

Integrability properties of Hurwitz partition functions. II. Multiplication of cut-and-join operators and WDVV equations

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ABSTRACT: Correlators in topological theories are given by the values of a linear form on the products of operators from a commutative associative algebra (CAA). As a corollary, partition functions of topological theory always satisfy the generalized WDVV equations of [42]. We consider the Hurwitz partition functions, associated in this way with the CAA of cut-and-join operators. The ordinary Hurwitz numbers for a given number of sheets in the covering provide trivial (sums of exponentials) solutions to the WDVV equations, with finite number of time-variables. The generalized Hurwitz numbers from [32, 33] provide a non-trivial solution with infinite number of times. The simplest solution of this type is associated with a subring, generated by the dilatation operators $\hat{W}_1 = \text{tr } D = \text{tr } X\partial/\partial X$.

KEYWORDS: Topological Field Theories, Integrable Hierarchies

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1 Introduction

Hurwitz theory is one of the rapidly developing branches of modern mathematical physics [1]–[34]. It has its origins in the enumeration problem of ramified coverings of CP^1 , and it is brought into modern context with the formula due to Frobenius, [4]

$$\text{Cover}_n(\Delta_1, \dots, \Delta_m) = \sum_R d_R^2 \varphi_R(\Delta_1) \dots \varphi_R(\Delta_m) \delta_{|R|,n} \quad (1.1)$$

expressing the covering multiplicities through the quantities $\varphi_R(\Delta)$ proportional to the characters of the symmetric group S_n [35–37]:¹

$$\text{Char}_R(\Delta) = d_R z(\Delta) \varphi_R(\Delta) \quad (1.2)$$

The combinatorial coefficient $z(\Delta) = \prod_k \frac{1}{k^{m_k} m_k!}$ here that counts the order of the automorphism group of the Young diagram, appears everywhere in the theory of symmetric functions and symmetric group $S(\infty)$, Δ_i is the Young diagram (integer partition), characterizing the type (conjugation class) of ramification point, all sizes of the diagrams being the same and equal to the number of sheets in the covering, $|\Delta_1| = \dots = |\Delta_m| = n$, and the sum runs over the Young diagrams R of the same size $|R| = n$, but this time they label representations of the symmetric group. The symmetric group characters are among the most important objects in combinatorics, more sophisticated than the $GL(\infty)$ characters $\chi_R(t)$, very well-known and used in physical applications [38]. Still the two sets of characters are directly related by the Frobenius formula, which is a sort of Fourier transform [35–37],

$$\chi_R(t) = \sum_{\Delta} d_R \varphi_R(\Delta) p(\Delta) \delta_{|\Delta|,|R|} \quad (1.3)$$

¹They are generated by the command `Chi(R, Δ)` in Maple in the package `combinat`.

where $\Delta = [\dots \geq \delta_2 \geq \delta_1] = [\dots, 3^{m_3}, 2^{m_2}, 1^{m_1}]$, its size $|\Delta| = \sum_k \delta_k = \sum_k km_k$, the time monomial

$$p(\Delta) = \prod_k p_{\delta_k} = \prod_k p_k^{m_k} \tag{1.4}$$

and $p_k = kt_k$. These two character sets are further unified through the concept [10] of cut-and-join operators $\hat{W}(\Delta)$, commuting differential operators of degree $|\Delta|$ in t -variables, for which they serve as eigenvectors and eigenvalues respectively [32, 33]:

$$\hat{W}(\Delta)\chi_R(t) = \varphi_R(\Delta)\chi_R(t) \tag{1.5}$$

The problem with these operators is that they look rather complicated (they belong to the class of W -operators, made from powers of the $U(1)$ current). In [32, 33] we explained that the cut-and-join operators acquire a very simple form if expressed in terms of the matrix Miwa variables $p_k = \text{tr } X^k$, then

$$\hat{W}_\Delta = : \prod_k \frac{1}{k^{m_k} m_k!} (\text{tr } \hat{D}^k)^{m_k} : \tag{1.6}$$

for the $GL(\infty)$ matrix generator $D = X \frac{\partial}{\partial X_{\text{tr}}}$. This representation opens a constructive way to evaluation of the structure constants $C_{\Delta_1 \Delta_2}^\Delta$, which appear in the CAA of cut-and-join operators and, as a consequence, in the ordinary multiplication algebra of symmetric group characters (this algebra was earlier considered from combinatorial point of view in [39] where it is claimed to be equivalent to the algebra of shifted symmetric functions of [40]):

$$\begin{aligned} \hat{W}_{\Delta_1} \hat{W}_{\Delta_2} &= \sum_{\Delta} C_{\Delta_1 \Delta_2}^\Delta \hat{W}_\Delta, \\ \varphi_R(\Delta_1) \varphi_R(\Delta_2) &= \sum_{\Delta} C_{\Delta_1 \Delta_2}^\Delta \varphi_R(\Delta) \end{aligned} \tag{1.7}$$

It is important in these formulae that the sums over Δ are not restricted to $|\Delta| = |\Delta_{1,2}|$, moreover, $|\Delta_1|$ can be different from $|\Delta_2|$. Note that already in (1.5) there is no restriction to $|R| = |\Delta|$, and $\varphi_R(\Delta)$'s in this formula are more general than those in (1.3). They are defined for the Young with r unit rows, $[\Delta] = [\tilde{\Delta}, 1^r]$ by adding additional unit rows to reach $|R| = |\Delta|$ in accordance with the rule

$$\varphi_R([\Delta]) \equiv \begin{cases} 0 & \text{for } |\Delta| > |R| \\ \frac{(|R|-|\Delta|+r)!}{r!(|R|-|\Delta|)!} \varphi_R([\Delta, \underbrace{1, \dots, 1}_{|R|-|\Delta|}]) & \text{for } |\Delta| \leq |R| \end{cases} \tag{1.8}$$

This naturally leads to extension of (1.1) by removing the projector $\delta_{|R|,n}$ and lifting the restriction $|\Delta_1| = \dots = |\Delta_m| = n$, which defines what was called *generalized Hurwitz numbers* in [32, 33].

Note that the CAA of cut-and-join operators induces the multiplication on the Young diagrams:

$$\Delta_1 * \Delta_2 = \sum_{\Delta} C_{\Delta_1 \Delta_2}^\Delta \Delta \tag{1.9}$$

This multiplication can be considered as extension of another \circ -multiplication on the Young diagrams, given by the composition of permutations and related to the ordinary Hurwitz numbers. This latter is connected with the $*$ -multiplication by restricting onto diagrams of the same size:

$$\Delta_1 \circ \Delta_2 = \sum_{\Delta} C_{\Delta_1 \Delta_2}^{\Delta} \Delta \delta_{|\Delta_1|, |\Delta|} \tag{1.10}$$

with $|\Delta_1| = |\Delta_2|$. Inversely, one can construct $*$ -multiplication from the \circ -one by the procedure described in section 3.

In this paper we discuss one of the immediate implications of the extension to the generalized Hurwitz numbers: their generating function is the partition function of a topological field theory associated with the CAA (1.7) and, hence, satisfies the WDVV equations of [42]. The ordinary Hurwitz partition functions, based on the ordinary \circ -multiplication in symmetric group $S(n)$, also satisfy WDVV for each given n , but they provide only trivial solutions (which, however, were not discussed in the existing literature). The WDVV equations are imposed on "quasiclassical τ -functions", which are obtained by one or another kind of Whitham averaging procedure [43–46] from the KP/Toda hierarchy and a particular set of Riemann surfaces (background) with additional data. The quasiclassical hierarchies are well studied in the case when the background is a Riemann sphere, but in the case of Hurwitz theory it should be different (a Lambert curve, for example, [24, 26, 27]), and such hierarchies are not yet described (see, however, [47]). An advantage of the quasiclassical hierarchy would be that particular equations involve only derivatives w.r.t. the finite number of time-variables, while the WDVV equations involve inversion of an infinite size matrix. Derivation of such reducible equations for Hurwitz partition functions remains an open problem.

This paper is the second in the series which describes properties of the generating functions in Hurwitz theory. The first paper, [34] contains a summary of integrable properties, which will be described in detail in our next paper, [41].

In section 2 we begin with reminding the general construction of topological theory for any CAA and explain why its partition function always satisfy the WDVV equations (their original form is a little more general than in [42], and far more general than in [48–52], the triple derivative equations of [42] being a direct corollary, but not vice versa!). Then, in sections 3 we provide discuss the two multiplications: \circ and $*$ -products. The corresponding multiplication tables can be found in appendices I and II. Knowledge of these tables allows one to examine concrete examples illuminating following sections. In section 4 we construct two types of generating functions which are associated with two multiplications, and in section 5 the corresponding WDVV equations satisfied by these generating functions are discussed. As a particular example, in sections 6 and 7 we provide details about the $[1^p]$ subring of the $*$ -algebra, describing cut-and-join operators, associated with the single row diagrams (the "complementary" single column operators would instead generate the entire algebra).

2 Topological theories and WDVV

2.1 Topological theory on sphere (tree level)

At the tree (string) level (i.e. on sphere) a topological theory is defined by three ingredients:

- a vector space with the basis of "observables" $\{\phi_i\}$,
- an associative and commutative multiplication

$$\phi_i * \phi_j = \sum_k C_{ij}^k \phi_k \tag{2.1}$$

- and a linear form (c -valued function) on this space $\langle \phi_i \rangle = K_i$.

At the loop (string) level (i.e. on higher genera Riemann surfaces) one needs also to define traces and impose an additional constraint on the toric 1-point function (in addition to associativity and commutativity of the multiplication), but we do not need this in the present text, which is fully devoted to the tree level topological theories.

The tree correlators are defined as

$$K_{i_1 \dots i_n} = \langle \phi_{i_1}, \dots, \phi_{i_n} \rangle = \langle \phi_{i_1} * \dots * \phi_{i_n} \rangle = \sum_k C_{i_1 \dots i_n}^k \langle \phi_k \rangle = \sum_k C_{i_1 \dots i_n}^k K_k \tag{2.2}$$

where the coefficients C are products of the original 3-valent structure constants C_{ij}^k . These correlators are totally symmetric under permutations of i_1, \dots, i_n .

