By $\mathcal{F}(\Delta_i)$ (resp. $\mathcal{F}(\nabla_i)$) denote the class of finite-dimensial S_i-modules admitting a Δ_i-filtration (resp. ∇_i-filtration).

Theorem 2.1. *Let m and n be arbitrary nonnegative integers. There exists a $G_1(m) \times G_2(n)$-module $\mathbf{M} = \mathbf{M}(m,n)$ such that*

(1) *$\mathbf{M}$ is a full tilting S_1-module and a full tilting S_2-module;*
(2) *the natural homomorphisms $S_1 \to \operatorname{End}_{S_2} \mathbf{M}$ and $S_2 \to \operatorname{End}_{S_1} \mathbf{M}$ are isomorphisms;*
(3) *there is an S_1-isomorphism $\Delta_1(\lambda) \xrightarrow{\sim} \operatorname{Hom}_{S_2}(\mathbf{M}, \nabla_2(\lambda^\dagger))$ and an S_2-isomorphism $\Delta_2(\lambda^\dagger) \xrightarrow{\sim} \operatorname{Hom}_{S_1}(\mathbf{M}, \nabla_1(\lambda))$ for each $\lambda \in \Upsilon_1$.*

The proof of this theorem is to be found in the next two sections.

Let $F : \mathcal{K}_1 \to \mathcal{K}_2$ be the functor given by $F\mathbf{X} = \mathrm{Hom}(\mathbf{M}, \mathbf{X})$. By Theorem 2.1(3) we have $F\nabla_1(\lambda) \simeq \Delta_2(\lambda^\dagger)$.

Now we can apply to the S_1-module $\mathbf{M}$ Theorem 1.3 and Propositions 1.4, 1.5, 1.6. Since by Theorem 2.1(2) the conjugate algebra $S_1^\dagger$ is isomorphic to the generalized Schur algebra S_2, we obtain (with the help of Proposition 1.2) the following corollaries.

Corollary 2.2. *The functor $RF : D^b(\mathcal{K}_1) \to D^b(\mathcal{K}_2)$ yields an equivalence of triangulated categories.*

Corollary 2.3. *For any $\mathbf{X}, \mathbf{Y} \in \mathcal{F}(\nabla_1)$ and $k \geqslant 0$, we have*

$$\mathrm{Ext}^k_{G_2}(F\mathbf{X}, F\mathbf{Y}) = \mathrm{Ext}^k_{G_2}(\mathbf{X}, \mathbf{Y}).$$

In particular,

$$\mathrm{Ext}^k_{G_2}(\Delta_2(\lambda^\dagger), \Delta_2(\mu^\dagger)) = \mathrm{Ext}^k_{G_1}(\nabla_1(\lambda), \nabla_1(\mu))$$

for any $k \geqslant 0$ and $\lambda, \mu \in \Upsilon_1$. Similarly,

$$\mathrm{Ext}^k_{G_1}(\Delta_1(\lambda), \Delta_1(\mu)) = \mathrm{Ext}^k_{G_2}(\nabla_2(\lambda^\dagger), \nabla_2(\mu^\dagger))$$

for any $k \geqslant 0$ and $\lambda, \mu \in \Upsilon_1$.

Corollary 2.4. *For $\lambda, \mu \in \Upsilon_1$ we have*

$$(\mathbf{T}_1(\lambda) : \nabla_1(\mu)) = [\nabla_2(\mu^\dagger) : \mathbf{L}_2(\lambda^\dagger)].$$

Similarly,

$$(\mathbf{T}_2(\mu^\dagger) : \nabla_2(\lambda^\dagger)) = [\nabla_1(\lambda) : \mathbf{L}_1(\mu)].$$

Remark. In [7, Section 3] Donkin obtained similar results for $G_1 = GL_m$. He considered the set $\Upsilon(r) \subset \mathfrak{X}_1^+$ consisting of all weights with Young diagrams of r squares. In this case the corresponding generalized Schur algebra $S(G_1, \Upsilon(r))$ is the classical Schur algebra $S = S(m, r)$. Donkin proved that $\Lambda^r(\mathbf{V}^n)$ (where $\mathbf{V}$ is the natural m-dimensional G_1-module and $n \geqslant r, m$) is a full tilting S-module and that the conjugate algebra $S^\dagger$ is isomorphic to a generalized Schur algebra of the group $G_2 = GL_n$. In his approach the corresponding map $\lambda \mapsto \lambda^\dagger$ from Υ to $\Upsilon^\dagger \subset \mathfrak{X}_2^+$ is the transposition of the Young diagram.

3. Constructions

In this section we construct the $G_1 \times G_2$-module $\mathbf{M}$ for different pairs of classical groups G_1 and G_2 listed in the previous section. We also obtain some elementary properties of $\mathbf{M}$.

Let $\mathbf{V}$ be the natural G_1-module and let $\mathbf{W}$ be the natural G_2-module (recall that for a spinor group by the natural module we mean the natural module for the corresponding orthogonal group). We denote the standard bases of the vector spaces $\mathbf{V}$ and $\mathbf{W}$ as follows:

$\mathbf{V}$	Standard basis	$\mathbf{W}$	Standard basis
$\mathbf{k}^m$	$(\mathbf{e}_1, \ldots, \mathbf{e}_m)$	$\mathbf{k}^n$	$(\mathbf{f}_1, \ldots, \mathbf{f}_n)$
$\mathbf{k}^{2m}$	$(\mathbf{e}_1, \ldots, \mathbf{e}_m, \mathbf{e}_{-m}, \ldots, \mathbf{e}_{-1})$	$\mathbf{k}^{2n}$	$(\mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{f}_{-n}, \ldots, \mathbf{f}_{-1})$
$\mathbf{k}^{2m+1}$	$(\mathbf{e}_1, \ldots, \mathbf{e}_m, \mathbf{e}_0, \mathbf{e}_{-m}, \ldots, \mathbf{e}_{-1})$	$\mathbf{k}^{2n+1}$	$(\mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{f}_0, \mathbf{f}_{-n}, \ldots, \mathbf{f}_{-1})$

If $G_1 = GL_m$ (thus $G_2 = GL_n$), then we denote by $\mathbf{e}_{-i}$ the i-th element of the dual basis for $\mathbf{V}^*$ and by $\mathbf{f}_{-j}$ the j-th element of the dual basis for $\mathbf{W}^*$.

If $\mathbf{V} = \mathbf{k}^{2m}, \mathbf{k}^{2m+1}$, then we denote the linear span of the vectors $\mathbf{e}_{-m}, \ldots, \mathbf{e}_{-1}$ by $\mathbf{V}_-$. Similarly, if $\mathbf{W} = \mathbf{k}^{2n}, \mathbf{k}^{2n+1}$, then we denote the linear span of the vectors $\mathbf{f}_{-n}, \ldots, \mathbf{f}_{-1}$ by $\mathbf{W}_-$.

Let the vector space $\mathbf{N}$ and its subspaces $\mathbf{N}_1, \mathbf{N}_2$ be defined by the following table.

G_1	G_2	$\mathbf{N}$	$\mathbf{N}_1$	$\mathbf{N}_2$
GL_m	GL_n	$\mathbf{V}\otimes\mathbf{W}^* \oplus \mathbf{V}^*\otimes\mathbf{W}$	$\mathbf{V}\otimes\mathbf{W}^*$	$\mathbf{V}^*\otimes\mathbf{W}$
Sp_{2m}	Sp_{2n}	$\mathbf{V}\otimes\mathbf{W}$	$\mathbf{V}\otimes\mathbf{W}_-$	$\mathbf{V}_-\otimes\mathbf{W}$
O_{2m}	SO_{2n}	$\mathbf{V}\otimes\mathbf{W}$	$\mathbf{V}\otimes\mathbf{W}_-$	$\mathbf{V}_-\otimes\mathbf{W}$
O_{2m+1}	$Spin_{2n}$	$\mathbf{V}\otimes\mathbf{W}$	$\mathbf{V}\otimes\mathbf{W}_-$	$\mathbf{V}_-\otimes\mathbf{W} \oplus \mathbf{e}_0\otimes\mathbf{W}_-$
Pin_{2m}	SO_{2n+1}	$\mathbf{V}\otimes\mathbf{W}$	$\mathbf{V}\otimes\mathbf{W}_- \oplus \mathbf{V}_-\otimes\mathbf{f}_0$	$\mathbf{V}_-\otimes\mathbf{W}$
$Spin_{2m+1}$	$Spin_{2n+1}$	$\mathbf{V}\otimes\mathbf{W}$	$\mathbf{V}\otimes\mathbf{W}_- \oplus \mathbf{V}_-\otimes\mathbf{f}_0$	$\mathbf{V}_-\otimes\mathbf{W} \oplus \mathbf{e}_0\otimes\mathbf{W}_-$

The elements of the form $\mathbf{t}_{i,j} = \mathbf{e}_i \otimes \mathbf{f}_j$ constitute a basis for $\mathbf{N}$.

We introduce a quadratic form Φ on $\mathbf{N}$ as follows:

$$\Phi\Big(\sum_{j=-n}^{n} \mathbf{v}_j \otimes \mathbf{f}_j\Big) = \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{v}_{-j}\rangle_1 - \langle \mathbf{v}_0, \mathbf{v}_0\rangle_1,$$

where $\langle\ ,\ \rangle_1$ is the pairing of $\mathbf{V}^*$ with $\mathbf{V}$ in the general linear case, the standard G_1-invariant skew-symmetric bilinear form on $\mathbf{V}$ in the symplectic case, and the polarization of the standard G_1-invariant quadratic form on $\mathbf{V}$ in the orthogonal (or spinor) case. Clearly, the last term $\langle \mathbf{v}_0, \mathbf{v}_0\rangle_1$ appears only for $\mathbf{W} = \mathbf{k}^{2n+1}$. On the other hand, it is readily seen that

$$\Phi\Big(\sum_{i=-m}^{m} \mathbf{e}_i \otimes \mathbf{w}_i\Big) = \sum_{i=1}^{m} \langle \mathbf{w}_i, \mathbf{w}_{-i}\rangle_2 - \langle \mathbf{w}_0, \mathbf{w}_0\rangle_2.$$

Therefore the natural action of each of the groups G_1 and G_2 on $\mathbf{N}$ preserves Φ. The actions of these two groups on $\mathbf{N}$ are permutable, hence the group $G_1 \times G_2$ acts on $\mathbf{N}$ by orthogonal (w.r.t. Φ) transformations. This action of $G_1 \times G_2$ on $\mathbf{N}$ extends uniquely to the action on the Clifford algebra $C(\Phi)$ by automorphisms.

We put $\mathbf{M}_1 = \Lambda^\bullet \mathbf{N}_1$ and $\mathbf{M}_2 = \Lambda^\bullet \mathbf{N}_2$. If $G_1 = Pin_{2m}$, $Spin_{2m+1}$, then we have $\mathbf{M}_1 \simeq (\Lambda^\bullet \mathbf{V})^{\otimes n} \otimes (\Lambda^\bullet \mathbf{V}_-)$. Recall that $\Lambda^\bullet \mathbf{V}_-$ is the spinor G_1-module. For all other cases we have $\mathbf{M}_1 \simeq (\Lambda^\bullet \mathbf{V})^{\otimes n}$. Thus $\mathbf{M}_1$ has natural structure of G_1-module. Similarly, $\mathbf{M}_2$ has natural structure of G_2-module.

