Tables in Signals and Systems

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Definitions

$$\operatorname{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t} \qquad \Omega_o \triangleq \frac{2\pi}{T_0}$$

I. Continuous-time Fourier series

A. Properties of Fourier series

Periodic signal	*Fourier serie coefficient*
$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_o t}$	$a_k \triangleq \frac{1}{T_o}\int_{T_o} x(t) e^{-jk\Omega_o t} dt$
$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with periodT_0	$\begin{matrix} a_k \\ b_k \end{matrix}$
$Ax(t) + By(t)$	$Aa_k + Bb_k$
$x(t - t_0)$	$a_k e^{-jk(2\pi/T_0)t_0}$
$e^{jM(2\pi/T_0)t} x(t)$	a_{k-M}
$x^*(t)$	a^*_{-k}
$x(-t)$	a_{-k}
$x(\alpha t), \alpha > 0$ (Periodic with period T_0/α)	a_k
$\int_{T_0} x(\tau) y(t-\tau) d\tau$	$T_0 a_k b_k$
$x(t) y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
$\frac{d}{dt} x(t)$	$jk\frac{2\pi}{T_0} a_k$
$\int_{-\infty}^{t} x(\tau) d\tau$ (Bounded and periodic only if $a_0 = 0$)	$\frac{1}{jk(2\pi/T_0)} a_k$
If $x(t)$ is real valued then	
$x(t)$	$\left\{\begin{matrix} a_k = a^*_{-k} \\ \Re\{a_k\} = \Re\{a_{-k}\} \\ \Im\{a_k\} = -\Im\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \arg\{a_k\} = -\arg\{a_{-k}\} \end{matrix}\right.$
$x_e(t) = \mathcal{E}\{x(t)\}$ $x_o(t) = \mathcal{O}\{x(t)\}$	$\Re\{a_k\}$ $j\Im\{a_k\}$

$$a_k e^{jk\Omega_0 t} + a_{-k} e^{-jk\Omega_0 t} = 2\Re\{a_k\}\cos(k\Omega_0 t) - 2\Im\{a_k\}\sin(k\Omega_0 t)$$

Parsevals relation for periodic signals

$$\frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

B. *Fourier series table*

	$x(t)$	a_k or the Fourier series expansion
a)	$\sum_{n=-\infty}^{\infty} \delta(t-nT)$	$a_k = \frac{1}{T}$, all k
b)	1	$(a_0 = 1,\ a_k = 0 \text{ otherwise}), \quad \forall\ T_0 > 0$
c)	$e^{j\Omega_o t}$	$a_1 = 1,\ a_k = 0$ otherwise
d)	$\cos \Omega_o t$	$a_1 = a_{-1} = \frac{1}{2},\ a_k = 0$ otherwise
e)	$\sin \Omega_o t$	$a_1 = -a_{-1} = \frac{1}{2j},\ a_k = 0$ otherwise
f)	$\begin{cases} 1, & \mid t \mid < T_1 \\ 0, & T_1 < \mid t \mid \leq \frac{T_o}{2} \end{cases}$ period T_0	$a_k = \frac{\Omega_o T_1}{\pi} \text{sinc} \frac{k\Omega_o T_1}{\pi} = \frac{\sin k\Omega_o T_1}{k\pi}$
g)	$\begin{cases} 1, & 0 < t < \pi \\ -1, & -\pi < t < 0 \end{cases}$	$\frac{4}{\pi}\left(\frac{\sin t}{1} + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots\right)$
h)	$\lvert t \rvert = \begin{cases} t, & 0 < t < \pi \\ -t, & -\pi < t < 0 \end{cases}$	$\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos t}{1^2} + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \cdots\right)$
i)	$t, \quad -\pi < t < \pi$	$2\left(\frac{\sin t}{1} - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \cdots\right)$
j)	$t, \quad 0 < t < 2\pi$	$\pi - 2\left(\frac{\sin t}{1} + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \cdots\right)$
k)	$\lvert \sin t \rvert, \quad -\pi < t < \pi$	$\frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2t}{1\cdot 3} + \frac{\cos 4t}{3\cdot 5} + \frac{\cos 6t}{5\cdot 7} + \cdots\right)$
l)	$\begin{cases} 0, & 0 < t < \pi - a \\ 1, & \pi - a < t < \pi + a \\ 0, & \pi + a < t < 2\pi \end{cases}$	$\frac{a}{\pi} - \frac{2}{\pi}\left(\frac{\sin a \cos t}{1} - \frac{\sin 2a \cos 2t}{2} + \frac{\sin 3a \cos 3t}{3} - \cdots\right)$