It is also convenient to introduce "the bare metric"

$$G_{ij} \equiv \langle \phi_i, \phi_j \rangle = \langle \phi_i * \phi_j \rangle = \sum_k C_{ij}^k \langle \phi_k \rangle \tag{2.3}$$

and use it and its inverse G^{ij} to raise and lower indices, in particular, to construct the totally symmetric tensors

$$C_{ijk} \equiv \langle \phi_i, \phi_j, \phi_k \rangle = \sum_m C_{ij}^m G_{mk} = \sum_m C_{jk}^m G_{mi} = \sum_m C_{ik}^m G_{mj} \tag{2.4}$$

and

$$\langle \phi_i, \phi_j, \phi_k, \phi_l \rangle = \sum_m C_{ij}^m C_{mkl} = \sum_m C_{ik}^m C_{mj l} = \sum_m C_{il}^m C_{mjk} = \dots \tag{2.5}$$

Next one defines the tree partition function

$$Z[\beta] = \left\langle e_{*}^{\sum_i \beta_i \phi_i} \right\rangle = \langle E[\beta] \rangle \equiv \langle \langle 1 \rangle \rangle \tag{2.6}$$

where $e_{*}(\phi) = \sum_n \frac{1}{n!} \underbrace{\phi * \phi * \dots * \phi}_n$. Then,

$$C_{ijk} = \left. \frac{\partial^3 Z[\beta]}{\partial \beta_i \partial \beta_j \partial \beta_k} \right|_{\beta=0} \tag{2.7}$$

2.2 Deformation by coupling constants β and WDVV equations

One can now introduce deformed, β -dependent algebra with a β -dependent multiplication

$$\phi_i \hat{*} \phi_j \equiv \phi_i * \phi_j * E[\beta] \equiv \sum_k \hat{C}_{ij}^k \phi_k \quad (2.8)$$

where $E[\beta]$ is a family of elements of the algebra. The new multiplication is still commutative and associative:

$$(\phi_i \hat{*} \phi_j) \hat{*} \phi_k = \phi_i * \phi_j * \phi_k * E[\beta] * E[\beta] = \phi_i \hat{*} (\phi_j \hat{*} \phi_k) \quad (2.9)$$

It is also possible to introduce the deformed observables $\hat{\phi}_i = \phi_i * E[\beta]$, then

$$\hat{\phi}_i * \hat{\phi}_j = \phi_i * \phi_j * E[\beta] * E[\beta] = (\phi_i \hat{*} \phi_j) * E[\beta] = \widehat{(\phi_i \hat{*} \phi_j)} = \sum_k \hat{C}_{ij}^k \hat{\phi}_k \quad (2.10)$$

is also a commutative associative algebra.

Now one can introduces the β -dependent correlators:

$$\langle\langle \phi_{i_1}, \dots, \phi_{i_n} \rangle\rangle \equiv \langle \phi_{i_1} * \dots * \phi_{i_n} * E[\beta] \rangle \quad (2.11)$$

Then the triple correlator possesses *two* alternative representations (the last two sums in this formula):

$$\begin{aligned} \hat{C}_{ijk} &\equiv \frac{\partial^3 Z[\beta]}{\partial \beta_i \partial \beta_j \partial \beta_k} = \langle\langle \phi_i, \phi_j, \phi_k \rangle\rangle = \langle \phi_i * \phi_j * \phi_k * E[\beta] \rangle \\ &= \sum_m \hat{C}_{ij}^m G_{mk} = \sum_m C_{ij}^m \hat{G}_{mk} \end{aligned} \quad (2.12)$$

The first representation is in terms of deformed \hat{C}_{ij}^k and the *bare* metric $G_{mk} = \langle \phi_k * \phi_m \rangle$, while the second one is in terms of the *bare* (undeformed) C_{ij}^k and the deformed metric

$$\hat{G}_{mk} \equiv \langle\langle \phi_m, \phi_k \rangle\rangle = \langle \phi_m * \phi_k * E[\beta] \rangle = \frac{\partial^2 Z[\beta]}{\partial \beta_k \partial \beta_m} \quad (2.13)$$

Associativity of original and deformed algebras is expressed in the commutativity condition of the structure constants $(\check{C}_i)_j^k \equiv C_{ij}^k$

$$\check{C}_i \check{C}_j = \check{C}_j \check{C}_i \quad (2.14)$$

In its turn, this implies that

$$\hat{C}_{ijm} G^{mn} \hat{C}_{kln} = \hat{C}_{ikm} G^{mn} \hat{C}_{jln} \quad (2.15)$$

and

$$\hat{C}_{ijm} \hat{G}^{mn} \hat{C}_{kln} = \hat{C}_{ikm} \hat{G}^{mn} \hat{C}_{jln} \quad (2.16)$$

which we respectively call as *bare* and *full* [42, 52, 54–60] WDVV equations for the partition function $Z[\beta]$ (in the case of the *full* WDVV equation all ingredients are triple

derivatives of $Z[\beta]$, the deformed metric is the triple correlator with the unity operator $\phi_0 = I: \hat{G}_{mn} = \hat{C}_{0mn}$).

In some cases the choice of the vector space and observables ϕ_i is not unique. The same algebra may possess different realizations (representations) and one can ask if the β -deformed topological theory respects this freedom. The problem is very similar to the representation theory of Lie algebras and to the concept of the universal group elements etc.

2.3 Hurwitz topological theory

Consider here an explicit example when ϕ_i depends on an additional label, R such that the structure constants in the product

$$\phi_i(R) * \phi_j(R) = \sum_k C_{ij}^k \phi_k(R) \tag{2.17}$$

do not depend on R and the mean value is defined by averaging over R (this is much similar to an ordinary functional integral over the field $\phi(x)$ with the variable x substituted by the discrete label R). An example of such topological theory is provided by the theory of Hurwitz numbers, and the role of index R can be played by different structures, for instance, the Young diagrams in the Frobenius formula (1.1). In this case, we define the correlators involving the sum over R :

$$\langle i_1, \dots, i_n \rangle = \langle \phi_{i_1}(R) * \dots * \phi_{i_n}(R) \rangle = \sum_R d_R^2 \phi_{i_1}(R) \dots \phi_{i_n}(R) \tag{2.18}$$

Now, it is clear that the equality

$$\begin{aligned} \hat{C}_{ijk} &\equiv \langle \langle i, j, k \rangle \rangle = \langle \phi_i(R) * \phi_j(R) * \phi_k(R) * E[\beta, R] \rangle = \\ &= \sum_m C_{ij}^m \langle \phi_m(R) * \phi_k(R) * E[\beta, R] \rangle = \sum_m C_{ij}^m \hat{G}_{mk} = \sum_m \hat{C}_{ij}^m G_{mk} \end{aligned} \tag{2.19}$$

with

$$E[\beta, R] = e_{*}^{\sum_i \beta_i \phi_i(R)} \tag{2.20}$$

continues to hold in this case.

As follows from the discussion above, the Hurwitz partition function as a function of β satisfies the *full* WDVV equations (2.16). Sometime, for restricted sets of β -variables, it happen to be also KP τ -functions [34] but this is beyond our consideration here. Instead we note that the weight $E[\beta, R]$ can be made more general than (2.20), without changing anything in the content of the previous consideration. Namely, one can change (2.20) for

$$P_*(\phi(R)) * e_{*}^{\sum_i \beta_i \phi_i(R)} \tag{2.21}$$

with an arbitrary $*$ -polynomial of observables $\phi_i(R)$ with the same R . Sometimes new integrability properties can occur for the partition function as a function on these additional parameters [34].

In Hurwitz theory *per se*, the role of observables $\phi_i(R)$ is played by the characters of the symmetric group S_∞ , denoted by $\varphi_R(\Delta)$, where the label $i \rightarrow \Delta$ is now the Young diagram. The most interesting choice for $P_*(\phi(R))$ is a product of several $GL(\infty)$ characters, $\chi_R(t)\chi_R(t')\chi_R(t'')\dots$, where $\chi_R(t)$ is related to $\varphi_R(\Delta)$ by formula (1.3). With so modified $E[\beta, R]$ the Hurwitz partition function becomes a function of both t - and β -variables. While in β -variables (when considering their complete set, not a subset) it is usually a "quasiclassical τ -function", i.e. a solution to the full WDVV equations, in t -variables it can be a KP τ -function. This is, indeed, the case when there is one (t) and two (t, t') sets of t -variables, see [11, 34, 41, 61]. Surprisingly or not, the KP integrability in t disappears for three (t, t', t'') or more sets of t -variables. This peculiar pattern of (non)-integrability structures is discussed in the next paper of this series [41].

In variance with our construction in this section, in [53] there was suggested another, polynomial class of solutions to the WDVV equations. In the Hurwitz theory context this would correspond to a power series instead of polynomial solutions.

3 Two multiplications of Young diagrams

As we mentioned above, there are two natural multiplications on the Young diagrams: one, \circ -multiplication given by the composition of permutations, and the other one, $*$ -multiplication induced by the algebra of cut-and-join operators (1.6).

3.1 \circ -multiplication of Young diagrams from composition of permutations

The \circ -multiplication is given on the Young diagrams (integer partitions) labeling elements of the group algebra of the symmetric group, i.e. the sum of all permutations² from the corresponding conjugation class:

$$[211] = (12) + (13) + (14) + (23) + (24) + (34)$$

etc. The number of items is denoted by $\|\Delta\|$.

The naive \circ -multiplication of Young diagrams, induced by the (non-commutative but associative) composition of permutations is commutative and associative. Of course, $\|\Delta_1 \circ \Delta_2\| = \|\Delta_1\| \cdot \|\Delta_2\|$.

Examples of the multiplication tables for different symmetric groups can be found in appendix II.

- For any k the \circ -multiplication by $[1^k]$ acts like unity:

$$[1^k] \circ \Delta = \Delta \quad \forall \Delta : |\Delta| = k \tag{3.1}$$

- Multiplication by $[2, 1^k]$ can be deduced from the cut-and-join property. Namely, if permutations are written in the cyclic notations, then permutation $(12) \in S_{k+2}$ acts as follows:

$$(12) \circ (12)K = K,$$

²The composition of permutations is done in Maple by the command *mulperms* of the package *group*.

$$\begin{aligned}
 (12) \circ (12C)K &= (1C)K, \\
 (12) \circ (21C)K &= (2C)K, \\
 (12) \circ (1C)K &= (12C)K, \\
 (12) \circ (C)K &= (12)(C)K, \\
 (12) \circ (1C_1)(2C_2)K &= (1C_22C_1)K
 \end{aligned} \tag{3.2}$$

where C denotes any set of elements, and K any set of non-intersecting cycles (of course, it is assumed that $1, 2 \notin C, C_1, C_2, K$ and $C, C_1, C_2 \notin K$). If for a given level k all the "powers" $||\Delta||$ are known, this is enough to construct all the entries $[2, 1^k] \circ \Delta$ in the \circ -multiplication table: the coefficient for each line in (3.2) is given by the ratio of items of the given type at the l.h.s. of (3.2) to the "power" at the r.h.s., multiplied by $||2, 1^k||$. These rules are illustrated in manifest examples of appendix I.

3.2 *-multiplication of Young diagrams

The *-multiplication of Young diagrams is associated with the product of the differential cut-and-join operators (1.6): if

$$\hat{W}[\Delta_1]\hat{W}[\Delta_2] = \sum_{\Delta} C_{\Delta_1\Delta_2}^{\Delta} \hat{W}[\Delta], \tag{3.3}$$

then

$$\Delta_1 * \Delta_2 = \sum_{\Delta} C_{\Delta_1\Delta_2}^{\Delta} \Delta \tag{3.4}$$

and thus it is commutative and associative. The sums are actually finite: the size of the diagrams Δ is restricted to

$$\max(|\Delta_1|, |\Delta_2|) \leq |\Delta| \leq |\Delta_1| + |\Delta_2| \tag{3.5}$$

As it was already mentioned in the Introduction, these commuting \hat{W} -operators have all the $GL(\infty)$ characters as common eigenfunctions, while S_{∞} characters are the corresponding eigenvalues (1.5). Representations of $GL(\infty)$ characters through the first and second Weyl formulas are associated to representations of the cut-and-join operators in time and matrix variables. It follows from (1.5) that the symmetric group characters form the same commutative associative algebra:

$$\varphi_R(\Delta_1)\varphi_R(\Delta_2) = \sum_{\Delta} C_{\Delta_1\Delta_2}^{\Delta} \varphi_R(\Delta) \tag{3.6}$$

with the same R -independent structure constants $C_{\Delta_1\Delta_2}^{\Delta}$.