The standard G_1-contravariant form on $\mathbf{V}$ was introduced in Section 1. It naturally extends to the contravariant form on $\Lambda^\bullet \mathbf{V}$. If $G_1 = Pin_{2m}$, $Spin_{2m+1}$, then we have also the standard G_1-contravariant form on the spinor module. Thus we get the non-singular G_1-contravariant symmetric bilinear form $(\ ,\ )_1$ on $\mathbf{M}_1$. In the same way we get the nonsingular G_2-contravariant symmetric bilinear form $(\ ,\ )_2$ on $\mathbf{M}_2$.

Let

$$C = \begin{cases} C(\Phi), & \text{if } \dim \mathbf{N} \text{ is even;} \\ C_+(\Phi), & \text{if } \dim \mathbf{N} \text{ is odd.} \end{cases}$$

Since $\mathbf{N}_1$ and $\mathbf{N}_2$ are maximal isotropic subspaces of $\mathbf{N}$, we see that there are natural isomorphisms $\pi_1 : C \to \operatorname{End}_{\mathbf{k}} \mathbf{M}_1$ and $\pi_2 : C \to \operatorname{End}_{\mathbf{k}} \mathbf{M}_2$.

There is a unique C-isomorphism $\varphi : \mathbf{M}_1 \to \mathbf{M}_2$ such that

$$\varphi(\mathbf{1}) = \bigwedge_{i=-m}^{-1} \bigwedge_{j=1}^{n} \mathbf{t}_{i,j}.$$

Let us identify the C-modules $\mathbf{M}_1$ and $\mathbf{M}_2$ via the isomorphism φ. Clearly, the forms $(\ ,\ )_1$ and $(\ ,\ )_2$ coincide after this identification. We shall write $\mathbf{M}$ instead of $\mathbf{M}_1$ and $\mathbf{M}_2$; and $(\ ,\ )$ instead of $(\ ,\ )_1$ and $(\ ,\ )_2$. Thus $\mathbf{M}$ has the structures of G_1-module and G_2-module. In addition, the bilinear form $(\ ,\ )$ on $\mathbf{M}$ is G_1- and G_2-contravariant.

Proposition 3.1. *The actions of G_1 and G_2 on $\mathbf{M}$ are permutable. In other words, $\mathbf{M}$ is a $G_1 \times G_2$-module.*

Proof. We have the commutative diagram

$$\begin{array}{ccccc} G_1 & \xrightarrow{\rho_1} & GL(\mathbf{M}) & \xleftarrow{\rho_2} & G_2 \\ \downarrow & & \downarrow{\scriptstyle \zeta} & & \downarrow \\ \mathrm{Aut}(C) & \xrightarrow{\sim} & PGL(\mathbf{M}) & \xleftarrow{\sim} & \mathrm{Aut}(C) \end{array}$$

Here ζ is the canonical projection. The images of G_1 and G_2 in $PGL(\mathbf{M}) \simeq \mathrm{Aut}(C)$ commute. Let us show that the same is true for the images of G_1 and G_2 in $GL(\mathbf{M})$.

Suppose $g_1 \in G_1$ and $g_2 \in G_2$. Since their images $\zeta(\rho_1(g_1))$ and $\zeta(\rho_2(g_2))$ in $PGL(\mathbf{M})$ commute, we see that $\rho_1(g_1)^{-1}\rho_2(g_2)^{-1}\rho_1(g_1)\rho_2(g_2) \in GL(\mathbf{M})$ is a scalar operator. Clearly, its determinant is equal to 1, hence it is an element of the finite group of scalar operators with determinant 1. But the group G_2 is connected, therefore for all $g_1 \in G_1$, $g_2 \in G_2$ we have

$$\rho_1(g_1)^{-1}\rho_2(g_2)^{-1}\rho_1(g_1)\rho_2(g_2) = \rho_1(g_1)^{-1}\rho_2(e_2)^{-1}\rho_1(g_1)\rho_2(e_2) = \mathrm{id},$$

where e_2 is the unity element of the group G_2. Thus $\rho_1(g_1)$ and $\rho_2(g_2)$ commute for all $g_1 \in G_1$, $g_2 \in G_2$. $\square$

Further on we shall use the realization of $\mathbf{M}$ as $\mathbf{M}_1 = \Lambda^\bullet \mathbf{N}_1$ (resp. $\mathbf{M}_2 = \Lambda^\bullet \mathbf{N}_2$) to study $\mathbf{M}$ as a G_1-module (resp. G_2-module).

It is readily seen that all the dominant weights of the G_1-module $\mathbf{M}_1$ belong to Υ_1, and all the dominant weights of the G_2-module $\mathbf{M}_2$ belong to Υ_2.

Suppose $\lambda \in \Upsilon_1$. Let $k_1, \dots, k_m$ be the lengths of rows of the Young diagram $\mathfrak{Y}(\lambda)$ and let $k_1^\dagger, \dots, k_n^\dagger$ be the lengths of rows of the Young diagram $\mathfrak{Y}(\lambda^\dagger)$. Let $l = m - k_n^\dagger$ and $l^\dagger = n - k_m$. In other words, l is the number of nonempty rows of the diagram $\mathfrak{Y}(\lambda)$, and $l^\dagger$ is the number of nonempty rows of the diagram $\mathfrak{Y}(\lambda^\dagger)$.

Let us define the "row products" $\mathbf{r}_1^{(\lambda)}(i) \in \mathbf{M}_1$ and $\mathbf{r}_2^{(\lambda^\dagger)}(j) \in \mathbf{M}_2$, where $i = 1, \dots, m$ and $j = 1, \dots, n$. We put

$$\mathbf{r}_1^{(\lambda)}(i) = \mathbf{t}_{i,-n} \wedge \mathbf{t}_{i,-n+1} \wedge \dots \wedge \mathbf{t}_{i,-n+k_i-1}$$

and

$$\mathbf{r}_2^{(\lambda^\dagger)}(j) = \mathbf{t}_{-m,j} \wedge \dots \wedge \mathbf{t}_{-m+k_j^\dagger-1,j}.$$

If $G_1 = O_{2m}, O_{2m+1}$, then we put also

$$\mathbf{r}_2^{(\lambda^\dagger)}(-n) = \begin{cases} \mathbf{t}_{-m,-n} \wedge \dots \wedge \mathbf{t}_{-m+k_n^\dagger-1,-n}, & \text{if } G_1 = O_{2m}, \\ \mathbf{t}_{0,-n} \wedge \mathbf{t}_{-m,-n} \wedge \dots \wedge \mathbf{t}_{-m+k_n^\dagger-1,-n}, & \text{if } G_1 = O_{2m+1}. \end{cases}$$

Consider the following element of $\mathbf{M}_1$:

$$\mathbf{m}_1^\lambda = \begin{cases} \bigwedge_{i=1}^{l} \mathbf{r}_1^{(\lambda)}(i), & \text{if } G_1 \neq O_{2m}, O_{2m+1} \text{ or } \mathfrak{S}(\lambda) \neq -1, \\ \left(\bigwedge_{i=1}^{l} \mathbf{r}_1^{(\lambda)}(i)\right) \wedge \mathbf{q}(l), & \text{if } G_1 = O_{2m}, O_{2m+1} \text{ and } \mathfrak{S}(\lambda) = -1, \end{cases}$$

where

$$\mathbf{q}(l) = \begin{cases} \bigwedge_{i=l+1}^{m} (\mathbf{t}_{i,-n} \wedge \mathbf{t}_{-i,-n}), & \text{if } G_1 = O_{2m}, \\ \mathbf{t}_{0,-n} \wedge \bigwedge_{i=l+1}^{m} (\mathbf{t}_{i,-n} \wedge \mathbf{t}_{-i,-n}), & \text{if } G_1 = O_{2m+1}. \end{cases}$$

It is easily checked that $\mathbf{m}_1^\lambda$ is a G_1-extremal λ-weight vector.

Let $\mathbf{m}_2^\lambda = \varphi(\mathbf{m}_1^\lambda)$. Then

$$\mathbf{m}_2^\lambda = \begin{cases} \pm \bigwedge_{i=1}^{l^\dagger} \mathbf{r}_2^{(\lambda^\dagger)}(i), & \text{if } G_1 \neq O_{2m}, O_{2m+1} \text{ or } \mathfrak{S}(\lambda) \neq -1, \\ \pm\left(\bigwedge_{i=1}^{n-1} \mathbf{r}_2^{(\lambda^\dagger)}(i)\right) \wedge \mathbf{r}_2^{(\lambda^\dagger)}(-n), & \text{if } G_1 = O_{2m}, O_{2m+1} \text{ and } \mathfrak{S}(\lambda) = -1. \end{cases}$$

Clearly, $\mathbf{m}_2^\lambda$ is a G_2-extremal $\lambda^\dagger$-weight vector.

Denote by $\mathbf{m}(\lambda)$ the vector in $\mathbf{M}$ that corresponds to $\mathbf{m}_1^\lambda \in \mathbf{M}_1$ and to $\mathbf{m}_2^\lambda \in \mathbf{M}_2$ under the identification $\mathbf{M} = \mathbf{M}_1 = \mathbf{M}_2$.

Proposition 3.2. *The character of* $\mathbf{M}$ *as* $G_1 \times G_2$*-module is given by the formula*

$$\operatorname{ch}_{G_1 \times G_2} \mathbf{M} = \sum_{\lambda \in \Upsilon_1} \chi_1(\lambda)\chi_2(\lambda^\dagger),$$

where $\chi_1(\lambda)$ (*resp.* $\chi_2(\lambda^\dagger)$) *is the Weyl character for the group* G_1 (*resp.* G_2).

Proof. Suppose $\operatorname{char} \mathbf{k} = 0$. Then the well-known description of the ring of vector invariants for the classical group G_1 (see [14]) is essentially equivalent to the fact that the centralizer of the image of G_1 in $\operatorname{End}_{\mathbf{k}} \mathbf{M} \simeq C$ is generated by the image of the Lie algebra of the group G_2 (see [9]). It follows that as a $G_1 \times G_2$-module

$$\mathbf{M} \simeq \bigoplus_{i=1}^{r} \mathbf{L}_1^{(i)} \otimes \mathbf{L}_2^{(i)},$$

where $\mathbf{L}_1^{(i)}$'s (resp. $\mathbf{L}_2^{(i)}$'s) are simple G_1-modules (resp. G_2-modules), $\mathbf{L}_1^{(i)} \not\simeq \mathbf{L}_1^{(j)}$ and $\mathbf{L}_2^{(i)} \not\simeq \mathbf{L}_2^{(j)}$ for $i \neq j$.

We have shown that for each $\lambda \in \Upsilon_1$ the vector $\mathbf{m}(\lambda) \in \mathbf{M}$ is G_1- and G_2-extremal and belongs to the λ-weight space w.r.t. G_1 and to the $\lambda^\dagger$-weight space w.r.t. G_2. Hence for some $i \in [1, r]$ we have $\mathbf{L}_1^{(i)} = \mathbf{L}_1(\lambda)$ and $\mathbf{L}_2^{(i)} = \mathbf{L}_2(\lambda^\dagger)$. Since all the dominant weights of the G_1-module $\mathbf{M}$ belong to Υ_1, we see that for any $j \in [1, r]$ there is $\mu \in \Upsilon_1$ such that $\mathbf{L}_1^{(j)} = \mathbf{L}_1(\mu)$. Hence,

$$\mathbf{M} \simeq \bigoplus_{\lambda \in \Upsilon_1} \mathbf{L}_1(\lambda) \otimes \mathbf{L}_2(\lambda^\dagger).$$

and thus the character formula is true for $\operatorname{char}\Bbbk = 0$. But the character of $\mathbf{M}$ doesn't depend on $\operatorname{char}\Bbbk$. $\square$

Suppose $\mathbf{K}$ is a $G_1 \times G_2$-module; then by $\mathbf{K}^{\nu,\bullet}$ we denote the ν-weight space of $\mathbf{K}$ w.r.t. G_1 and by $\mathbf{K}^{\bullet,\beta}$ we denote the β-weight space of $\mathbf{K}$ w.r.t. G_2. Evidently, $\mathbf{K}^{\nu,\bullet}$ is a G_2-module and $\mathbf{K}^{\bullet,\beta}$ is a G_1-module. Let $\mathbf{K}^{\nu,\beta} = \mathbf{K}^{\nu,\bullet} \cap \mathbf{K}^{\bullet,\beta}$. Since $\lambda \mapsto \lambda^\dagger$ is an order-reversing bijection of Υ_1 onto Υ_2, we get from Proposition 3.2

Corollary 3.3. *If $\mu, \nu \in \Upsilon_1$ and $\nu \nleq \mu$, then* $\dim \mathbf{M}^{\nu,\mu^\dagger} = 0$.

Corollary 3.4. *For any $\lambda \in \Upsilon_1$ the $T_1 \times T_2$-module $\mathbf{M}^{\lambda,\lambda^\dagger}$ is generated by* $\mathbf{m}(\lambda)$.

Let $\mathbf{Y}_1(\lambda) = \langle G_1\mathbf{m}(\lambda)\rangle$ and $\mathbf{Y}_2(\lambda^\dagger) = \langle G_2\mathbf{m}(\lambda)\rangle$.

Proposition 3.5. *The G_1-module $\mathbf{Y}_1(\lambda)$ is isomorphic to $\Delta_1(\lambda)$ and the G_2-module $\mathbf{Y}_2(\lambda^\dagger)$ is isomorphic to $\Delta_2(\lambda^\dagger)$.*

Proof. Let us show that $\mathbf{Y}_1(\lambda) \simeq \Delta_1(\lambda)$.

Denote by l_j the height of j-th column of the Young diagram $\mathfrak{Y}(\lambda)$ ($j = 1, \dots, n$). Clearly, $l_1 = l$.

The highest weight vector $\mathbf{m}_1^\lambda$, which was defined in terms of the "row products" $\mathbf{r}_1^{(\lambda)}(i)$, may also be expressed in terms of the "column products"

$$\mathbf{c}_1^{(\lambda)}(j) = \mathbf{t}_{1,-n+j-1} \wedge \mathbf{t}_{2,-n+j-1} \wedge \dots \wedge \mathbf{t}_{l_j,-n+j-1}.$$

It is readily seen that

$$\mathbf{m}_1^\lambda = \begin{cases} \pm \bigwedge\limits_{j=1}^{n} \mathbf{c}_1^{(\lambda)}(j), & \text{if } G_1 \neq O_{2m}, O_{2m+1} \text{ or } \mathfrak{S}(\lambda) \neq -1, \\ \pm \mathbf{c}_1^{(\lambda)}(1) \wedge \mathbf{q}(l) \wedge \bigwedge\limits_{j=2}^{n} \mathbf{c}_1^{(\lambda)}(j), & \text{if } G_1 = O_{2m}, O_{2m+1} \text{ and } \mathfrak{S}(\lambda) = -1. \end{cases}$$

Let

$$q = \begin{cases} l, & \text{if } G_1 \neq O_{2m}, O_{2m+1} \text{ or } \mathfrak{S}(\lambda) \neq -1, \\ 2m - l, & \text{if } G_1 = O_{2m} \text{ and } \mathfrak{S}(\lambda) = -1, \\ 2m + 1 - l, & \text{if } G_1 = O_{2m+1} \text{ and } \mathfrak{S}(\lambda) = -1. \end{cases}$$

Now it is clear that $\mathbf{m}_1^\lambda$ is unique (up to action of T_1) highest weight vector in the G-module

$$\Lambda^q(\mathbf{V}\otimes \mathbf{f}_{-n}) \wedge \Lambda^{l_2}(\mathbf{V}\otimes \mathbf{f}_{-n+1}) \wedge \dots \wedge \Lambda^{l_n}(\mathbf{V}\otimes \mathbf{f}_{-1}) \simeq \Lambda^q\mathbf{V} \otimes \Lambda^{l_2}\mathbf{V} \otimes \dots \otimes \Lambda^{l_n}\mathbf{V}$$

(if $G_1 \neq Pin_{2m}, Spin_{2m+1}$) or in

$$\Lambda^{l_1}(\mathbf{V}\otimes \mathbf{f}_{-n}) \wedge \dots \wedge \Lambda^{l_n}(\mathbf{V}\otimes \mathbf{f}_{-1}) \wedge \Lambda^\bullet(\mathbf{V}_- \otimes \mathbf{f}_0) \simeq \Lambda^{l_1}\mathbf{V} \otimes \dots \otimes \Lambda^{l_n}\mathbf{V} \otimes \Lambda^\bullet\mathbf{V}_-$$

(if $G_1 = Pin_{2m}, Spin_{2m+1}$). By Corollary 1.9 it remains to show that the highest weight vector of $\Lambda^\bullet\mathbf{V}_-$ (for spinor groups) and the highest weight vector of $\Lambda^k\mathbf{V}$ (with any k for orthogonal groups and with $0 \leqslant k \leqslant m$ for all other groups) generate submodules that are isomorphic to Weyl modules.

For $G_1 = Pin_{2m}, Spin_{2m+1}$ the unique dominant weight of the spinor G_1-module is minimal in $\mathfrak{X}_1^+$, hence the spinor G_1-module is isomorphic to the Weyl module (which is simple). The same argument applies to $\Lambda^k\mathbf{V}$ for $G_1 = GL_m$.

For orthogonal groups (and thus for spinor groups) it can easily be checked that $\Lambda^k\mathbf{V}$ is generated by its highest weight vector. For example, we may apply the argument from [2, § 13], which is valid not only for $\operatorname{char}\mathbb{k} = 0$ (as stated) but for $\operatorname{char}\mathbb{k} \neq 2$.

For $G_1 = Sp_{2m}$ we may use [11, Lemma 2 and Proposition 1]. The authors of [11] assume that $\operatorname{char}\mathbb{k} \neq 2$, but the proof of the fact that we need doesn't depend on this assumption.

Thus we have proved that $\mathbf{Y}_1(\lambda) \simeq \Delta_1(\lambda)$.

The isomorphism $\mathbf{Y}_2(\lambda^\dagger) \simeq \Delta_2(\lambda^\dagger)$ is established in the same way. □

4. The proof of the main theorem

Suppose G_1, G_2, and $\mathbf{M}$ are as in the previous section. We continue to study the $G_1 \times G_2$-module $\mathbf{M}$.

Let $\Upsilon_1 = \{\lambda(1), \dots, \lambda(r)\}$, where $i < j$ whenever $\lambda(i) < \lambda(j)$. Then $\Upsilon_2 = \{\lambda^\dagger(1), \dots, \lambda^\dagger(r)\}$, and $i < j$ whenever $\lambda^\dagger(j) < \lambda^\dagger(i)$. For each $i = 1, \dots, r$ we put

$$\mathbf{M}_i = \sum_{j=1}^{i} \langle G_2 \mathbf{M}^{\bullet, \lambda^\dagger(j)} \rangle.$$

It is evident that $\mathbf{M}_i$ is a $G_1 \times G_2$-module and that $\mathbf{0} = \mathbf{M}_0 \subset \mathbf{M}_1 \subset \dots \subset \mathbf{M}_r = \mathbf{M}$. We put $\overline{\mathbf{M}}_i = \mathbf{M}_i / \mathbf{M}_{i-1}$.

For $\lambda \in \Upsilon_1$ we put $\mathbf{Z}_1(\lambda) = \{\mathbf{m} \in \mathbf{M}^{\bullet,\lambda^\dagger} \mid u\mathbf{m} = \mathbf{m} \text{ for all } u \in U_2^+\}$. In other words, $\mathbf{Z}_1(\lambda)$ is the subspace of all G_2-extremal vectors in $\mathbf{M}^{\bullet,\lambda^\dagger}$. Since by Corollaries 3.3 and 3.4 $\lambda^\dagger$ is the highest weight of the G_2-module $\mathbf{M}^{\lambda,\bullet}$, we have $\mathbf{M}^{\lambda,\lambda^\dagger} \subseteq \mathbf{Z}_1(\lambda)$, thus $\mathbf{m}(\lambda) \in \mathbf{Z}_1(\lambda)$. Moreover, since the action of G_1 on $\mathbf{M}$ is permutable with the action of G_2, we see that the subspace $\mathbf{Z}_1(\lambda)$ is G_1-invariant. In particular, $\mathbf{Y}_1(\lambda) = \langle G_1\mathbf{m}(\lambda)\rangle \subseteq \mathbf{Z}_1(\lambda)$.

Proposition 4.1. *Suppose $\lambda = \lambda(i) \in \Upsilon_1$. Then*

(1) *the $G_1 \times T_2$-module $\mathbf{Z}_1(\lambda)$ is isomorphic to $\Delta_1(\lambda) \otimes L_{\lambda^\dagger}$;*
(2) *the $G_1 \times G_2$-module $\overline{\mathbf{M}}_i$ is isomorphic to $\nabla_1(\lambda) \otimes \Delta_2(\lambda^\dagger)$.*

Remark. The special case $G_1 = Sp_{2m}$, $G_2 = SL_2$ was considered in [1].

Proof. Let us show that the first assertion of the proposition implies the second one. We need two lemmas.