3.3 Connection between the two multiplications

The *-multiplication (3.4) can be expressed through the \circ -multiplication of Young diagrams. It is a rather long recursive formula, but actually a very constructive one:

$$\Delta_1 * \Delta_2 = \sum_{n=\max(|\Delta_1|, |\Delta_2|)}^{|\Delta_1|+|\Delta_2|} \{\Delta_1, \Delta_2\}_n$$

$$\{\Delta_1, \Delta_2\}_n = \sum_{\Delta: |\Delta|=n} C_{\Delta_1 \Delta_2}^\Delta \Delta = \rho_{n-|\Delta_1|}(\Delta_1) \circ \rho_{n-|\Delta_2|}(\Delta_2) - \sum_{k=\max(|\Delta_1|, |\Delta_2|)}^{n-1} \rho_{n-k}(\{\Delta_1, \Delta_2\}_k) \quad (3.7)$$

and ρ_k is a lift of the Young diagram to the size $|\Delta| + k$, achieved by adding k unit length rows with additional numeric factor: if Δ already has r rows of the length 1, then

$$\rho_k([\Delta]) = \frac{(r+k)!}{r!k!} [\Delta, \underbrace{1, \dots, 1}_k], \quad \text{or} \quad \rho_k([\tilde{\Delta}, 1^r]) = \frac{(r+k)!}{r!k!} [\tilde{\Delta}, 1^{r+k}] \quad (3.8)$$

where $[\Delta] \equiv [\tilde{\Delta}, 1^r]$ and $\tilde{\Delta}$ does not contains unit rows. According to this definition, $\rho_0(\Delta) = \Delta$. Note that

$$\rho_k(\rho_l(\Delta)) = \frac{(k+l+r)!}{k!l!r!} [\Delta, 1^{k+l}] \neq \rho_{k+l}(\Delta) = \frac{(k+l+r)!}{(k+l)!r!} [\Delta, 1^{k+l}] \quad (3.9)$$

The highest term in the product (3.7) is

$$\{\Delta_1, \Delta_2\}_{|\Delta_1|+|\Delta_2|} = C_{\Delta_1 \Delta_2}^{[\Delta_1, \Delta_2]} [\Delta_1, \Delta_2] \quad (3.10)$$

and for $\Delta_1 = [k^{m_k}]$, $\Delta_2 = [k^{n_k}]$, $[\Delta_1, \Delta_2] = [k^{m_k+n_k}]$ the combinatorial coefficient is

$$C_{\Delta_1 \Delta_2}^{[\Delta_1, \Delta_2]} = \prod_k \frac{(m_k + n_k)!}{m_k! n_k!} \quad (3.11)$$

This follows from the definition (1.6) of the \hat{W} operator [32, 33].

If $\Delta_1 * \Delta_2 = \sum_{\Delta} C_{\Delta_1 \Delta_2}^\Delta \Delta$, formula (3.7) can be rewritten as

$$\rho_{n-|\Delta_1|}(\Delta_1) \circ \rho_{n-|\Delta_2|}(\Delta_2) = \sum_m \rho_{n-m}(\{\Delta_1, \Delta_2\}_m) = \sum_{\Delta} C_{\Delta_1 \Delta_2}^\Delta \rho_{n-|\Delta|}(\Delta) \quad (3.12)$$

Expressed in terms of the generating functions $J_\Delta(u) = \sum_{m=0}^{\infty} u^{|\Delta|+m} \rho_m(\Delta)$ this multiplication formula becomes

$$\oint J_{\Delta_1}(u) \circ J_{\Delta_2}\left(\frac{v}{u}\right) \frac{du}{u} = \sum_{\Delta} C_{\Delta_1 \Delta_2}^\Delta J_\Delta(v) = J_{\Delta_1 * \Delta_2}(v) \quad (3.13)$$

Note that the contour integral over u at the l.h.s. selects diagrams of the same weight, so that the operation \circ is well defined.

Examples of the $*$ -multiplication tables can be found in section 6 and in appendix II, here we consider only the case of product $[1] * [\Delta]$ which will be of use for our further consideration.

3.4 Example of level (1, m)

For Δ of the size $|\Delta| = m$, which already has r columns of height 1, one gets

$$\begin{aligned} [1] * [\Delta] &= \{1, \Delta\}_m + \{1, \Delta\}_{m+1}, \\ \{1, \Delta\}_m &= \rho_{m-1}[1] \circ \Delta = m[1^m] \circ \Delta = m\Delta, \\ \{1, \Delta\}_{m+1} &= \rho_m[1] \circ \rho_1[\Delta] - \rho_1(\{1, \Delta\}_m) = (m+1)[1^{m+1}] \circ (r+1)[\Delta, 1] - m(r+1)[\Delta, 1] = (r+1)[\Delta, 1], \\ [1] * \Delta &= |\Delta| \Delta + (r+1)[\Delta, 1] \end{aligned} \quad (3.14)$$

In other words, if $\Delta = [\tilde{\Delta}, 1^r]$, where $\tilde{\Delta}$ contains no more units, then

$$[1] * [\tilde{\Delta}, 1^r] = (|\tilde{\Delta}| + r)[\tilde{\Delta}, 1^r] + (r+1)[\tilde{\Delta}, 1^{r+1}] \quad (3.15)$$

4 Generating functions

Introduce now a linear form (average) on the Young diagrams:

$$\langle \Delta \rangle = \frac{\delta(\Delta, [1^{|\Delta|}])}{|\Delta|!} = \sum_R d_R^2 \varphi_R(\Delta) \delta_{|R|, |\Delta|} \quad (4.1)$$

It can be used to construct a variety of generating functions for averages of Young diagrams products (correlators).

In fact, the projection to $|R| = |\Delta|$ in the sum over R in (4.1) can be eliminated, and the *infinite* sum

$$\sum_R d_R^2 \varphi_R(\Delta) = \sum_{R: |R| \geq |\Delta|} d_R^2 \varphi_R(\Delta) = e \langle \Delta \rangle \quad (4.2)$$

where $e = 2.718\dots$. This formula is important for evaluation of the partition functions in the case of *-products.

4.1 The standard Hurwitz partition function

The standard generating function of Hurwitz numbers is

$$\begin{aligned} Z_\circ\{\beta_{\tilde{\Delta}}\} &= \sum_n q^n Z_\circ^{(n)}\{\beta_\Delta\} = \sum_n q^n \left\langle \exp_\circ \left(\sum_{\tilde{\Delta}: |\tilde{\Delta}| \leq n} \beta_{\tilde{\Delta}} \rho_{n-|\tilde{\Delta}|}(\tilde{\Delta}) \right) \right\rangle \\ &= \sum_n q^n \left\langle \exp_\circ \left(\sum_{\tilde{\Delta}: |\tilde{\Delta}| \leq n} \beta_{\tilde{\Delta}} [\tilde{\Delta}, 1^{n-|\tilde{\Delta}|}] \right) \right\rangle \end{aligned} \quad (4.3)$$

where, as before, $\tilde{\Delta}$ denotes the Young diagram without units and the \circ -multiplication of the Young diagram of different sizes is defined to be zero. Actually, $q = e^{\beta_1}$. Of course, one can also introduce a whole infinite tower of β -variables for each $\tilde{\Delta}$: $\beta_{\tilde{\Delta}, p} = \beta_{[\tilde{\Delta}, 1^p]}$, but we prefer not to do it. Again, for each given n the component $Z_\circ^{(n)}\{\beta_\Delta\}$ satisfies the WDVV equations, but after summation over n , (4.3) is not an average of any CAA exponential and does not need to satisfy WDVV equations. And, indeed, it does not.

With the help of (1.3) one can rewrite (4.3) in terms of symmetric group characters $\varphi_R(\Delta)$, (1.8). Indeed, because of (1.5) they form a representation of the CAA algebra with *-product:

$$\varphi_R(\Delta_1) \varphi_R(\Delta_2) = \varphi_R(\Delta_1 * \Delta_2) = \sum_{\substack{\tilde{\Delta} \\ \max(|\Delta_1|, |\Delta_2|) \leq |\tilde{\Delta}| \leq |\Delta_1| + |\Delta_2|}} C_{\Delta_1 \Delta_2}^{\tilde{\Delta}} \varphi_R(\tilde{\Delta}) \quad \forall R, \Delta_1, \Delta_2 \quad (4.4)$$

For $|\Delta_1| = |\Delta_2| = n$ and for $|R| = n$ the property (1.8) implies that also

$$\varphi_R(\Delta_1 \circ \Delta_2) = \varphi_R(\Delta_1) \varphi_R(\Delta_2), \quad \text{for } |R| = |\Delta_1| = |\Delta_2| \quad (4.5)$$

and this allows one to express (4.3) through $\varphi_R(\Delta)$: if all the sizes $|\Delta_1| = \dots = |\Delta_k| = n$ are the same, then

$$\begin{aligned} \left\langle \circ_{i=1}^k \Delta_i \right\rangle &= \sum_{\Delta} c_{\Delta_1, \dots, \Delta_k}^{\Delta} \langle \Delta \rangle \delta_{|\Delta|, n} = \sum_{\Delta} c_{\Delta_1, \dots, \Delta_k}^{\Delta} \sum_R d_R^2 \varphi_R(\Delta) \delta_{|\Delta|, n} \delta_{|R|, n} \\ &= \sum_R d_R^2 \prod_{i=1}^k \varphi_R(\Delta_i) \delta_{|R|, n} \end{aligned} \quad (4.6)$$

Here

$$c_{\Delta_1, \dots, \Delta_k}^{\Delta} = \sum_{\Delta'_1, \dots, \Delta'_{k-2}} C_{\Delta_1 \Delta'_1}^{\Delta} C_{\Delta_2 \Delta'_2}^{\Delta'_1} \dots C_{\Delta_{k-1} \Delta_k}^{\Delta'_{k-2}} \delta_{|\Delta'_1|, n} \dots \delta_{|\Delta'_{k-2}|, n} \quad (4.7)$$

we denote it by *small* letter c to emphasize that all sums are restricted (projected) to diagrams of the same size n . Restriction to $|R| = n$ is important for the last transition in (4.6), where *small* c stand at the l.h.s.: still equality takes place because of (4.5).

From (4.6) it follows directly that partition function (4.3) can be rewritten as

$$Z_{\circ} \{ \beta_{\tilde{\Delta}} \} = \sum_n q^n \sum_{R: |R|=n} d_R^2 \exp \left(\sum_{\tilde{\Delta}: |\tilde{\Delta}| \leq n} \beta_{\tilde{\Delta}} \varphi_R(\tilde{\Delta}, 1^{n-|\tilde{\Delta}|}) \right) \quad (4.8)$$

or simply

$$Z_{\circ} \{ \beta_{\Delta} \} = \sum_n q^n Z_{\circ}^{(n)} \{ \beta_{\Delta} \} = \sum_n q^n \sum_{R: |R|=n} d_R^2 \exp \left(\sum_{\Delta: |\Delta|=n} \beta_{\Delta} \varphi_R(\Delta) \right) \quad (4.9)$$

where, in principle, one can either impose the restriction

$$\beta_{[\tilde{\Delta}, 1^r]} = \beta_{\tilde{\Delta}} \quad (4.10)$$

or not.

If (4.10) is not imposed, then, making use of (1.3), one can further perform a Fourier transform of its m -th derivative into t -variables:

$$\begin{aligned} \sum_n q^n \sum_{\substack{\Delta_1, \dots, \Delta_m \\ |\Delta_1| = \dots = |\Delta_m| = n}} \frac{\partial^m Z_{\circ}^{(n)} \{ \beta_{\Delta} \}}{\partial \beta_{\Delta_1} \dots \partial \beta_{\Delta_m}} p^{(1)}(\Delta_1) \dots p^{(m)}(\Delta_m) &= \\ = \sum_n q^n \sum_{R: |R|=n} d_R^{2-m} \chi_R(t^{(1)}) \dots \chi_R(t^{(m)}) \exp \left(\sum_{\tilde{\Delta}: |\tilde{\Delta}|=n} \beta_{\tilde{\Delta}} \varphi_R(\tilde{\Delta}) \right) \end{aligned} \quad (4.11)$$

4.2 Extension to *-product

With the *-product one can associate another, generalized Hurwitz partition function [32, 33]:

$$Z_* \{ \beta_{\Delta} \} = \left\langle \exp_* \left(\sum_{\Delta} \beta_{\Delta} \Delta \right) \right\rangle \quad (4.12)$$

In variance with $Z_\circ\{\beta_\Delta\}$ in (4.3), it satisfies the WDVV equations, but in variance with individual components $Z_\circ^{(n)}\{\beta_\Delta\}$ (which also satisfy WDVV) it involves infinitely many time-variables β_Δ .

One can also rewrite Z_* in terms of $\varphi_R(\Delta)$ characters, but this time, in the case of *-products, there would be no restriction on the sizes $|\Delta|$ in (4.6). Then, in this case the sum over R will be restricted not to $|R| = n$, but to $|R| = |\Delta|$:

$$\left\langle *_{i=1}^k \Delta_i \right\rangle = \sum_{\Delta} C_{\Delta_1, \dots, \Delta_k}^{\Delta} \langle \Delta \rangle = \sum_{\Delta} C_{\Delta_1, \dots, \Delta_k}^{\Delta} \sum_R d_R^2 \varphi_R(\Delta) \delta_{|R|, |\Delta|} \quad (4.13)$$

and, because of the Δ -dependent projector, in this formula one can not make any direct use of the relation

$$\sum_{\Delta} C_{\Delta_1, \dots, \Delta_k}^{\Delta} \varphi_R(\Delta) = \prod_{i=1}^k \varphi_R(\Delta_i) \quad (4.14)$$

However, one can actually get rid of the projector! The reason is that (1.3) has important generalizations:

$$\sum_{\Delta} d_R \varphi_R(\Delta) p(\Delta) \delta_{|\Delta|, |R|} = \chi_R(t), \quad (4.15)$$

$$\sum_{\Delta} d_R \varphi_R(\Delta) p(\Delta) = \chi_R(t_k + \delta_{k1}), \quad (4.16)$$

$$\sum_{\tilde{\Delta}} d_R \varphi_R(\tilde{\Delta}) p(\tilde{\Delta}) = \chi_R(1, t_2, t_3, \dots), \quad (4.17)$$

$$d_R = \chi_R(1, 0, 0, \dots) = \chi_R(\delta_{k1}) \quad (4.18)$$

Note that because of the property (1.8) all the sums over Δ are finite, and these formulas are elementary, not transcendental.

Combining (4.16) and (4.18) with the celebrated Cauchy completeness formula

$$\sum_R \chi_R(t) \chi_R(t') = \exp \left(\sum_k k t_k t'_k \right) \quad (4.19)$$

one obtains

$$\begin{aligned} \sum_{R, \Delta} d_R^2 \varphi_R(\Delta) p(\Delta) &= \sum_R \chi_R(t_k + \delta_{k1}) \chi_R(\delta_{k1}) = e^{1+t_1} \\ &= e \sum_{\Delta} \langle \Delta \rangle p(\Delta) = e \sum_{R, \Delta} d_R^2 \varphi_R(\Delta) p(\Delta) \delta_{|R|, |\Delta|} \end{aligned} \quad (4.20)$$

In other words, we obtain the already-mentioned statement (4.2):

$$\langle \Delta \rangle = \sum_R d_R^2 \varphi_R(\Delta) \delta_{|R|, |\Delta|} = \frac{1}{e} \sum_R d_R^2 \varphi_R(\Delta) \quad (4.21)$$

i.e. one can simply substitute everywhere the average (4.1) by the alternative one,

$$\langle\langle \Delta \rangle\rangle = \sum_R d_R^2 \varphi_R(\Delta) = e \langle \Delta \rangle \quad (4.22)$$

where sum goes over Young diagrams R of all sizes, not restricted to $|R| = |\Delta|$. The difference is actually exhausted by a factor of $e = 2.718\dots$

Coming back to (4.13), one now knows how to eliminate the unwanted projector from the r.h.s. and apply (4.14):

$$\begin{aligned} \langle *_{i=1}^k \Delta_i \rangle &= \frac{1}{e} \langle \langle *_{i=1}^k \Delta_i \rangle \rangle = \frac{1}{e} \sum_{\Delta} C_{\Delta_1 \dots \Delta_k}^{\Delta} \langle \langle \Delta \rangle \rangle \\ &= \frac{1}{e} \sum_{\Delta} C_{\Delta_1 \dots \Delta_k}^{\Delta} \sum_R d_R^2 \varphi_R(\Delta) = \frac{1}{e} \sum_R d_R^2 \prod_{i=1}^k \varphi_R(\Delta_i) \end{aligned} \quad (4.23)$$

In particular, the pair correlator is equal to

$$\begin{aligned} \sum_{\Delta_1, \Delta_2} \langle \Delta_1 * \Delta_2 \rangle p^{\Delta_1} \bar{p}^{\Delta_2} &= \sum_{\Delta_1, \Delta_2} \frac{1}{e} \sum_R d_R^2 \varphi_R(\Delta_1) \varphi_R(\Delta_2) p^{\Delta_1} \bar{p}^{\Delta_2} \stackrel{(4.16)}{=} \\ &= \frac{1}{e} \sum_R \chi_R(t_k + \delta_{k1}) \chi_R(\bar{t}_k + \delta_{k1}) \stackrel{(4.19)}{=} \exp \left((t_1 + 1)(\bar{t}_1 + 1) - 1 + \sum_{k \geq 2} t_k \bar{t}_k \right) \end{aligned} \quad (4.24)$$

From formula (4.23) one obtains a character expansion of Z_* and a much better counterpart of (4.11):

$$Z_*\{\beta_{\Delta}\} = \left\langle \exp_* \left(\sum_{\Delta} \beta_{\Delta} \Delta \right) \right\rangle = \frac{1}{e} \sum_R d_R^2 \exp \left(\sum_{\Delta} \beta_{\Delta} \varphi_R(\Delta) \right) \quad (4.25)$$

and

$$\begin{aligned} \sum_{\Delta_1, \dots, \Delta_m} \frac{\partial^m Z_*\{\beta_{\Delta}\}}{\partial \beta_{\Delta_1} \dots \partial \beta_{\Delta_m}} p^{(1)}(\Delta_1) \dots p^{(m)}(\Delta_m) &= \\ = \frac{1}{e} \sum_n q^n \sum_{R: |R|=n} d_R^{2-m} \chi_R(t_k^{(1)} + \delta_{k1}) \dots \chi_R(t_k^{(m)} + \delta_{k1}) \exp \left(\sum_{\Delta} \beta_{\Delta} \varphi_R(\Delta) \right) \end{aligned} \quad (4.26)$$

We emphasize once again that there are no restrictions on the sizes of any Δ_i and of R in the sum. In (4.26) we do not impose (4.10), otherwise one should just write correlators instead of β -derivatives at the l.h.s.

5 WDVV equations

Whenever the generating function is an average of the CAA exponential, it satisfies the WDVV equations. This happens, at least, in two cases. The first one is for the \circ -product at the given level $|\Delta|$: then one gets a trivial WDVV solution in form of a finite linear combination of ordinary exponentials. The second case is that of the $*$ -product: then the number of time-variables is infinite, even in the simplest case of the $[1^P]$ -subring, an interesting open question being to find an adequate quasiclassical (dispersionless) hierarchy, which is associated with this particular solution to the WDVV equations: it is probably related to the KP-Whitham hierarchy over the Lambert curve [24, 26, 27, 47].

We start with the trivial case of the \circ -partition function, then comment on the case of the $*$ -partition function when the WDVV equations are awaited on common grounds but one tediously check them directly. We postpone until the next two sections a discussion of the case of $[1^p]$ -subring when some less involved direct checks of the WDVV equations can be done.

5.1 \circ -products at given level

For a given n one introduces a function of $|S_n|$ variables β_Δ :

$$Z_n(\beta) = \left\langle \exp_\circ \left(\sum_{\Delta: |\Delta|=n} \beta_\Delta \Delta \right) \right\rangle = \sum_{R: |R|=n} d_R^2 \prod_{\Delta: |\Delta|=n} e^{\beta_\Delta \varphi_R(\Delta)} \quad (5.1)$$

Each Z_n satisfies the set of WDVV equations. In this case this follows not only from the general arguments, true for any topological field theory, but also from a much simpler consideration: one can obtain by a linear transform from $\{\beta_\Delta\}$ the separated variables

$$\xi_R = \sum_{\Delta: |\Delta|=n} \beta_\Delta \varphi_R(\Delta) \quad (5.2)$$

so that

$$\tilde{Z}_n(\xi) = \sum_R d_R^2 e^{\xi_R} = Z_n(\beta) \quad (5.3)$$

Then the WDVV equations are trivially satisfied and, since this is no more than a linear (and non-degenerate) change of beta-variables, the original WDVV equations for $Z_n(\beta)$ are also true.

It is also evident that if one inserts into the sum (5.1) an arbitrary product of characters $\chi_R(t_k^{(1)}) \dots \chi_R(t_k^{(m)})$, it does not spoil the argument and, hence, the WDVV equations are still satisfied.

5.2 The $*$ -product case

In this case one has to consider the WDVV equations with infinite matrices: there is no any simple finite truncation for them. For instance, the simplest WDVV equation $\check{C}_2 \check{C}_3 = \check{C}_3 \check{C}_2$ (2.14) is

$$\frac{\langle 112 \rangle \langle 123 \rangle}{\langle 11 \rangle} + \frac{\langle 122 \rangle \langle 223 \rangle}{\langle 22 \rangle} + \frac{\langle 123 \rangle \langle 233 \rangle}{\langle 33 \rangle} = \frac{\langle 113 \rangle \langle 122 \rangle}{\langle 11 \rangle} + \frac{\langle 123 \rangle \langle 222 \rangle}{\langle 22 \rangle} + \frac{\langle 133 \rangle \langle 223 \rangle}{\langle 33 \rangle} \quad (5.4)$$

and it does *not* hold with

$$\begin{aligned} \langle 123 \rangle &= \langle [1] * [2] * [3] \rangle = 0, & \langle 113 \rangle &= 0, \\ \langle 122 \rangle &= 3/2, & \langle 223 \rangle &= 1, & \langle 133 \rangle &= 4/3, \\ \langle 11 \rangle &= 1, & \langle 22 \rangle &= 1/2, & \langle 33 \rangle &= 1/3 \end{aligned} \quad (5.5)$$

Indeed, vanishing entries in the first line reduce the equation just to

$$\frac{\langle 122 \rangle \langle 223 \rangle}{\langle 22 \rangle} = \frac{\langle 133 \rangle \langle 223 \rangle}{\langle 33 \rangle} \tag{5.6}$$

which is not true.