Lemma 4.1.1. *G_1-module $\mathbf{Z}_1(\lambda)^\sigma$ is isomorphic to $\overline{\mathbf{M}}_i^{\bullet,\lambda^\dagger}$.*

Proof. Let $(\ ,\ )_{\mathbf{M}^{\bullet,\lambda^\dagger}}$ be the restriction of the form $(\ ,\ )$ to $\mathbf{M}^{\bullet,\lambda^\dagger}$, let $\mathbf{Z}_1(\lambda)^\perp$ be the orthogonal complement of $\mathbf{Z}_1(\lambda)$ w.r.t. $(\ ,\ )_{\mathbf{M}^{\bullet,\lambda^\dagger}}$, and let $\pi^{\bullet,\lambda^\dagger}$ be the orthoprojector of $\mathbf{M}$ onto $\mathbf{M}^{\bullet,\lambda^\dagger}$. We have

$$\mathbf{Z}_1(\lambda) = \Big(\bigcap_{u\in U_2^+} \operatorname{Ker}(u-\mathrm{id})\Big) \cap \mathbf{M}^{\bullet,\lambda^\dagger} = \pi^{\bullet,\lambda^\dagger}\Big(\bigcap_{u\in U_2^+} \operatorname{Ker}(u-\mathrm{id})\Big).$$

Here and further on we use the same notation for elements of G_2 and the corresponding elements of $\operatorname{End}\mathbf{M}$. Since the operators u and u^σ are adjoint w.r.t. the nonsingular G_2-contravariant form and since the anti-automorphism σ maps bijectively U_2^+ onto U_2^-, we get

$$\mathbf{Z}_1(\lambda)^\perp = \pi^{\bullet,\lambda^\dagger}\Big(\sum_{u'\in U_2^-} \operatorname{Im}(u'-\mathrm{id})\Big) = \Big(\sum_{u'\in U_2^-} \operatorname{Im}(u'-\mathrm{id})\Big) \cap \mathbf{M}^{\bullet,\lambda^\dagger}.$$

Let us prove that $\mathbf{Z}_1(\lambda)^\perp = \mathbf{M}_{i-1}^{\bullet,\lambda^\dagger}$. For any $u' \in U_2^-$, $\beta \in X_2^+$, and $\mathbf{m} \in \mathbf{M}^{\bullet,\beta}$ we have

$$u'\mathbf{m} - \mathbf{m} \in \bigoplus_{\gamma<\beta} \mathbf{M}^{\bullet,\gamma}.$$

Hence

$$\sum_{u'\in U_2^-} \operatorname{Im}(u'-\mathrm{id}) \cap \mathbf{M}^{\bullet,\lambda^\dagger} = \sum_{u'\in U_2^-} \Big((u'-\mathrm{id})\Big(\bigoplus_{\delta>\lambda^\dagger} \mathbf{M}^{\bullet,\delta}\Big)\Big) \cap \mathbf{M}^{\bullet,\lambda^\dagger}.$$

Further, if $\delta \in X_2$ is such that $\delta > \lambda^\dagger$ and $\mathbf{M}^{\bullet,\delta} \neq 0$, then $\delta \in W_2\gamma$, where $\gamma \in \Upsilon_2$, $\gamma > \lambda^\dagger$, and W_2 is the Weyl group for G_2. Denote by $\varkappa$ the preimage of γ under the bijection $\Upsilon_1 \to \Upsilon_2$, then $\gamma = \varkappa^\dagger$. Obviously, $\varkappa < \lambda$, thus $\varkappa = \lambda(j)$ for some $j < i$. Hence,

$$\mathbf{M}^{\bullet,\delta} \subseteq \langle G_2 \mathbf{M}^{\bullet,\varkappa^\dagger}\rangle \subseteq \mathbf{M}_j \subseteq \mathbf{M}_{i-1}.$$

Therefore,

$$\sum_{u'\in U_2^-} \Big((u'-\mathrm{id})\Big(\bigoplus_{\delta>\lambda^\dagger} \mathbf{M}^{\bullet,\delta}\Big)\Big) \cap \mathbf{M}^{\bullet,\lambda^\dagger} \subseteq \mathbf{M}_{i-1}^{\bullet,\lambda^\dagger}$$

and

$$\mathbf{Z}_1(\lambda)^\perp \subseteq \mathbf{M}_{i-1}^{\bullet,\lambda^\dagger}.$$

On the other hand,

$$\mathbf{M}_{i-1}^{\bullet,\lambda^\dagger} = \pi^{\bullet,\lambda^\dagger}\left(\sum_{j=1}^{i-1} \left\langle G_2 \mathbf{M}^{\bullet,\lambda^\dagger(j)}\right\rangle\right) = \sum_{j=1}^{i-1} \pi^{\bullet,\lambda^\dagger} \left\langle G_2 \mathbf{M}^{\bullet,\lambda^\dagger(j)}\right\rangle.$$

Let us show that the last sum lies in $\mathbf{Z}_1(\lambda)^\perp$.

Suppose $g \in G_2$. Consider the Bruhat decomposition $g = u'wu$, where $u' \in U_2^-$, $w \in N_{G_2}(T_2)$, and $u \in U_2^+$. For any $\mathbf{m} \in \mathbf{M}^{\bullet,\lambda^\dagger(j)}$ with $j < i$ we have

$$w u \mathbf{m} \in \bigoplus_{\gamma > \lambda^\dagger(j)} \mathbf{M}^{\bullet,\bar{w}\gamma},$$

where $\bar{w}$ is the image of w in W_2. Note that if $\lambda^\dagger = \bar{w}\gamma$, then $\lambda^\dagger \geqslant \gamma$; besides, $\lambda^\dagger \not\geqslant \lambda^\dagger(j)$. Hence for any $\gamma \geqslant \lambda^\dagger(j)$ we have $\lambda^\dagger \neq \bar{w}\gamma$.

Therefore $\pi^{\bullet,\lambda^\dagger} wu\mathbf{m} = 0$. It follows that

$$\pi^{\bullet,\lambda^\dagger} u'wu\mathbf{m} = \pi^{\bullet,\lambda^\dagger}(u' - \mathrm{id})wu\mathbf{m}.$$

Hence,

$$\pi^{\bullet,\lambda^\dagger} \left\langle G_2 \mathbf{M}^{\bullet,\lambda^\dagger(j)} \right\rangle \subseteq \pi^{\bullet,\lambda^\dagger}\Big(\sum_{u' \in U_2^-} \mathrm{Im}\big(u' - \mathrm{id}\big)\Big) = \mathbf{Z}_1(\lambda)^\perp.$$

and

$$\sum_{j=1}^{i-1} \pi^{\bullet,\lambda^\dagger} \left\langle G_2 \mathbf{M}^{\bullet,\lambda^\dagger(j)} \right\rangle \subseteq \mathbf{Z}_1(\lambda)^\perp.$$

Thus,

$$\mathbf{M}_{i-1}^{\bullet,\lambda^\dagger} = \mathbf{Z}_1(\lambda)^\perp.$$

Since the form $(\ ,\ )_{\mathbf{M}^{\bullet,\lambda^\dagger}}$ is nonsingular and G_1-contravariant, we get

$$\mathbf{Z}_1(\lambda)^\sigma \simeq \mathbf{M}^{\bullet,\lambda^\dagger}/\mathbf{M}_{i-1}^{\bullet,\lambda^\dagger} \simeq \overline{\mathbf{M}}_i^{\bullet,\lambda^\dagger}. \quad \square$$

Lemma 4.1.2. *The $\lambda^\dagger$-weight space $\overline{\mathbf{M}}_i^{\bullet,\lambda^\dagger} \subseteq \overline{\mathbf{M}}_i$ consists of G_2-extremal vectors.*

Proof. Let $\bar{\mathbf{m}} \in \overline{\mathbf{M}}_i^{\bullet,\lambda^\dagger}$, that is $\bar{\mathbf{m}} = \mathbf{m} + \mathbf{M}_{i-1}$, where $\mathbf{m} \in \mathbf{M}_i^{\bullet,\lambda^\dagger} = \mathbf{M}^{\bullet,\lambda^\dagger}$. Then for any $u \in U_2^+$ we have $u\mathbf{m} = \mathbf{m} + \mathbf{n}$ with $\mathbf{n} \in \bigoplus_{\delta > \lambda^\dagger} \mathbf{M}^{\bullet,\delta}$. Besides, if $\delta > \lambda^\dagger$ and $\mathbf{M}^{\bullet,\delta} \neq 0$, then $\mathbf{M}^{\bullet,\delta} \subseteq \mathbf{M}_{i-1}$ (see the proof of Lemma 4.1.1). Hence $u\bar{\mathbf{m}} = \bar{\mathbf{m}}$, thus $\bar{\mathbf{m}}$ is a G_2-extremal vector. $\square$

Suppose the first assertion of the proposition is true. Then by Lemma 4.1.1 there exists a G_1 -isomorphism $\eta : \nabla_1(\lambda) \to \overline{\mathbf{M}}_i^{\bullet,\lambda^\dagger}$. For any $\mathbf{v} \in \nabla_1(\lambda)$ its image $\eta(\mathbf{v}) \in \overline{\mathbf{M}}_i^{\bullet,\lambda^\dagger}$ is a G_2-extremal vector in $\overline{\mathbf{M}}_i$ (see Lemma 4.1.2), therefore $\eta(\mathbf{v})$ is the highest weight vector of $\langle G_2\eta(\mathbf{v})\rangle$. Hence there is a G_2-epimorphism $\theta_{\mathbf{v}} : \Delta_2(\lambda^\dagger) \to \langle G_2\eta(\mathbf{v})\rangle \subseteq \overline{\mathbf{M}}_i$ that takes the fixed highest weight vector $\mathbf{h} \in \Delta_2(\lambda^\dagger)$ to $\eta(\mathbf{v})$. Let τ be a $G_1 \times G_2$-homomorphism $\nabla_1(\lambda) \otimes \Delta_2(\lambda^\dagger) \to \overline{\mathbf{M}}_i$ such that $\tau(\sum \mathbf{v} \otimes \mathbf{w}) = \sum \theta_{\mathbf{v}}(\mathbf{w})$ for all $\mathbf{v} \in \nabla_1(\lambda)$, $\mathbf{w} \in \Delta_2(\lambda^\dagger)$. By definition we have $\overline{\mathbf{M}}_i = \langle G_2\overline{\mathbf{M}}_i^{\bullet,\lambda^\dagger}\rangle$. Hence τ is an epimorphism.

Let us prove that τ is an isomorphism. Assume the converse: $\operatorname{Ker}\tau \neq 0$. Let $\mathbf{L}$ be a simple $G_1 \times G_2$-submodule of $\operatorname{Ker}\tau$. Then $\mathbf{L} \simeq \mathbf{L}_1(\mu) \otimes \mathbf{L}_2(\nu^\dagger)$ for some $\mu, \nu \in \Upsilon_1$. We have the embeddings

$$\begin{aligned}\mathbf{L} &\hookrightarrow \operatorname{Soc}_{G_1\times G_2}(\operatorname{Ker}\tau) \hookrightarrow \operatorname{Soc}_{G_1\times G_2}(\nabla_1(\lambda)\otimes\Delta_2(\lambda^\dagger)) \\ &\hookrightarrow (\operatorname{Soc}_{G_1}\nabla_1(\lambda))\otimes\Delta_2(\lambda^\dagger).\end{aligned}$$

Since $\operatorname{Soc}_{G_1}\nabla_1(\lambda) \simeq \mathbf{L}_1(\lambda)$, we see that $\mu = \lambda$, thus $\mathbf{L} \simeq \mathbf{L}_1(\lambda)\otimes\mathbf{L}_2(\nu^\dagger)$.

Let $\mathbf{z}$ be the highest weight vector of $\mathbf{L}$ (w.r.t. $G_1 \times G_2$). Then $\mathbf{z}$ belongs to the tensor product of the weight spaces $\nabla_1(\lambda)^\lambda \otimes \Delta_2(\lambda^\dagger)^{\nu^\dagger}$. Since $\dim\nabla_1(\lambda)^\lambda = 1$, we have $\mathbf{z} = \mathbf{x}\otimes\mathbf{y}$ with $\mathbf{x}\in\nabla_1(\lambda)^\lambda$, $\mathbf{y}\in\Delta_2(\lambda^\dagger)^{\nu^\dagger}$. By definition $\mathbf{z}\in\mathbf{L}\subseteq\operatorname{Ker}\tau$, hence $\tau(\mathbf{z}) = \theta_{\mathbf{x}}(\mathbf{y}) = 0$. So the epimorphism $\theta_{\mathbf{x}}$ has nonzero kernel, therefore $\Delta_2(\lambda^\dagger) \not\simeq \langle G_2\eta(\mathbf{x})\rangle$.

Since $\mathbf{x}\in\nabla_1(\lambda)^\lambda$ and η is a G_1-isomorphism, we see that $\eta(\mathbf{x})$ is a unique (up to scalar multiple) nonzero λ-weight vector in the G_1-module $\overline{\mathbf{M}}_i^{\bullet,\lambda^\dagger}$. Hence $\langle\eta(\mathbf{x})\rangle = \mathbf{M}^{\lambda,\lambda^\dagger} + \mathbf{M}_{i-1}$ and

$$\langle G_2\eta(\mathbf{x})\rangle = \langle G_2\mathbf{M}^{\lambda,\lambda^\dagger}\rangle + \mathbf{M}_{i-1} = \mathbf{Y}_2(\lambda^\dagger) + \mathbf{M}_{i-1}.$$

Further, we have $\mathbf{Y}_2(\lambda^\dagger) \subseteq \mathbf{M}^{\lambda,\bullet}$ and

$$\mathbf{M}_{i-1}^{\lambda,\bullet} = \sum_{j=1}^{i-1}\langle G_2\mathbf{M}^{\bullet,\lambda^\dagger(j)}\rangle^{\lambda,\bullet} = \sum_{j=1}^{i-1}\langle G_2\mathbf{M}^{\lambda,\lambda^\dagger(j)}\rangle = 0$$

(the last equality follows from Corollary 3.3). Thus $\mathbf{Y}_2(\lambda^\dagger)\cap\mathbf{M}_{i-1} = 0$ and

$$\langle G_2\eta(\mathbf{x})\rangle \simeq \mathbf{Y}_2(\lambda^\dagger) \simeq \Delta_2(\lambda^\dagger)$$

(we use Proposition 3.5). But we saw that $\operatorname{Ker}\tau \neq 0$ implies

$$\langle G_2\eta(\mathbf{x})\rangle \not\simeq \Delta_2(\lambda^\dagger).$$

This contradiction proves that τ is an isomorphism.

It remains to prove the first assertion of the proposition. Let us do it by induction on i.

Let $i = 1$. Then $\lambda = \lambda(i)$ is the minimal weight in Υ_1 and $\lambda^\dagger = \lambda^\dagger(i)$ is the maximal weight in Υ_2. Using Proposition 3.2, we get $\operatorname{ch}_{G_1}\mathbf{M}^{\bullet,\lambda^\dagger} = \chi_1(\lambda)$. Since λ is minimal, we see that this character is irreducible. Hence the G_1-modules $\mathbf{M}^{\bullet,\lambda^\dagger}$, $\mathbf{L}_1(\lambda)$, and $\Delta_1(\lambda)$ are isomorphic. At the same time $\mathbf{Z}_1(\lambda)$ is a nonzero submodule of $\mathbf{M}^{\bullet,\lambda^\dagger}$, therefore $\mathbf{Z}_1(\lambda) = \mathbf{M}^{\bullet,\lambda^\dagger} \simeq \Delta_1(\lambda)$. Thus for $i = 1$ the first assertion of the proposition is true.

Let $j > 1$ and $\varkappa = \lambda(j) \in \Upsilon_1$. Suppose the first assertion of the proposition is true for all $i < j$. Then the second assertion is also true for all $i < j$. Let us prove the first assertion for $i = j$.

Lemma 4.1.3. $\operatorname{ch}_{G_1}\overline{\mathbf{M}}_j^{\bullet,\varkappa^\dagger} = \chi_1(\varkappa)$.

Proof. By Proposition 3.2 we have

$$\mathrm{ch}_{G_1\times G_2}\mathbf{M}=\sum_{\lambda\in\Upsilon_1}\chi_1(\lambda)\chi_2(\lambda^\dagger).$$

Hence,

$$\mathrm{ch}_{G_1}\mathbf{M}^{\bullet,\varkappa^\dagger}=\sum_{\lambda\in\Upsilon_1}\chi_1(\lambda)h_\lambda(\varkappa^\dagger)=\sum_{i=1}^{j}\chi_1(\lambda(i))h_{\lambda(i)}(\varkappa^\dagger),$$

where $h_\lambda(\varkappa^\dagger)$ is the dimension of $\varkappa^\dagger$-weight space for the Weyl module $\Delta_2(\lambda^\dagger)$.

On the other hand, the second assertion of the proposition for $i<j$ implies that

$$\mathrm{ch}_{G_1\times G_2}\mathbf{M}_{j-1}=\mathrm{ch}_{G_1\times G_2}\bigoplus_{i=1}^{j-1}\overline{\mathbf{M}}_i=\sum_{i=1}^{j-1}\chi_1(\lambda(i))\chi_2(\lambda^\dagger(i)).$$

Therefore

$$\mathrm{ch}_{G_1}\mathbf{M}_{j-1}^{\bullet,\varkappa^\dagger}=\sum_{i=1}^{j-1}\chi_1(\lambda(i))h_{\lambda(i)}(\varkappa^\dagger).$$

Since $\overline{\mathbf{M}}_j^{\bullet,\varkappa^\dagger}\simeq\mathbf{M}^{\bullet,\varkappa^\dagger}/\mathbf{M}_{j-1}^{\bullet,\varkappa^\dagger}$, we have

$$\begin{aligned}\mathrm{ch}_{G_1}\overline{\mathbf{M}}_j^{\bullet,\varkappa^\dagger}&=\sum_{i=1}^{j}\chi_1(\lambda(i))h_{\lambda(i)}(\varkappa^\dagger)-\sum_{i=1}^{j-1}\chi_1(\lambda(i))h_{\lambda(i)}(\varkappa^\dagger)\\&=\chi_1(\varkappa)h_\varkappa(\varkappa^\dagger)=\chi_1(\varkappa).\quad\square\end{aligned}$$

By Proposition 3.5 we have $\mathrm{ch}\,\mathbf{Y}_1(\varkappa)=\mathrm{ch}\,\Delta_1(\varkappa)=\chi_1(\varkappa)$. Combining Lemmas 4.1.1 and 4.1.3, we get $\mathrm{ch}_{G_1}\mathbf{Z}_1(\varkappa)=\chi_1(\varkappa)$. But $\mathbf{Y}_1(\varkappa)$ is a submodule of $\mathbf{Z}_1(\varkappa)$, hence as a G_1-module $\mathbf{Z}_1(\varkappa)=\mathbf{Y}_1(\varkappa)\simeq\Delta_1(\varkappa)$. $\square$

Let $\mathbf{Z}_2(\lambda^\dagger)=\{\mathbf{m}\in\mathbf{M}^{\lambda,\bullet}\mid u\mathbf{m}=\mathbf{m}\text{ for all }u\in U_1^+\}$. Clearly, it is a $T_1\times G_2$-module, and $\mathbf{Y}_2(\lambda^\dagger)\subseteq\mathbf{Z}_2(\lambda^\dagger)$.

Proposition 4.2. *Suppose $\lambda=\lambda(i)\in\Upsilon_1$. Then the $T_1\times G_2$-module $\mathbf{Z}_2(\lambda^\dagger)$ is isomorphic to $l_\lambda\otimes\Delta_2(\lambda^\dagger)$.*

Proof. We must interchange G_1 with G_2 and repeat the proof of Proposition 4.1 with filtration

$$\cdots\supset\sum_{j=i}^{r}\langle G_2\mathbf{M}^{\bullet,\lambda^\dagger(j)}\rangle\supset\sum_{j=i+1}^{r}\langle G_2\mathbf{M}^{\bullet,\lambda^\dagger(j)}\rangle\supset\ldots$$

instead of filtration $\cdots\subset\mathbf{M}_i\subset\mathbf{M}_{i+1}\subset\ldots.$

Since the group G_1 is not always connected, it is necessary to make some changes in the proofs of the lemmas. For instance, if G_1 is not a connected group, then its Weyl group doesn't act on X_1, but acts on $\check{X}_1$ (the weight lattice for the connected group $\breve{G}_1$); hence we must use the weight decomposition w.r.t. $\breve{G}_1$ as well as the weight decomposition w.r.t. G_1. The details are left to the reader. □

Consider the generalized Schur algebras $S_1 = S(G_1, \Upsilon_1)$ and $S_2 = S(G_2, \Upsilon_2)$. Recall that the G_1-module $\mathbf{M}$ belongs to Υ_1, hence it is an S_1-module; similarly, G_2-module $\mathbf{M}$ belongs to Υ_2, hence it is an S_2-module.

Proposition 4.3. $\mathbf{M}$ *is a full tilting S_1-module and a full tilting S_2-module.*

Proof. First let us show that $\mathbf{M} \in \mathcal{T}_{S_1}$. By the second assertion of Proposition 4.1 the G_1-module $\mathbf{M}$ admits a ∇_1-filtration. Since the G_1-contravariant form $(\ ,\ )$ is nonsingular, we see that $\mathbf{M}^\sigma \simeq \mathbf{M}$, hence $\mathbf{M}$ also admits a Δ_1-filtration.

Now let us show that $\mathbf{T}_1(\lambda)$ occurs as a direct summand of the G_1-module $\mathbf{M}$ for each $\lambda \in \Upsilon_1$. Indeed, the G_1-submodule $\mathbf{M}^{\bullet,\lambda^\dagger}$ is a direct summand of $\mathbf{M}$, hence $\mathbf{M}^{\bullet,\lambda^\dagger} \in \mathcal{T}_{S_1}$ and $\mathbf{M}^{\bullet,\lambda^\dagger}$ is a sum of $\mathbf{T}_1(\mu)$'s. Moreover, by Corollary 3.3 we see that λ is highest weight of the G_1-module $\mathbf{M}^{\bullet,\lambda^\dagger}$. Thus $\mathbf{T}_1(\lambda)$ occurs as a direct summand of $\mathbf{M}^{\bullet,\lambda^\dagger}$ and hence of $\mathbf{M}$. The same argument show that $\mathbf{M}$ is a full tilting S_2-module. □

Since $\mathbf{M}$ is an S_2-module, we have a homomorphism $\zeta : S_2 \to \operatorname{End}_{\mathbf{k}} \mathbf{M}$. The action of S_2 is permutable with the action of S_1, hence $\zeta S_2 \subseteq \operatorname{End}_{S_1} \mathbf{M}$.