What is the reason? In fact, $\check{C}_2 \check{C}_3 = \check{C}_3 \check{C}_2$ holds in the following way:

$$\begin{aligned} \langle (2 * 2) * 3 \rangle &= C_{22}^3 \langle 3 * 3 \rangle = C_{22}^3 C_{33}^{111} \langle 111 \rangle = 3 \cdot 2 \cdot \frac{1}{6}, \\ \langle 2 * (2 * 3) \rangle &= C_{23}^{[21]} \langle 2 * [21] \rangle = C_{32}^{[21]} C_{[21],2}^{111} \langle 111 \rangle = 3 \cdot 2 \cdot \frac{1}{6} \end{aligned} \tag{5.7}$$

i.e. as ${}_2(\check{C}_2 \check{C}_3 = \check{C}_3 \check{C}_2)^{[111]}$:

$$C_{22}^3 C_{33}^{111} = C_{32}^{[21]} C_{[21],2}^{111} \tag{5.8}$$

However, while

$$C_{22}^3 C_{33}^{111} = \frac{\langle 2 * 2 * 3 \rangle \langle 3 * 3 * 111 \rangle}{\langle 3 * 3 \rangle \langle 111 * 111 \rangle} = \frac{\langle 3 * 2 * 21 \rangle \langle 2 * 21 * 111 \rangle}{\langle 21 * 21 \rangle \langle 111 * 111 \rangle} = C_{32}^{[21]} C_{[21],2}^{111} \tag{5.9}$$

the symbolical expression neglecting the number of units does not hold:

$$C_{22}^3 C_{33}^1 \stackrel{?}{=} \frac{\langle 223 \rangle \langle 133 \rangle}{\langle 33 \rangle \langle 11 \rangle} \neq \frac{\langle 223 \rangle \langle 122 \rangle}{\langle 22 \rangle \langle 11 \rangle} \stackrel{?}{=} C_{32}^2 C_{22}^1 \tag{5.10}$$

Thus, reduction does not take place in the 1-sector.

Thus, one has to deal with infinite matrices. Though the WDVV equations still have to be satisfied, checking them directly is a non-trivial problem. This is easier to do in the simpler case of the $[1^p]$ -subring case, which we discuss in the next two sections.

6 Sub-ring of $[1^p]$ operators and its action on entire algebra

In this section we discuss multiplication by the $[1^p]$ operators, which form a closed sub-algebra of the entire algebra. Moreover, in this case, it is possible to write down general formulas.

6.1 *-subring of $[1^p]$ operators (single-line diagrams)

The multiplication of $[1^p]$ operators is given by the formula

$$\begin{aligned} [1^p] * [1^q] &= \sum_{i=\max(0,p-q)}^p \frac{(q+i)!}{i!(p-i)!(q-p+i)!} [1^{q+i}] \\ &= \sum_{s=\max(p,q)}^{p+q} \frac{s!}{(s-p)!(s-q)!(p+q-s)!} [1^s] \end{aligned} \tag{6.1}$$

Introduce $I(x) = \sum_p x^p [1^p]$. Then it follows that

$$I(x) * I(y) = I(x + y + xy),$$

$$\begin{aligned}
 I(x) * I(y) * I(z) &= I(x + y + xy) * I(z) = I(x + y + z + xy + yz + zx + xyz), \\
 &\dots, \\
 *_{i=1}^m I(x_i) &= I\left(\sum_i x_i + \sum_{i<j} x_i x_j + \sum_{i<j<k} x_i x_j x_k + \dots + \prod_{i=1}^m x_i\right) \\
 &= I\left(-1 + \prod_i (x_i + 1)\right) \tag{6.2}
 \end{aligned}$$

6.2 Action of $[1^p]$ operators on the entire algebra

The multiplication of any operator by $[1^p]$ operators is described with the following formulas. Let $\Delta = [\tilde{\Delta}, 1^r]$, with $1 \notin \tilde{\Delta}$. Let first $p \leq m = |\Delta| = |\tilde{\Delta}| + r$:

$$\begin{aligned}
 [1^p] * [\Delta] &= \sum_{i=0}^p \{1^p, \Delta\}_{m+i}, \\
 \{1^p, \Delta\}_m &= \rho_{m-p} [1^p] \circ \Delta = \frac{m!}{p!(m-p)!} [1^m] \circ \Delta = C_m^p \Delta, \\
 \{1^p, \Delta\}_{m+1} &= \rho_{m-p+1} [1^p] \circ \rho_1 [\Delta] - \rho_1 (\{1^p, \Delta\}_m) = \\
 &= C_{m+1}^p [1^{m+1}] \circ (r+1) [\Delta, 1] - C_m^p \cdot (r+1) [\Delta, 1] = (r+1) C_m^{p-1} [\Delta, 1], \\
 \{1^p, \Delta\}_{m+2} &= \rho_{m-p+2} [1^p] \circ \rho_2 [\Delta] - \rho_2 (\{1^p, \Delta\}_m) - \rho_1 (\{1^p, \Delta\}_{m+1}) = \\
 &= C_{m+2}^p [1^{m+2}] \circ \frac{(r+1)(r+2)}{2} [\Delta, 1, 1] - \frac{(r+1)(r+2)}{2} C_m^p [\Delta, 1, 1] \\
 &\quad - (r+2) \cdot (r+1) C_m^{p-1} [\Delta, 1, 1] = \\
 &= \frac{(r+1)(r+2)}{2} (C_{m+2}^p - C_m^p - 2C_m^{p-1}) [\Delta, 1, 1] = C_{r+2}^2 C_m^{p-2} [\Delta, 1, 1], \\
 &\dots \\
 [1^p] * \Delta &= C_m^p \Delta + (r+1) C_m^{p-1} [\Delta, 1] + C_{r+2}^2 C_m^{p-2} [\Delta, 1, 1] + \dots \\
 &= \sum_{i=0}^p C_{r+i}^i C_m^{p-i} [\Delta, 1^i] \tag{6.3}
 \end{aligned}$$

In this form the formula holds also for $p > m$, just the sum actually goes from $i = \max(0, p - m)$.

Eq. (6.3) can be rewritten also as

$$[1^p] * [\tilde{\Delta}, 1^r] = \sum_{i=0}^p C_{r+i}^i C_{|\tilde{\Delta}|+r}^{p-i} [\tilde{\Delta}, 1^{r+i}] \tag{6.4}$$

and, further,

$$I(x) * [\tilde{\Delta}, 1^r] = \sum_{p,i} x^p C_{|\tilde{\Delta}|}^{p-i} C_{r+i}^i [\tilde{\Delta}, 1^{i+r}] = (1+x)^{|\tilde{\Delta}|} \sum_i x^i C_{r+i}^i [\tilde{\Delta}, 1^{i+r}] \tag{6.5}$$

If we introduce now a new generating function $I_{\tilde{\Delta}}(x) = \sum_p x^p [\tilde{\Delta}, 1^p]$, then

$$I(x) * I_{\tilde{\Delta}}(y) = (1+x)^{|\tilde{\Delta}|} \sum_{i,r} C_{r+i}^i x^i (1+x)^r y^r [1^{r+i}] = (1+x)^{|\tilde{\Delta}|} I_{\tilde{\Delta}}(x+y+xy) \tag{6.6}$$

This formula describes the action of the $[1^p]$ -subring on the entire algebra of cut-and-join operators.

6.3 \circ -correlators in the $[1^p]$ subring

The \circ -multiplication is much simpler and in the $[1^p]$ sector can be written by the two generating functions. The first one is

$$\sum_{n=0}^{\infty} q^n \langle [1^n] \circ [1^n] \rangle t_1^n \bar{t}_1^n = \sum_{n=0}^{\infty} q^n \langle [1^n] \rangle t_1^n \bar{t}_1^n = \sum_{n=0}^{\infty} \frac{q^n t_1^n \bar{t}_1^n}{n!} = \exp(qt_1 \bar{t}_1) \quad (6.7)$$

Similarly,

$$\sum_{n=0}^{\infty} q^n \langle [1^n]^{\circ m} \rangle (t_1^{(1)} \dots t_1^{(m)})^n = \exp\left(qt_1^{(1)} \dots t_1^{(m)}\right) \quad (6.8)$$

The second generating function is

$$\begin{aligned} \sum_{n=0}^{\infty} q^n \left\langle \exp_{\circ}(\beta_1 [1^n]) \right\rangle &= \sum_{n=0}^{\infty} q^n \left\langle \delta_{n,0} + \beta_1 [1^n] + \frac{\beta_1^2}{2!} [1^n] \circ [1^n] + \dots \right\rangle \\ &= \sum_{n=0}^{\infty} \frac{q^n e^{\beta_1 n}}{n!} \end{aligned} \quad (6.9)$$

While each item in the sum is an average of a \circ -exponential, the sum over n is not, and one can *not* expect that *such* partition functions satisfy the WDVV equations.

One can also consider a simplified version of (6.9), with all β^{1^n} equal: $\beta_1^n = \beta_1$. Then (6.9) turns into

$$\sum_{n=0}^{\infty} q^n \left\langle \exp_{\circ}(\beta_1 [1^n]) \right\rangle = \sum_{n=0}^{\infty} q^n \left\langle \exp_{\circ}(\beta_1 \rho_{n-1}[1]) \right\rangle = e^{\beta_1 + q} \quad (6.10)$$

The standard Hurwitz partition function is direct generalization of this formula.

6.4 Connecting two multiplications

Formula (3.13) connecting the two multiplications can be further specified for the $[1^p]$ subring. Define in this case one more generating function

$$\sum_p x^p J_{[1^p]}(u) = \sum_{p,m \geq 0} u^{p+m} x^p \rho_m([1^p]) = \sum_{m,p \geq 0} \frac{(m+p)!}{m!p!} u^{p+m} x^p [1^{p+m}] \quad (6.11)$$

Then, in terms of $I(u) = \sum_m u^m [1^m]$, one has: $J_{[1^p]}(u) = u^p \partial_u^p I(u)/p!$ and one can easily relate (3.13) to $I(x) * I(y) = I(x + y + xy)$, so that

$$\oint J_{I(x)}(u) \circ J_{I(y)}\left(\frac{v}{u}\right) \frac{du}{u} = J_{I(x+y+xy)}(v) \quad (6.12)$$

Note that in the generic case it would be interesting to consider a generating function with the full set of time variable $\{p_k\}$, $J(u|p) = \sum_{\Delta} J_{\Delta}(u)p_{\Delta}$, so that

$$\oint J(u|p) \circ J\left(\frac{v}{u}|\bar{p}\right) \frac{du}{u} = \sum_{\Delta_1, \Delta_2} J_{\Delta_1 * \Delta_2}(v) \quad (6.13)$$

and realize if there is an interesting expression for the r.h.s.