Proposition 4.4. $\zeta : S_2 \to \operatorname{End}_{S_1} \mathbf{M}$ *is an isomorphism.*

Proof. First let us show that $\operatorname{Ker} \zeta = 0$.

Consider S_2 as a left S_2-module. By [6, (3.2a)], S_2 admits a Δ_2-filtration. Combining this with [12, Lemma 6], we see that there exists an embedding $S_2 \hookrightarrow \mathbf{N}$, where $\mathbf{N} \in \mathcal{T}_{S_2}$. Since S_2 is an algebra with unit, we see that the representation of S_2 in S_2 is faithful, hence the representation of S_2 in $\mathbf{N}$ is also faithful. Therefore the representation of S_2 in any full tilting module is faithful. Thus ζ is a monomorphism.

Now let us prove that $\dim \operatorname{End}_{S_1} \mathbf{M} = \dim S_2$. We know that

$$\dim S_2 = \sum_{\lambda \in \Upsilon_1} (\dim \Delta_2(\lambda^\dagger))^2.$$

Note that

$$\operatorname{End}_{S_1} \mathbf{M} \simeq \operatorname{Hom}_{G_1}(\mathbf{M}, \mathbf{M}) \simeq \operatorname{Hom}_{G_1}(\mathbf{k}, \mathbf{M} \otimes \mathbf{M}^*) \simeq \operatorname{Hom}_{G_1}(\nabla_1(0), \mathbf{M} \otimes \mathbf{M}^*).$$

By Proposition 4.3 we have $\mathbf{M} \in \mathcal{T}_{G_1}$. Hence $\mathbf{M} \otimes \mathbf{M}^* \in \mathcal{T}_{G_1}$, by Proposition 1.10. In particular, $\mathbf{M} \otimes \mathbf{M}^*$ admits a ∇-filtration.

Lemma 4.4.1. *Suppose a G_1-module $\mathbf{N}$ admits a ∇_1-filtration. Then*

$$\dim \operatorname{Hom}_{G_1}(\nabla_1(0), \mathbf{N}) = (\mathbf{N} : \nabla_1(0)).$$

Proof. Suppose we have a short exact sequence of G_1-modules

$$\mathbf{0} \to \nabla_1(\lambda) \to \mathbf{N} \to \mathbf{K} \to \mathbf{0},$$

and $\mathbf{L}$ admits a ∇_1-filtration. Applying the functor $\mathrm{Hom}_{G_1}(\nabla_1(0), \bullet)$, we get the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{G_1}(\nabla_1(0), \nabla_1(\lambda)) \to \mathrm{Hom}_{G_1}(\nabla_1(0), \mathbf{N}) \to \mathrm{Hom}_{G_1}(\nabla_1(0), \mathbf{K}) \\ \to \mathrm{Ext}^1_{G_1}(\nabla_1(0), \nabla_1(\lambda)) \to \dots \end{aligned}$$

Since $0 \not> \lambda$, we have $\mathrm{Ext}^1_{G_1}(\nabla_1(0), \nabla_1(\lambda)) = 0$. Hence we obtain

$$\dim \mathrm{Hom}_{G_1}(\nabla_1(0), \mathbf{N}) = \dim \mathrm{Hom}_{G_1}(\nabla_1(0), \mathbf{K}) + \dim \mathrm{Hom}_{G_1}(\nabla_1(0), \nabla_1(\lambda)).$$

Since $\mathrm{Soc}_{G_1} \nabla_1(\lambda) \simeq \mathbf{L}_1(\lambda)$, we have

$$\mathrm{Hom}_{G_1}(\nabla_1(0), \nabla_1(\lambda)) \simeq \begin{cases} 0, & \text{if } \lambda \neq 0; \\ \mathbf{k}, & \text{if } \lambda = 0. \end{cases}$$

Thus,

$$\dim \mathrm{Hom}_{G_1}(\nabla_1(0), \mathbf{N}) = \dim \mathrm{Hom}_{G_1}(\nabla_1(0), \mathbf{K}) + \delta_{0,\lambda}.$$

Now the lemma follows by induction on the length of ∇_1-filtration. □

Using the lemma, we get

$$\dim \mathrm{End}_{S_1} \mathbf{M} = \dim \mathrm{Hom}_{G_1}(\nabla_1(0), \mathbf{M} \otimes \mathbf{M}^*) = (\mathbf{M} \otimes \mathbf{M}^* : \nabla_1(0)).$$

The last number equals $(\mathrm{ch}_1(\mathbf{M} \otimes \mathbf{M}^*) : \chi_1(0))$, that is the coefficient of $\chi_1(0)$ in the expansion of $\mathrm{ch}_1(\mathbf{M} \otimes \mathbf{M}^*)$ as a linear combination of Weyl characters. By Proposition 3.2 we have

$$\mathrm{ch}_1(\mathbf{M} \otimes \mathbf{M}^*) = \left(\sum_{\lambda \in \Upsilon_1} \chi_1(\lambda) \dim \Delta_2(\lambda^\dagger) \right) \cdot \left(\sum_{\lambda \in \Upsilon_1} \chi_1(-\lambda) \dim \Delta_2(\lambda^\dagger) \right).$$

Clearly,

$$(\chi_1(\lambda)\chi_1(-\mu) : \chi_1(0)) = \begin{cases} 1, & \text{if } \mu = \lambda, \\ 0, & \text{if } \mu \neq \lambda. \end{cases}$$

Thus,

$$\dim \mathrm{End}_{S_1} \mathbf{M} = \sum_{\lambda \in \Upsilon_1} (\dim \Delta_2(\lambda^\dagger))^2.$$

This completes the proof of the proposition. □

Proposition 4.5. *For each $\lambda \in \Upsilon_1$ there is an S_1-isomorphism $\Delta_1(\lambda) \xrightarrow{\sim} \mathrm{Hom}_{S_2}(\mathbf{M}, \nabla_2(\lambda^\dagger))$ and an S_2-isomorphism $\Delta_2(\lambda^\dagger) \xrightarrow{\sim} \mathrm{Hom}_{S_1}(\mathbf{M}, \nabla_1(\lambda))$.*

Proof. We have $\mathrm{Hom}_{S_2}(\mathbf{M}, \nabla_2(\lambda^\dagger)) = \mathrm{Hom}_{G_2}(\mathbf{M}, \nabla_2(\lambda^\dagger))$. Using the properties of the functor "Transpose", we obtain

$$\mathrm{Hom}_{G_2}(\mathbf{M}, \nabla_2(\lambda^\dagger)) \simeq \mathrm{Hom}_{G_2}(\nabla_2(\lambda^\dagger)^\sigma, \mathbf{M}^\sigma) \simeq \mathrm{Hom}_{G_2}(\Delta_2(\lambda^\dagger), \mathbf{M}).$$

The last module is naturally isomorphic to $\mathbf{Z}_1(\lambda)$, i.e. the subspace of all G_2-extremal $\lambda^\dagger$-weight vectors in $\mathbf{M}$. But $\mathbf{Z}_1(\lambda) \simeq \Delta_1(\lambda)$, by Proposition 4.1.

In the same way from Proposition 4.2 we obtain an S_2-isomorphism $\Delta_2(\lambda^\dagger) \xrightarrow{\sim} \mathrm{Hom}_{S_1}(\mathbf{M}, \nabla_1(\lambda))$. $\square$

Propositions 4.1, 4.2, 4.3, 4.4 and 4.5 imply Theorem 2.1.

5. Invariants

Suppose G is one of the groups GL_m, Sp_{2m}, or O_m. Let $\mathbf{V}$ be the natural G-module. For $G = GL_m$ (resp. $G = Sp_{2m}$) we denote by $(\mathbf{e}_1, \dots, \mathbf{e}_m)$ (resp. $(\mathbf{e}_1, \dots, \mathbf{e}_m, \mathbf{e}_{-m}, \dots, \mathbf{e}_{-1})$) the standard basis for the vector space $\mathbf{V}$. If $G = GL_m$, then we also need the dual basis $(\mathbf{e}_1^*, \dots, \mathbf{e}_m^*)$ for $\mathbf{V}^*$. If $G = O_m$, then we assume that $\operatorname{char} \Bbbk \neq 2$, and denote by $(\mathbf{e}_1, \dots, \mathbf{e}_m)$ an *orthonormal* basis of $\mathbf{V}$ (thus we are changing our notation).

Let us consider the space

$$\underbrace{\mathbf{V} \oplus \cdots \oplus \mathbf{V}}_{n_1} \oplus \underbrace{\mathbf{V}^* \oplus \cdots \oplus \mathbf{V}^*}_{n_2}. \tag{1}$$

We put

$$A = \Lambda^\bullet(\underbrace{\mathbf{V} \oplus \cdots \oplus \mathbf{V}}_{n_1} \oplus \underbrace{\mathbf{V}^* \oplus \cdots \oplus \mathbf{V}^*}_{n_2}).$$

Our goal is to describe the subalgebra A^G of all G-invariants in A.

Let $n = n_1 + n_2$. For $G = Sp_{2m}$ or $G = O_m$ the G-modules $\mathbf{V}$ and $\mathbf{V}^*$ are isomorphic, so we shall consider only the case $n_2 = 0$ (thus $n = n_1$).

Denote the basis of the j-th summand in (1) by $(\mathbf{e}_{1j}, \dots, \mathbf{e}_{mj})$ (if $1 \leqslant j \leqslant n_1$) or by $(\mathbf{e}_{1j}^*, \dots, \mathbf{e}_{mj}^*)$ (if $n_1 + 1 \leqslant j \leqslant n$).

Consider the following elements $\psi_{rs}^{(k)} \in A$:

(1) Suppose $G = GL_m$. Then we put

$$\psi_{rs}^{(k)} = \sum_{i_1=1}^{m-k+1} \sum_{i_2=i_1+1}^{m-k+2} \cdots \sum_{i_k=i_{k-1}+1}^{m} \mathbf{e}_{i_1 r} \wedge \mathbf{e}_{i_1 s}^* \wedge \mathbf{e}_{i_2 r} \wedge \mathbf{e}_{i_2 s}^* \wedge \cdots \wedge \mathbf{e}_{i_k r} \wedge \mathbf{e}_{i_k s}^*$$

for $1 \leqslant r \leqslant n_1$, $n_1 + 1 \leqslant s \leqslant n$, and $1 \leqslant k \leqslant m$.

(2) Suppose $G = Sp_{2m}$. Then we put

$$\psi_{rr}^{(k)} = \sum_{i_1=1}^{m-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{m} \mathbf{e}_{i_1 r} \wedge \mathbf{e}_{-i_1 r} \wedge \cdots \wedge \mathbf{e}_{i_k r} \wedge \mathbf{e}_{-i_k r}$$

for $1 \leqslant r \leqslant n$ and $1 \leqslant k \leqslant m$. We also put

$$\psi_{rs}^{(k)} = \sum_{i_1=1}^{m-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{m} (\mathbf{e}_{i_1 r} \wedge \mathbf{e}_{-i_1 s} - \mathbf{e}_{-i_1 r} \wedge \mathbf{e}_{i_1 s}) \wedge \ldots \wedge (\mathbf{e}_{i_k r} \wedge \mathbf{e}_{-i_k s} - \mathbf{e}_{-i_k r} \wedge \mathbf{e}_{i_k s})$$

for $1 \leqslant r < s \leqslant n$ and $1 \leqslant k \leqslant m$.

(3) Suppose $G = O_m$. Then we put

$$\psi_{rs}^{(k)} = \sum_{i_1=1}^{m-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{m} \mathbf{e}_{i_1 r} \wedge \mathbf{e}_{i_1 s} \wedge \cdots \wedge \mathbf{e}_{i_k r} \wedge \mathbf{e}_{i_k s}$$

for $1 \leqslant r < s \leqslant n$ and $1 \leqslant k \leqslant m$.

In all the cases we put $\psi_{rs}^{(0)} = 1$ and $\psi_{rs}^{(k)} = 0$ for $k > m$.

It is readily seen that $\psi_{rs}^{(1)}$ is the classical basic invariant of the group G (see [14]) and $\psi_{rs}^{(k)}$ is its k-th divided power.

Theorem 5.1. *The algebra A^G is generated by the elements $\psi_{rs}^{(k)}$ listed above.*

Proof. We put $G_1 = G$ and

$$G_2 = \begin{cases} GL_n & \text{for } G = GL_m, \\ Sp_{2n} & \text{for } G = Sp_{2m}, \\ SO_{2n} & \text{for } G = O_m \text{ with } m \text{ even}, \\ Spin_{2n} & \text{for } G = O_m \text{ with } m \text{ odd}, \end{cases}$$

and consider the $G_1 \times G_2$-module $\mathbf{M} = \Lambda^\bullet(\mathbf{V} \otimes \mathbf{W}_-)$ defined in Section 3. As G_1-module

$$A \simeq \begin{cases} \mathbf{M} \otimes (\Lambda^m \mathbf{V}^*)^{\otimes n_2} & \text{for } G = GL_m, \\ \mathbf{M} & \text{for } G = Sp_{2m}, O_m. \end{cases}$$

Such an isomorphism is constructed as follows. For $G = GL_m$ we have

$$\begin{aligned} A &= \Lambda^\bullet(\underbrace{\mathbf{V} \oplus \cdots \oplus \mathbf{V}}_{n_1} \oplus \underbrace{\mathbf{V}^* \oplus \cdots \oplus \mathbf{V}^*}_{n_2}) \\ &\simeq \underbrace{\Lambda^\bullet \mathbf{V} \otimes \cdots \otimes \Lambda^\bullet \mathbf{V}}_{n_1} \otimes \underbrace{\Lambda^\bullet \mathbf{V}^* \otimes \cdots \otimes \Lambda^\bullet \mathbf{V}^*}_{n_2} \\ &\simeq \underbrace{\Lambda^\bullet \mathbf{V} \otimes \cdots \otimes \Lambda^\bullet \mathbf{V}}_{n_1} \otimes \underbrace{(\Lambda^\bullet \mathbf{V} \otimes \Lambda^m \mathbf{V}^*) \otimes \cdots \otimes (\Lambda^\bullet \mathbf{V} \otimes \Lambda^m \mathbf{V}^*)}_{n_2} \\ &\simeq \underbrace{\Lambda^\bullet \mathbf{V} \otimes \cdots \otimes \Lambda^\bullet \mathbf{V}}_{n} \otimes (\Lambda^m \mathbf{V}^*)^{\otimes n_2} \simeq \Lambda^\bullet(\underbrace{\mathbf{V} \oplus \cdots \oplus \mathbf{V}}_{n}) \otimes (\Lambda^m \mathbf{V}^*)^{\otimes n_2} \\ &\simeq \Lambda^\bullet(\mathbf{V} \otimes \mathbf{f}_{-1} \oplus \cdots \oplus \mathbf{V} \otimes \mathbf{f}_{-n}) \otimes (\Lambda^m \mathbf{V}^*)^{\otimes n_2} = \mathbf{M} \otimes (\Lambda^m \mathbf{V}^*)^{\otimes n_2}. \end{aligned}$$

For other groups we just have

$$A = \Lambda^\bullet(\underbrace{\mathbf{V} \oplus \cdots \oplus \mathbf{V}}_{n}) \simeq \Lambda^\bullet(\mathbf{V} \otimes \mathbf{f}_{-1} \oplus \cdots \oplus \mathbf{V} \otimes \mathbf{f}_{-n}) = \mathbf{M}.$$

Recall that $(\mathbf{f}_{-m}, \dots, \mathbf{f}_{-1})$ is the basis of $\mathbf{W}_-$.

We see that A has a natural structure of $G_1 \times G_2$-module (for $G = GL_m$ we regard $(\Lambda^m \mathbf{V}^*)^{\otimes n_1}$ as a trivial G_2-module). The subalgebra A^G coincides with the subspace of G_1-extremal 0-weight vectors in A, hence

$$A^G \simeq \begin{cases} \mathbf{M}^{\lambda,\bullet} \otimes (\Lambda^m \mathbf{V}^*)^{\otimes n_2}, \quad \lambda = \underbrace{(n_2, \dots, n_2)}_{m} & \text{for } G = GL_m, \\ \mathbf{M}^{\lambda,\bullet}, \quad \lambda = 0 & \text{for } G = Sp_{2m}, O_m. \end{cases}$$

By Proposition 4.1 the G_2-module A^G is isomorphic to the Weyl module $\Delta_2(\lambda^\dagger)$, where

$$\lambda^\dagger = \begin{cases} (\underbrace{m, \dots, m}_{n_1}, \underbrace{0, \dots, 0}_{n_2}) & \text{for } G = GL_m, \\ \underbrace{(m, \dots, m)}_{n} & \text{for } G = Sp_{2m}, \\ \underbrace{(\frac{m}{2}, \dots, \frac{m}{2})}_{n} & \text{for } G = O_m. \end{cases}$$

The unity element $\mathbf{1}_A$ of the algebra A belongs to A^G and has weight $\lambda^\dagger$ w.r.t. G_2. Hence $\mathbf{1}_A$ is highest weight vector in $A^G \simeq \Delta_2(\lambda^\dagger)$, thus $A^G = \langle U_2^- \mathbf{1}_A \rangle$.

For any algebraic group K we denote by $\mathcal{U}(K)$ its hyperalgebra (algebra of distributions). By [8, Lemma 7.15], we see that $\langle U_2^- \mathbf{1}_A \rangle = \mathcal{U}(U_2^-)\mathbf{1}_A$.

Suppose α is a root of G_2; then by X_α we denote the corresponding element of the Chevalley basis for the Lie algebra $\mathfrak{g}_2$. We choose the Chevalley basis for the classical Lie algebra $\mathfrak{g}_2$ as in [2, § 13]. The hyperalgebra $\mathcal{U}(U_2^-)$ is generated by the devided powers $X_\alpha^{(k)}$ for $\alpha \in R_2^-$.

Consider the following subset of R_2^-:

$$\Theta = \begin{cases} \{\theta(r,s) = \varepsilon_s - \varepsilon_r \mid 1 \leqslant r \leqslant n_1, n_1 + 1 \leqslant s \leqslant n\} & \text{for } G = GL_m, \\ \{\theta(r,s) = -\varepsilon_s - \varepsilon_r \mid 1 \leqslant r \leqslant s \leqslant n\} & \text{for } G = Sp_{2m}, \\ \{\theta(r,s) = -\varepsilon_s - \varepsilon_r \mid 1 \leqslant r < s \leqslant n\} & \text{for } G = O_m. \end{cases}$$

It is clear that $X_\alpha^{(k)} \mathbf{1}_A = 0$ unless $k = 0$ or $\alpha \in \Theta$.

Suppose $\Xi \subseteq R_2$; then by $\mathcal{U}_\Xi$ we denote the subalgebra of $\mathcal{U}(G_2)$ that is generated by all $X_\alpha^{(k)}$ with $\alpha \in \Xi$ and $k \in \mathbb{Z}_+$. By the Poincaré-Birkhoff-Witt theorem for $\mathcal{U}(G_2)$ (see [13, theorem 2]), we have $\mathcal{U}(U_2^-) = \mathcal{U}_\Theta \cdot \mathcal{U}_{R_2^- \setminus \Theta}$. Since $A^G = \mathcal{U}(U_2^-)\mathbf{1}_A$ and $\mathcal{U}_{R_2^- \setminus \Theta} \mathbf{1}_A = \mathbb{k}\mathbf{1}_A$, we see that $A^G = \mathcal{U}_\Theta \mathbf{1}_A$.

But for any $\mathbf{a} \in A$, $\theta = \theta(r,s) \in \Theta$, and $k \in \mathbb{Z}_+$ we have $X_\theta^{(k)}\mathbf{a} = \psi_{rs}^{(k)} \wedge \mathbf{a}$. This completes the proof. $\square$

Let us describe relations for the generators $\psi_{rs}^{(k)}$ of the ring A^G.

First of all we have standard relations for divided powers

$$\psi_{rs}^{(k_1)} \wedge \psi_{rs}^{(k_2)} = \binom{k_1+k_2}{k_1} \psi_{rs}^{(k_1+k_2)}. \tag{2}$$

Assume that $G = GL_m$.

Suppose a_r (for $r \in [1, n_1]$) and b_s (for $s \in [n_1+1, n]$) are nonnegative integers such that $\sum_{r=1}^{n_1} a_r = \sum_{s=n_1+1}^{n} b_s > m$. Then

$$\sum_{(k_{rs})} (-1)^{l((k_{rs}))} \bigwedge_{r=1}^{n_1} \bigwedge_{s=n_1+1}^{n} \psi_{rs}^{(k_{rs})} = 0, \tag{3}$$

where the summation is taken over all $(k_{rs})_{r\in[1,n_1],s\in[n_1+1,n]}$ with $k_{rs} \in \mathbb{Z}_+$ such that $\sum_{s=n_1+1}^{n} k_{rs} = a_r$, $\sum_{r=1}^{n_1} k_{rs} = b_s$, and

$$l((k_{rs})) = \sum_{\substack{r_1<r_2 \\ s_1>s_2}} k_{r_1 s_1} k_{r_2 s_2}.$$

Assume that $G = Sp_{2m}$.

Suppose a_r (for $r \in [1, n]$) are nonnegative integers such that $\sum_{r=1}^{n} a_r = 2l$, $l \in \mathbb{Z}_+$, and $l > m$. Then

$$\sum_{(k_{rs})} \bigwedge_{r=1}^{n} \bigwedge_{s=r}^{n} \psi_{rs}^{(k_{rs})} = 0, \tag{4}$$

where the summation is taken over all $(k_{rs})_{r\in[1,n],s\in[r,n]}$ with $k_{rs} \in \mathbb{Z}_+$ such that $\sum_{r=1}^{j-1} k_{rj} + 2k_{jj} + \sum_{s=j+1}^{n} k_{js} = a_j$ for any $j \in [1, n]$.