7 Generalized Hurwitz partition function for the $[1^p]$ subring

7.1 *-correlators

Averaging converts $I(x)$ into the exponential:

$$\begin{aligned} \langle I(x) \rangle &= \sum_p \frac{x^p}{p!} = e^x, \\ \langle *_i I(x_i) \rangle &= \exp\left(-1 + \prod_i (x_i + 1)\right) \end{aligned} \quad (7.1)$$

or

$$1 + \log \langle *_i I(x_i) \rangle = \prod_i (1 + x_i) \quad (7.2)$$

In particular,

$$\sum_{k,l=0}^{\infty} \langle [1^k] * [1^l] \rangle t_1^k \bar{t}_1^l = \langle I(t_1 + \bar{t}_1 + t_1 \bar{t}_1) \rangle = \exp(t_1 + \bar{t}_1 + t_1 \bar{t}_1) \quad (7.3)$$

Similarly

$$\sum_{k_1, \dots, k_m=0}^{\infty} \langle [1^{k_1}] * \dots * [1^{k_m}] \rangle (t_1^{(1)})^{k_1} \dots (t_1^{(m)})^{k_m} = \exp\left(-1 + \prod_{i=1}^m \left(1 + t_m^{(i)}\right)\right) \quad (7.4)$$

7.2 Correlators of *-exponentials

It is convenient to introduce a grading of the diagram by its rescaling with a formal parameter q : $[1^p] \rightarrow q^p [1^p]$. The rescaled diagrams (operators $q^p \hat{W}[1^p]$) also form a *-ring, but with rescaled structure constants. In terms of the generating function $I_q(x) = \sum_p x^p q^p [1^p] = I(qx)$ one has $I_q(x) * I_q(y) = I(qx) * I(qy) = I_q(x + y + qxy)$. Note that for the \circ -multiplication $[1]^p \circ [1]^p = [1]^p$ and logarithm of the average $\log \langle \sum_{p,q} x^p y^q [1]^p \circ [1]^q \delta_{p,q} \rangle = xy$ is obtained from $\frac{1}{q} \log \langle I_q(x) * I_q(y) \rangle = \frac{x+y}{q} + xy$ in the limit of $q \rightarrow \infty$.

In order to check the associativity, one has to define the partition function

$$\begin{aligned} Z_*\{\beta|q\} &= \left\langle \exp_* \left(\sum_p \beta_{[1^p]} q^p [1^p] \right) \right\rangle \\ &= 1 + \sum_p \frac{1}{p!} q^p \beta_p + \frac{1}{2!} \sum_{p_1, p_2} \beta_{p_1} \beta_{p_2} \oint \frac{dx}{x^{p_1+1}} \oint \frac{dy}{y^{p_2+1}} e^{x+y+qxy} + \dots \end{aligned} \quad (7.5)$$

and check if its third derivatives w.r.t. β 's satisfy the WDVV equations.

This quantity can be studied using the technique developed above. For checks of the equation, one can use the perturbative expansion of $Z_*\{\beta|q\}$ into the power series in q . Let us see how this works in the leading order.

7.3 WDVV equations for the $[1^p]$ subring

Despite even for this subring the partition function depends on infinitely many variables and, hence, the matrices of the third derivatives

$$\left(\hat{C}_i\right)_{jk} \equiv \hat{C}_{ijk} = C_{ijk} = \frac{\partial^3 Z_*\{\beta|q\}}{\partial\beta_i\partial\beta_j\partial\beta_k} \quad (7.6)$$

is infinite-dimensional, the associativity equations can be explicitly checked.

In the leading order approximation one has to check the associativity of the non-deformed structure constants, i.e. (7.6) calculated at all $\beta_k = 0$. The generating functions for the structure constants A_{pq}^s of the subring,

$$[1^p] * [1^q] = \sum_{s=\max(p,q)}^{p+q} A_{pq}^s [1^s] \quad (7.7)$$

are given by

$$A^s(x, y) = \sum_{p,q} A_{pq}^s x^p y^q = (x + y + xy)^s \quad (7.8)$$

or

$$A(x, y; u) = \sum_{p,q,s} A_{pq}^s \frac{x^p y^q}{u^{s+1}} = \frac{1}{u - (x + y + xy)} \quad (7.9)$$

The associativity is guaranteed by the symmetricity of

$$\sum_{p,q,t} x^p y^q z^t \sum_s A_{pq}^s A_{st}^r = \oint A(x, y; u) A^r(u, z) du = (x + y + z + xy + yz + xz + xyz)^r \quad (7.10)$$

w.r.t. $x \leftrightarrow z$ and $y \leftrightarrow z$.

Associativity condition (7.10) should be complemented by

$$\begin{aligned} \langle [1^p] * [1^q] * [1^r] \rangle &= \sum_s A_{pq}^s \langle [1^s] * [1^r] \rangle, \\ e^{x+y+z+xy+yz+xz+xyz} &= \oint A(x, y; u) e^{u+z+uz} du \end{aligned} \quad (7.11)$$

which proves the WDVV equations in the leading order.

Further, one can switch on perturbations and check the WDVV equations explicitly in higher orders. We certainly know that they are correct basing on the general grounds, however, the procedure described here allows one to check this iteratively.

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A Multiplications of compositions

We illustrate the use of formulae of s.3 with the following example: $[2, 1^r] \circ [2, 1^r]$. Here $[2, 1^r] = (12) + \dots$ is a sum of $\|2, 1^r\| = C_{r+2}^2 = \frac{(r+1)(r+2)}{2}$ permutations. In the \circ -product each of them act on the permutations in another $[2, 1^r]$, and the action of, say, (12) by the rules (3.2) is different on (12), (13) and (34). Then, on (14) it is the same as on (13) and on (56) — the same as on (34) — the same in the sense that it produces an element in the same conjugation class. According to (3.2),

$$\begin{aligned} (12) \circ (12) &= () \in [1^{r+2}], \\ (12) \circ (13) &= (123) \in [3, 1^{r-1}], \\ (12) \circ (34) &= (12)(34) \in [22, 1^{r-2}] \end{aligned} \tag{A.1}$$

It remains to calculate the multiplicities:

$$\begin{aligned} \|2, 1^{r-1}\| \cdot 1 / \|1^{r+2}\| &= \|2, 1^r\| = C_{r+2}^2 = \frac{(r+1)(r+2)}{2}, \\ \|2, 1^{r-1}\| \cdot 2r / \|3, 1^{r-1}\| &= C_{r+2}^2 \cdot 2r / 2C_{r+2}^3 = 3, \\ \|2, 1^{r-1}\| \cdot C_r^2 / \|22, 1^{r-2}\| &= C_{r+2}^2 \cdot C_r^2 / (C_{r+2}^2 C_r^2 / 2) = 2 \end{aligned} \tag{A.2}$$

Here 1, $2r$ and C_r^2 are the numbers of permutations of the types (12), $(1k)$ or $(2k)$ with $k \neq 1, 2$ and (kl) with $k, l \neq 1, 2$ in the conjugation class $[2, 1^r]$. Of course, $1 + 2r + C_r^2 = \|2, 1^r\| = C_{r+2}^2$. Thus we obtain:

$$[2, 1^r] \circ [2, 1^r] = \frac{(r+1)(r+2)}{2} [1^{r+2}] + 3[3, 1^{r-1}] + 2[22, 1^{r-2}] \tag{A.3}$$

Similarly,

$$[2, 1^r] \circ [3, 1^{r-1}] = 2r[2, 1^r] + [32, 1^{r-3}] + 4[4, 1^{r-2}] \tag{A.4}$$

and so on.

- Similarly, for $[3, 1^m]$:

$$\begin{aligned} (123) \circ (123)K &= (132)K, \\ (123) \circ (132)K &= K, \\ (123) \circ (12C)K &= (1C)(23)K, \\ (123) \circ (13C)K &= (12C)K, \\ (123) \circ (1C)K &= (123C)K, \\ (123) \circ (C)K &= (123)CK, \\ (123) \circ (12C_1)(3C_2)K &= (12C_1 3C_2)K \\ (123) \circ (13C_1)(2C_2)K &= (13C_1 2C_2)K \\ (123) \circ (1C_1)(2C_2)(3C_3)K &= (1C_1 2C_2 3C_3)K \end{aligned} \tag{A.5}$$

Therefore

$$[3111] \circ [3111] = 40[1^6] + 8[2211] + 10[3111] + 2[33] + 5[51] \tag{A.6}$$

[11]	[2]
[2]	[11]

Table 1. Table of multiplication at level 2. It can be present in this form, since $[1^k] \circ \Delta = \Delta$.

and in general

$$\begin{aligned}
[3, 1^r] \circ [3, 1^r] &= \frac{2C_{r+3}^3}{2C_{r+3}^3} [3, 1^r] + \frac{2C_{r+3}^3}{1} [1^{r+3}] + \frac{2C_{r+3}^3 \cdot 3r}{C_{r+3}^2 C_{r+1}^2 / 2} [22, 1^{r-1}] + \\
&\quad + \frac{2C_{r+3}^3 \cdot 3r}{2C_{r+3}^3} [3, 1^r] + \frac{2C_{r+3}^3 \cdot 6C_r^2}{24C_{r+3}^5} [5, 1^{r-2}] + \frac{2C_{r+3}^3 \cdot 2C_r^3}{2C_{r+3}^3 \cdot 2C_r^3 / 2} [33, 1^{r-3}] = \\
&= \frac{(r+3)(r+2)(r+1)}{3} [1^{r+3}] + 8[22, 1^{r-1}] + (3r+1)[3, 1^r] + 8[33, 1^{r-3}] + 5[5, 1^{r-2}]
\end{aligned} \tag{A.7}$$

A check of multiplicities in this formula:

$$1 + 1 + 2 \cdot 3r + 6C_r^2 + 2C_r^3 = 2C_{r+3}^3 \tag{A.8}$$

• The same formulas (A.5) can be used to handle more general \circ -products involving the triple cycles. In particular,

$$\begin{aligned}
[3111] \circ [33] &= 2[311] + 2[33] + 12[6], \\
[3, 1^4] \circ [331] &= 8[3, 1^4] + 2[331] + 8[43] + 18[61]
\end{aligned} \tag{A.9}$$

and in general

$$\begin{aligned}
[3, 1^{r+3}] \circ [33, 1^r] &= \frac{(r+1)(r+2)(r+3)}{3} [3, 1^{r+3}] + 2[33, 1^r] + 3[333, 1^{r-3}] + \\
&\quad + 8[43, 1^{r-1}] + 5[531^{r-2}] + 6(r+2)[61^r]
\end{aligned} \tag{A.10}$$

B Multiplication tables

B.1 Group $S(2)$

$$|[11]| = 1, \quad |[2]| = 1 \tag{B.1}$$

See table 1.

B.2 Group $S(3)$

$$\begin{aligned}
[111] &= (), & |[111]| &= 1 \\
[21] &= (12) + (13) + (23), & |[21]| &= 3 \\
[3] &= (123) + (132), & |[3]| &= 2
\end{aligned} \tag{B.2}$$

See table 2.