Assume that $G = O_m$.

Suppose a_r (for $r \in [1, n]$) and b_s (for $s \in [1, n]$) are nonnegative integers such that $\sum_{r=1}^{n} a_r = \sum_{s=1}^{n} b_s > m$. Then

$$\sum_{(k_{rs})} (-1)^{l((k_{rs}))} \bigwedge_{r=1}^{n} \left(\bigwedge_{s=1}^{r-1} (-1)^{k_{rs}} \psi_{sr}^{(k_{rs})} \wedge \bigwedge_{s=r+1}^{n} \psi_{rs}^{(k_{rs})} \right) = 0, \tag{5}$$

where the summation is taken over all $(k_{rs})_{r\in[1,n],s\in[1,n]}$ with $k_{rs} \in \mathbb{Z}_+$ such that $k_{rr} = 0$ for any $r \in [1, n]$, $\sum_{s=1}^{n} k_{rs} = a_r$, $\sum_{r=1}^{n} k_{rs} = b_s$, and

$$l((k_{rs})) = \sum_{\substack{r_1<r_2 \\ s_1>s_2}} k_{r_1 s_1} k_{r_2 s_2}.$$

Clearly, it is enough to verify these relations over $\mathbb{Z}$ or, equivalently, over a field of characteristic zero, in which case they follow from the fact that the rank of nonzero skew-symmetric tensor can't exceed the dimension of vector space (cf. the relations in Weyl's book [14]).

Theorem 5.2. *All relations for the generators $\psi_{rs}^{(k)}$ of the algebra A^G follow from the relations listed above.*

Proof. As we mentioned in the proof of the previous theorem, the G_2-module A^G is isomorphic to the Weyl module $\Delta_2(\lambda^\dagger)$. The latter is a universal finite-dimensional $\mathcal{U}(G_2)$-module generated by a vector of highest weight $\lambda^\dagger$. In other words, $\Delta_2(\lambda^\dagger)$ is isomorphic to the maximal finite-dimensional quotient module of the Verma module $V(\lambda^\dagger)$.

Let P be the quotient module of $V(\lambda^\dagger)$ by the submodule generated by $X_\alpha^{(k)}v$ with $\alpha \in R_2^- \setminus \Theta$ and $k > 0$, where v is the highest weight vector of $V(\lambda^\dagger)$. Clearly, the projection $V(\lambda^\dagger) \to \Delta_2(\lambda^\dagger)$ factors through P. By the Poincaré-Birkhoff-Witt theorem for $\mathcal{U}(G_2)$, we see that P is a free $\mathcal{U}_\Theta$-module of rank 1.

Consider an arbitrary U_Θ-isomorphism $\tau : U_\Theta \to P$. With the help of this isomorphism we introduce a structure of associative algebra on P. Put $x_\theta^{(k)} = \tau(X_\theta^{(k)})$ for all $\theta \in \Theta$ and $k \in \mathbb{Z}_+$. Since the algebra U_Θ is commutative, we see that P is also commutative. The algebra P may be defined as the algebra with generators $x_\theta^{(k)}$ ($\theta \in \Theta$, $k \in \mathbb{Z}_+$) and relations

$$x_\theta^{(k_1)} x_\theta^{(k_2)} = \binom{k_1+k_2}{k_1} x_\theta^{(k_1+k_2)}. \tag{6}$$

Consider the $\mathcal{U}(G_2)$-homomorphism $\xi : P \to A^G$ that takes the unity element $\mathbf{1}_P \in P$ to $\mathbf{1}_A \in A^G$. It is easily shown that it is an algebra homomorphism. Besides, for any $\theta = \theta(r,s) \in \Theta$ and $k \in \mathbb{Z}_+$ we have $\xi(x_\theta^{(k)}) = \psi_{rs}^{(k)}$. The relations (6) for the generators $x_\theta^{(k)}$ of P correspond to the relations (2) for the generators $\psi_{rs}^{(k)}$ of A^G.

Let $I = \operatorname{Ker}\xi$. We must prove that the ideal I is generated by the elements corresponding to the relations of the form (3), (4), or (5) (depending on the group G).

(1) Let $G = GL_m$. The relation (3) correspond to

$$\mathcal{X}(a_1, \ldots, a_{n_1}; b_{(n_1+1)}, \ldots, b_n) = \sum_{(k_{rs})} (-1)^{l((k_{rs}))} \prod_{r=1}^{n_1} \prod_{s=n_1+1}^{n} x_{\varepsilon_s - \varepsilon_r}^{(k_{rs})} \in I.$$

Let $\alpha_0 = \varepsilon_{n_1} - \varepsilon_{(n_1+1)}$. Note that

$$\mathcal{X}(a_1, \ldots, a_{n_1}; b_{(n_1+1)}, \ldots, b_n) = \prod_{r=1}^{n_1-1} X_{\varepsilon_{n_1}-\varepsilon_r}^{(a_r)} \prod_{s=n_1+2}^{n} X_{\varepsilon_s-\varepsilon_{(n_1+1)}}^{(b_s)} x_{-\alpha_0}^{(k)}$$

for $k = a_1 + \cdots + a_{n_1}$.

(2) Let $G = Sp_{2m}$. The relation (4) corresponds to

$$\mathcal{X}(a_1, \dots, a_n) = \sum_{(k_{rs})} \prod_{r=1}^{n} \prod_{s=r}^{n} x^{(k_{rs})}_{-\varepsilon_s - \varepsilon_r} \in I.$$

Let $\alpha_0 = 2\varepsilon_n$. We have

$$\mathcal{X}(a_1, \dots, a_n) = \prod_{r=1}^{n-1} X^{(a_r)}_{\varepsilon_n - \varepsilon_r} x^{(k)}_{-\alpha_0}$$

for $k = (a_1 + \cdots + a_n)/2$.

(3) Let $G = O_m$. The relation (5) corresponds to

$$\mathcal{X}(a_1, \dots, a_n; b_1, \dots, b_n) =$$
$$\sum_{(k_{rs})} (-1)^{l((k_{rs}))} \prod_{r=1}^{n} \left(\prod_{s=1}^{r-1} (-1)^{k_{rs}} x^{(k_{rs})}_{-\varepsilon_s - \varepsilon_r} \prod_{s=r+1}^{n} x^{(k_{rs})}_{-\varepsilon_s - \varepsilon_r} \right) \in I.$$

Let $\alpha_0 = \varepsilon_{n-1} + \varepsilon_n$. Suppose $a_{n-1} = b_n = 0$; then we have

$$\mathcal{X}(a_1, \dots, a_n; b_1, \dots, b_n) = \prod_{r=1}^{n-2} X^{(a_r)}_{\varepsilon_n - \varepsilon_r} \prod_{s=1}^{n-2} X^{(b_s)}_{\varepsilon_{n-1} - \varepsilon_s} x^{(k)}_{-\alpha_0}$$

for $k = a_1 + \cdots + a_n$.

Let J be the ideal in P generated by $\mathcal{X}(a_1, \dots)$ (for $G = O_m$ we take for the generators of J only $\mathcal{X}(a_1, \dots, a_n; b_1, \dots, b_n)$ with $a_{n-1} = b_n = 0$). Clearly, we have $J \subseteq I$. It remains to show that $I \subseteq J$.

First we prove that J is a $\mathcal{U}(G_2)$-submodule of P. Let $k > m$. For $\alpha \in R_2^+ \setminus \{\alpha_0\}$ we have $X^{(l)}_\alpha x^{(k)}_{-\alpha_0} = 0$. At the same time,

$$X^{(l)}_{\alpha_0} x^{(k)}_{-\alpha_0} = \binom{k - m - 1}{l} x^{(k-l)}_{-\alpha_0}.$$

It follows that the linear span of the elements $x^{(k)}_{-\alpha_0}$ with $k > m$ is $\mathcal{U}(B_2^+)$-invariant. By the Poincaré-Birkhoff-Witt theorem for $\mathcal{U}(G_2)$, we see that the $\mathcal{U}(G_2)$-submodule of P generated by all $x^{(k)}_{-\alpha_0}$ with $k > m$ coincides with the $\mathcal{U}(U_2^-)$-submodule generated by the same elements.

Let

$$\Omega = \begin{cases} \{\varepsilon_{n_1} - \varepsilon_r \mid r < n_1\} \cup \{\varepsilon_s - \varepsilon_{(n_1+1)} \mid s > n_1 + 1\} & \text{for } G = GL_m, \\ \{\varepsilon_n - \varepsilon_r \mid r < n\} & \text{for } G = Sp_{2m}, \\ \{\varepsilon_n - \varepsilon_r \mid r < n - 1\} \cup \{\varepsilon_{n-1} - \varepsilon_r \mid r < n - 1\} & \text{for } G = O_m. \end{cases}$$

As we noted, the generators of J have the form

$$X^{(k_1)}_{\omega_1} \cdot X^{(k_2)}_{\omega_2} \cdot \ldots \cdot X^{(k_t)}_{\omega_t} x^{(k)}_{-\alpha_0}$$

with $\omega_1, \dots, \omega_t \in \Omega$, $k_1, \dots, k_t \in \mathbb{Z}_+$, and $k > m$.

By the Poincaré-Birkhoff-Witt theorem for $\mathcal{U}(G_2)$, we have $\mathcal{U}(U_2^-) = \mathcal{U}_\Theta \cdot \mathcal{U}_\Omega \cdot \mathcal{U}_{R_2^- \setminus (\Theta \cup \Omega)}$. Since $X_\alpha^{(l)}(x_{-\alpha_0}^{(k)}) = 0$ for any $\alpha \in R_2^- \setminus (\Theta \cup \Omega)$ and $l > 0$, we see that $\mathcal{U}_{R_2^- \setminus (\Theta \cup \Omega)} x_{-\alpha_0}^{(k)} = \mathbb{k} x_{-\alpha_0}^{(k)}$. Hence $\mathcal{U}(U_2^-) x_{-\alpha_0}^{(k)} = \mathcal{U}_\Theta \cdot \mathcal{U}_\Omega x_{-\alpha_0}^{(k)} \subseteq J$ for $k > m$. Therefore the $\mathcal{U}(G_2)$-submodule of P generated by all $x_{-\alpha_0}^{(k)}$ with $k > m$ coincides with J. Thus J is a $\mathcal{U}(G_2)$-submodule of P.

Since $x_\theta^{(k)} \in J$ for all $\theta \in \Theta$ and $k > m$, we see that the algebra P/J is finite-dimensional. Hence P/J is a finite-dimensional $\mathcal{U}(G_2)$-module. Since $J \subseteq I$ and $P/I \simeq \Delta_2(\lambda^\dagger)$, by universality of the Weyl module it follows that $J = I$. $\square$

By the same method it is possible to get a similar description of invariants in exterior algebra for the groups SL_m and SO_m.

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