[111]	[21]	[3]
[21]	3[111] + 3[3]	2[21]
[3]	2[21]	2[111] + [3]

Table 2. Table of multiplication at level 3.

[1111]	[211]	[22]	[31]	[4]
[211]	6[1111]+2[22]+3[31]	[211] + 2[4]	4[211] + 4[4]	4[22] + 3[31]
[22]	[211] + 2[4]	3[1111]+2[22]	3[31]	2[211] + [4]
[31]	4[211] + 4[4]	3[31]	8[1111]+8[22]+4[31]	4[211] + 4[4]
[4]	4[22] + 3[31]	2[211] + [4]	4[211] + 4[4]	6[1111]+2[22]+3[31]

Table 3. Table of multiplication at level 4.

B.3 Group $S(4)$

$$\begin{aligned}
 [1111] &= (), & ||1111|| &= 1 \\
 [211] &= (12) + (13) + (14) + (23) + (24) + (34), & ||211|| &= 6 \\
 [22] &= (12)(34) + (13)(24) + (14)(23), & ||22|| &= 3 \\
 [31] &= (123) + (132) + (124) + (142) + (134) + (143) + (234) + (243), & ||31|| &= 8 \\
 [4] &= (1234) + (1243) + (1324) + (1342) + (1423) + (1432), & ||4|| &= 6
 \end{aligned}$$

See table 3.

B.4 Group $S(5)$

$$\begin{aligned}
 [11111] &= (), & ||11111|| &= 1 \\
 [2111] &= (12) + \dots, & ||2111|| &= C_5^2 = 10 \\
 [221] &= (12)(34) + \dots, & ||221|| &= \frac{1}{2}C_5^2C_3^2 = 15 \\
 [311] &= (123) + (132) + \dots, & ||311|| &= 2C_5^3 = 20 \\
 [32] &= (123)(45) + \dots, & ||32|| &= ||311|| = 20 \\
 [41] &= (1234) + \dots, & ||41|| &= 3!C_5^1 = 30 \\
 [5] &= (12345) + \dots, & ||5|| &= 4! = 24
 \end{aligned} \tag{B.3}$$

$$\sum_{\Delta: |\Delta|=5} ||\Delta|| = 5! = 120 \tag{B.4}$$

See table 4.

[11111]	[2111]	[221]	[311]	[32]	[41]	[5]
[2111]	10[11111] +2[221] + 3[311]	3[2111] +3[32] + 2[41]	6[2111] +[32] + 4[41]	4[221] +[311] + 5[5]	9[311] +5[5]	8[41]
[221]	3[2111] +3[32] + 2[41]					
[311]	6[2111] +[32] + 4[41]		20[11111] + 8[221] +7[311] + 5[5]			
[32]	4[221] +[311] + 5[5]					
[41]	9[311] + 5[5]					
[5]	8[41]					

Table 4. Table of multiplication at level 5.

C Structure constants of *-multiplication

C.1 Level (1, 1)

When it does not cause confusion we omit square brackets in the notation of particular Young diagrams, to simplify the formulas.

$$[1] * [1] = \{1, 1\}_1 + \{1, 1\}_2, \tag{C.1}$$

$$\{1, 1\}_1 = [1] \circ [1] = [1],$$

$$\{1, 1\}_2 = \rho_1[1] \circ \rho_1[1] - \rho_1([1] \circ [1]) = \rho_1[1] \circ \rho_1[1] - \rho_1([1]) = 2[11] \circ 2[11] - 2 \circ [11] = 2[11]$$

Thus

$$[1] * [1] = \underline{[1]} + 2[11] \tag{C.2}$$

Here and below we underline the terms in the product, which belong to the \circ -product: they exist only when $|\Delta_1| = |\Delta_2|$ and are contained in the first term $\{\Delta_1, \Delta_2\}_{|\Delta_1|=|\Delta_2|} = \Delta_1 \circ \Delta_2$ with $n = |\Delta_1| = |\Delta_2|$ in the sum in (3.7).

C.2 Level (1, 2)

$$[1] * [11] = \{1, 11\}_2 + \{1, 11\}_3,$$

$$\{1, 11\}_2 = \rho_1[1] \circ [11] = 2[11] \circ [11] = 2[11],$$

$$\{1, 11\}_3 = \rho_2[1] \circ \rho_1[11] - \rho_1(\{1, 11\}_2) = 3[111] \circ 3[111] - 2 \cdot 3[111] = 3[111],$$

$$[1] * [11] = 2[11] + 3[111] \tag{C.3}$$

and

$$[1] * [2] = \{1, 2\}_2 + \{1, 2\}_3,$$

$$\{1, 2\}_2 = \rho_1[1] \circ [2] = 2[11] \circ [2] = 2[2],$$

$$\{1, 2\}_3 = \rho_2[1] \circ \rho_1[2] - \rho_1(\{1, 2\}_2) = 3[111] \circ [21] - 2[21] = [21],$$

$$[1] * [2] = 2[2] + [21] \tag{C.4}$$

C.3 Level (1, 3)

$$\begin{aligned}
 [1] * [111] &= \{1, 111\}_3 + \{1, 111\}_4, \\
 \{1, 111\}_3 &= \rho_2[1] \circ [111] = 3[111] \circ [111] = 3[111], \\
 \{1, 111\}_4 &= \rho_3[1] \circ \rho_1[111] - \rho_1(\{1, 111\}_3) = 4[1111] \circ 4[1111] - 4 \cdot 3[1111] = 4[1111], \\
 [1] * [111] &= 3[111] + 4[1111] \tag{C.5}
 \end{aligned}$$

$$\begin{aligned}
 [1] * [21] &= \{1, 21\}_3 + \{1, 21\}_4, \\
 \{1, 21\}_3 &= \rho_2[1] \circ [21] = 3[111] \circ [21] = 3[21], \\
 \{1, 21\}_4 &= \rho_3[1] \circ \rho_1[21] - \rho_1(\{1, 21\}_3) = 4[1111] \circ 2[211] - 2 \cdot 3[211] = 2[211], \\
 [1] * [21] &= 3[21] + 2[211] \tag{C.6}
 \end{aligned}$$

$$\begin{aligned}
 [1] * [3] &= \{1, 3\}_3 + \{1, 3\}_4, \\
 \{1, 3\}_3 &= \rho_2[1] \circ [3] = 3[111] \circ [3] = 3[3], \\
 \{1, 3\}_4 &= \rho_3[1] \circ \rho_1[3] - \rho_1(\{1, 3\}_3) = 4[1111] \circ [31] - 3[31] = [31], \\
 [1] * [3] &= 3[3] + [31] \tag{C.7}
 \end{aligned}$$

C.4 Level (2, 2)

$$\begin{aligned}
 [11] * [11] &= \{11, 11\}_2 + \{11, 11\}_3 + \{11, 11\}_4, \\
 \{11, 11\}_2 &= [11] \circ [11] = [11], \\
 \{11, 11\}_3 &= \rho_1[11] \circ \rho_1[11] - \rho_1(\{11, 11\}_2) = 3[111] \circ 3[111] - 3[111] = 6[111], \\
 \{11, 11\}_4 &= \rho_2[11] \circ \rho_2[11] - \rho_2(\{11, 11\}_2) - \rho_1(\{11, 11\}_3) \\
 &= 6[1111] \circ 6[1111] - 6[1111] - 4 \cdot 6[1111] = 6[1111], \\
 [11] * [11] &= \underline{[11]} + 6[111] + 6[1111] \tag{C.8}
 \end{aligned}$$

$$\begin{aligned}
 [11] * [2] &= \{11, 2\}_2 + \{11, 2\}_3 + \{11, 2\}_4, \\
 \{11, 2\}_2 &= [11] \circ [2] = [2], \\
 \{11, 2\}_3 &= \rho_1[11] \circ \rho_1[2] - \rho_1(\{11, 2\}_2) = 3[111] \circ [21] - [21] = 2[21], \\
 \{11, 2\}_4 &= \rho_2[11] \circ \rho_2[2] - \rho_2(\{11, 2\}_2) - \rho_1(\{11, 2\}_3) \\
 &= 6[1111] \circ [211] - [211] - 2 \cdot 2[211] = [211], \\
 [11] * [2] &= \underline{[2]} + 2[21] + [211] \tag{C.9}
 \end{aligned}$$

$$\begin{aligned}
 [2] * [2] &= \{2, 2\}_2 + \{2, 2\}_3 + \{2, 2\}_4, \\
 \{2, 2\}_2 &= [2] \circ [2] = [11], \\
 \{2, 2\}_3 &= \rho_1[2] \circ \rho_1[2] - \rho_1(\{2, 2\}_2) = [21] \circ [21] - 3[111] = 3[3], \\
 \{2, 2\}_4 &= \rho_2[2] \circ \rho_2[2] - \rho_2(\{2, 2\}_2) - \rho_1(\{2, 2\}_3) \\
 &= [211] \circ [211] - 6[1111] - 3[31] = 2[22], \\
 [2] * [2] &= \underline{[11]} + 3[3] + 2[22] \tag{C.10}
 \end{aligned}$$

C.5 Level (1, 4)

$$[1] * [1111] = \{1, 1111\}_4 + \{1, 1111\}_5,$$

$$\begin{aligned}
 \{1, 1111\}_4 &= \rho_3[1] \circ [1111] = 4[1111] \circ [1111] = 4[1111], \\
 \{1, 1111\}_5 &= \rho_4[1] \circ \rho_1[1111] - \rho_1(\{1, 1111\}_4) \\
 &= 5[1111] \circ 5[1111] - 5 \cdot 4[1111] = 5[1111], \\
 [1] * [1111] &= 4[1111] + 5[1111] \tag{C.11}
 \end{aligned}$$

$$\begin{aligned}
 [1] * [211] &= \{1, 211\}_4 + \{1, 211\}_5, \\
 \{1, 211\}_4 &= \rho_3[1] \circ [211] = 4[1111] \circ [211] = 4[211], \\
 \{1, 211\}_5 &= \rho_4[1] \circ \rho_1[211] - \rho_1(\{1, 211\}_4) \\
 &= 5[1111] \circ 3[211] - 3 \cdot 4[211] = 3[211], \\
 [1] * [211] &= 4[211] + 3[211] \tag{C.12}
 \end{aligned}$$

$$\begin{aligned}
 [1] * [22] &= \{1, 22\}_4 + \{1, 22\}_5, \\
 \{1, 22\}_4 &= \rho_3[1] \circ [22] = 4[1111] \circ [22] = 4[22], \\
 \{1, 22\}_5 &= \rho_4[1] \circ \rho_1[22] - \rho_1(\{1, 2\}_4) = 5[1111] \circ [22] - 4[22] = [22], \\
 [1] * [22] &= 4[22] + [22] \tag{C.13}
 \end{aligned}$$

$$\begin{aligned}
 [1] * [31] &= \{1, 31\}_4 + \{1, 31\}_5, \\
 \{1, 31\}_4 &= \rho_3[1] \circ [31] = 4[1111] \circ [31] = 4[31], \\
 \{1, 31\}_5 &= \rho_4[1] \circ \rho_1[31] - \rho_1(\{1, 31\}_4) = 5[1111] \circ 2[31] - 2 \cdot 4[31] = 2[31], \\
 [1] * [31] &= 4[31] + 2[31] \tag{C.14}
 \end{aligned}$$

$$\begin{aligned}
 [1] * [4] &= \{1, 4\}_4 + \{1, 4\}_5, \\
 \{1, 4\}_4 &= \rho_3[1] \circ [4] = 4[1111] \circ [4] = 4[4], \\
 \{1, 4\}_5 &= \rho_4[1] \circ \rho_1[4] - \rho_1(\{1, 4\}_4) = 5[1111] \circ [4] - 4[4] = [4], \\
 [1] * [4] &= 4[4] + [4] \tag{C.15}
 \end{aligned}$$

C.6 Level (2, 3)

$$\begin{aligned}
 [11] * [111] &= \{11, 111\}_3 + \{11, 111\}_4 + \{11, 111\}_5, \\
 \{11, 111\}_3 &= \rho_1[11] \circ [111] = 3[111], \\
 \{11, 111\}_4 &= \rho_2[11] \circ \rho_1[111] - \rho_1(\{11, 111\}_3) = 6[1111] \circ 4[111] - 4 \cdot 3[111] = 12[1111], \\
 \{11, 111\}_5 &= \rho_3[11] \circ \rho_2[111] - \rho_2(\{11, 111\}_3) - \rho_1(\{11, 111\}_4) = \\
 &= 10[11111] \circ 10[1111] - 10 \cdot 3[1111] - 5 \cdot 12[1111] = 10[11111], \\
 [11] * [111] &= 3[111] + 12[1111] + 10[11111] \tag{C.16}
 \end{aligned}$$

$$\begin{aligned}
 [11] * [21] &= \{11, 21\}_3 + \{11, 21\}_4 + \{11, 21\}_5, \\
 \{11, 21\}_3 &= \rho_1[11] \circ [21] = 3[21], \\
 \{11, 21\}_4 &= \rho_2[11] \circ \rho_1[21] - \rho_1(\{11, 21\}_3) = 6[1111] \circ 2[21] - 2 \cdot 3[21] = 6[211], \\
 \{11, 21\}_5 &= \rho_3[11] \circ \rho_2[21] - \rho_2(\{11, 21\}_3) - \rho_1(\{11, 21\}_4) = \\
 &= 10[11111] \circ 3[211] - 3 \cdot 3[211] - 3 \cdot 6[211] = 3[2111], \\
 [11] * [21] &= 3[21] + 6[211] + 3[2111] \tag{C.17}
 \end{aligned}$$

$$[11] * [3] = \{11, 3\}_3 + \{11, 3\}_4 + \{11, 3\}_5,$$

$$\begin{aligned}
 \{11, 3\}_3 &= \rho_1[11] \circ [3] = 3[3], \\
 \{11, 3\}_4 &= \rho_2[11] \circ \rho_1[3] - \rho_1(\{11, 3\}_3) = 6[1111] \circ [31] - 3[31] = 3[31], \\
 \{11, 3\}_5 &= \rho_3[11] \circ \rho_2[3] - \rho_2(\{11, 3\}_3) - \rho_1(\{11, 3\}_4) = \\
 &= 10[11111] \circ [311] - 3[311] - 2 \cdot 3[311] = [311], \\
 [11] * [3] &= 3[3] + 3[31] + [311] \tag{C.18}
 \end{aligned}$$

$$\begin{aligned}
 [2] * [111] &= \{2, 111\}_3 + \{2, 111\}_4 + \{2, 111\}_5, \\
 \{2, 111\}_3 &= \rho_1[2] \circ [111] = [21], \\
 \{2, 111\}_4 &= \rho_2[2] \circ \rho_1[111] - \rho_1(\{2, 111\}_3) = [211] \circ 4[1111] - 2[211] = 2[211], \\
 \{2, 111\}_5 &= \rho_3[2] \circ \rho_2[111] - \rho_2(\{2, 111\}_3) - \rho_1(\{2, 111\}_4) = \\
 &= [2111] \circ 10[11111] - 3[2111] - 3 \cdot 2[2111] = [2111], \\
 [2] * [111] &= [21] + 2[211] + [2111] \tag{C.19}
 \end{aligned}$$

$$\begin{aligned}
 [2] * [21] &= \{2, 21\}_3 + \{2, 21\}_4 + \{2, 21\}_5, \\
 \{2, 21\}_3 &= \rho_1[2] \circ [21] = [21] \circ [21] = 3[111] + 3[3], \\
 \{2, 21\}_4 &= \rho_2[2] \circ \rho_1[21] - \rho_1(\{2, 21\}_3) = [211] \circ 2[211] - 4 \cdot 3[1111] - 3[31] = 4[22] + 3[31], \\
 \{2, 21\}_5 &= \rho_3[2] \circ \rho_2[21] - \rho_2(\{2, 21\}_3) - \rho_1(\{2, 21\}_4) = \\
 &= [2111] \circ 3[2111] - 10 \cdot 3[11111] - 3[311] - 4[221] - 2 \cdot 3[311] = 2[221], \\
 [2] * [21] &= 3[111] + 3[3] + 4[22] + 3[31] + 2[221] \tag{C.20}
 \end{aligned}$$

$$\begin{aligned}
 [2] * [3] &= \{2, 3\}_3 + \{2, 3\}_4 + \{2, 3\}_5, \\
 \{2, 3\}_3 &= \rho_1[2] \circ [3] = [21] \circ [3] = 2[21], \\
 \{2, 3\}_4 &= \rho_2[2] \circ \rho_1[3] - \rho_1(\{2, 3\}_3) = [211] \circ [31] - 2 \cdot 2[211] = 4[4], \\
 \{2, 3\}_5 &= \rho_3[2] \circ \rho_2[3] - \rho_2(\{2, 3\}_3) - \rho_1(\{2, 3\}_4) = \\
 &= [2111] \circ [311] - 3 \cdot 2[2111] - 4[41] = [32], \\
 [2] * [3] &= 2[21] + 4[4] + [32] \tag{C.21}
 \end{aligned}$$

Above tables reproduce the ones from s.2.4.4 of [32, 33]. What follows are some new pieces of the *-multiplication tables.

C.7 Level (1, 5)

See table 5.

C.8 Level (3, 3)

$$[111] * [111] = \underline{[111]} + 12[1111] + 30[11111] + 20[111111] \tag{C.22}$$

$$[111] * [21] = \underline{[21]} + 6[211] + 9[2111] + 4[21111] \tag{C.23}$$

$$[111] * [3] = \underline{[3]} + 3[31] + 3[311] + [3111] \tag{C.24}$$

$$[21] * [21] = \underline{3[111]} + \underline{3[3]} + 12[1111] + 8[22] + 9[31] + 10[221] + 6[311] + 4[2211] \tag{C.25}$$

$$[21] * [3] = \underline{2[21]} + 4[211] + 8[4] + 3[32] + 4[41] + [321] \tag{C.26}$$

$$[3] * [3] = \{3, 3\}_3 + \{3, 3\}_4 + \{3, 3\}_5 + \{3, 3\}_6,$$

$[1] * [5]$	$= 5[5] + [51]$
$[1] * [41]$	$= 5[41] + 2[411]$
$[1] * [32]$	$= 5[32] + [321]$
$[1] * [311]$	$= 5[311] + 3[3111]$
$[1] * [221]$	$= 5[221] + 2[2211]$
$[1] * [2111]$	$= 5[2111] + 4[21111]$
$[1] * [11111]$	$= 5[11111] + 6[111111]$

Table 5. Table of *-multiplication at level (1,5).

$$\begin{aligned}
\{3, 3\}_3 &= [3] \circ [3] = 2[111] + [3], \\
\{3, 3\}_4 &= \rho_1[3] \circ \rho_1[3] - \rho_1(\{3, 3\}_3) = [31] \circ [31] - 4 \cdot 2[1111] - [31] = 8[22] + 3[31], \\
\{3, 3\}_5 &= \rho_2[3] \circ \rho_2[3] - \rho_2(\{3, 3\}_3) - \rho_1(\{3, 3\}_4) = \\
&= [311] \circ [311] - 10 \cdot 2[11111] - [311] - 8[221] - 2 \cdot 3[311] = 5[5], \\
\{3, 3\}_6 &= \rho_3[3] \circ \rho_3[3] - \rho_3(\{3, 3\}_3) - \rho_2(\{3, 3\}_4) - \rho_1(\{3, 3\}_5) = \\
&= [3111] \circ [3111] - 20 \cdot 2[1^6] - [3111] - 8[2211] - 3 \cdot 3[3111] - 5[51] = 2[33], \\
[3] * [3] &= \underline{2[111] + [3]} + 8[22] + 3[31] + 5[5] + 2[33] \tag{C.27}
\end{aligned}$$

C.9 Selected products

$$\begin{aligned}
[3] * [33] &= \{3, 33\}_6 + \{3, 33\}_7 + \{3, 33\}_8 + \{3, 33\}_9, \\
\{3, 33\}_6 &= \rho_3[3] \circ [33] = [3111] \circ [33] = 2[3111] + 2[33] + 12[6], \\
\{3, 33\}_7 &= \rho_4[3] \circ \rho_1[33] - \rho_1(\{3, 33\}_6) = \\
&= [31^4] \circ [331] - 8[31^4] - 2[331] - 12[61] = 8[43] + 6[61], \\
\{3, 33\}_8 &= \rho_5[3] \circ \rho_2[33] - \rho_2(\{3, 33\}_6) - \rho_1(\{3, 33\}_7) = 5[53], \\
\{3, 33\}_9 &= \rho_6[3] \circ \rho_3[33] - \rho_3(\{3, 33\}_6) - \rho_2(\{3, 33\}_7) - \rho_1(\{3, 33\}_8) = 3[333], \\
[3] * [33] &= 2[33] + 2[3111] + 12[6] + 8[43] + 6[61] + 5[53] + 3[333] \tag{C.28}
\end{aligned}$$

D The table of symmetric characters $\varphi_R(\Delta)$ at level 6

See table 6.

$R \Delta$	d_R	[6]	[51]	[42]	[411]	[33]	[321]	[3111]	[222]	[2211]	[21111]	[111111]
[6]	$\frac{1}{720}$	120	144	90	90	40	120	40	15	45	15	1
[51]	$\frac{1}{144}$	-24	0	-18	18	-8	0	16	-3	9	9	1
[42]	$\frac{1}{80}$	0	-16	10	-10	0	0	0	5	5	5	1
[411]	$\frac{1}{72}$	12	0	0	0	4	-12	4	-3	-9	3	1
[33]	$\frac{1}{144}$	0	0	-18	-18	16	24	-8	-9	9	3	1
[321]	$\frac{1}{45}$	0	9	0	0	-5	0	-5	0	0	0	1
[3111]	$\frac{1}{72}$	-12	0	0	0	4	12	4	3	-9	-3	1
[222]	$\frac{1}{144}$	0	0	-18	18	16	-24	-8	9	9	-3	1
[2211]	$\frac{1}{80}$	0	-16	10	10	0	0	0	-5	5	-5	1
[21111]	$\frac{1}{144}$	24	0	-18	-18	-8	0	16	3	9	-9	1
[111111]	$\frac{1}{720}$	-120	144	90	-90	40	-120	40	-15	45	-15	1

Table 6. Table of characters at level 6.

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