II. Continuous-time Fourier transform

A. Properties of the Fourier transform

Non-periodic signal	*Fourier transform*
$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$	$X(j\Omega) \stackrel{\triangle}{=} \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$
Alternativly with frequency f instead of angular frequency Ω.	
$x(t) = \int_{-\infty}^{\infty} X_f(f)e^{j2\pi ft}df$	$X_f(f) \stackrel{\triangle}{=} \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = X(\omega)_{\{\Omega=2\pi f\}}$
$x(t)$ $y(t)$	$X(j\Omega)$ $Y(j\Omega)$
$ax(t) + by(t)$	$aX(j\Omega) + bY(j\Omega)$
$x(t - t_0)$	$e^{-j\Omega t_0}X(j\Omega)$
$e^{j\Omega_0 t}x(t)$	$X(j(\Omega - \Omega_0))$
$x^*(t)$	$X^*(j(-\Omega))$
$x(-t)$	$X(j(-\Omega))$
$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\Omega}{a}\right)$
$x(t) * y(t)$	$X(j\Omega)Y(j\Omega)$
$x(t)y(t)$	$\frac{1}{2\pi}X(j\Omega) * Y(j\Omega)$
$\frac{d}{dt}x(t)$	$j\Omega X(j\Omega)$
$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\Omega}X(j\Omega) + \pi X(0)\delta(\Omega)$
$tx(t)$	$j\frac{d}{d\Omega}X(j\Omega)$
If $x(t)$ is real valued then	
$x(t)$	$\begin{cases} X(j\Omega) = X^*(j(-\Omega)) \\ \Re\{X(j\Omega)\} = \Re\{X(j(-\Omega))\} \\ \Im\{X(j\Omega)\} = -\Im\{X(j(-\Omega))\} \\ \lvert X(j\Omega)\rvert = \lvert X(j(-\Omega))\rvert \\ \arg\{X(j\Omega)\} = -\arg\{X(j(-\Omega))\} \end{cases}$
$x_e(t) = \mathcal{E}\{x(t)\}$ $x_o(t) = \mathcal{O}\{x(t)\}$	$\Re\{X(j\Omega)\}$ $j\Im\{X(j\Omega)\}$

Duality

$$f(u) = \int_{-\infty}^{\infty} g(v)e^{-juv}dv, \qquad g(t) \stackrel{\mathcal{F}}{\longleftrightarrow} f(j\Omega), \quad f(t) \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi g(j(-\Omega))$$

Parsevals relation for non-periodic signals

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega$$

B. Fourier transform table

The table is valid for $\Re\{\alpha\} > 0$ and $\Re\{\beta\} > 0$

	$x(t)$	$X(j\Omega)$	$X(f)$
a)	$u\left(t+\frac{T}{2}\right)-u\left(t-\frac{T}{2}\right)$	$T\frac{\sin\Omega T/2}{\Omega T/2}$	$T\frac{\sin\pi fT}{\pi fT}=T\text{sinc}(fT)$
b)	$\frac{\sin Wt}{\pi t}=\frac{W}{\pi}\text{sinc}\frac{Wt}{\pi}$	$u(\Omega+W)-u(\Omega-W)$	$u\left(f+\frac{W}{2\pi}\right)-u\left(f-\frac{W}{2\pi}\right)$
c)	$\begin{cases} 1-2\frac{\mid t\mid}{T}, & \mid t\mid<\frac{T}{2} \\ 0, & \text{otherwise}\end{cases}$	$\frac{T}{2}\left[\frac{\sin\Omega T/4}{\Omega T/4}\right]^2$	$\frac{T}{2}\text{sinc}^2(Tf/2)$
d)	$e^{-\alpha t}u(t)$	$\frac{1}{j\Omega+\alpha}$	$\frac{1}{j2\pi f+\alpha}$
e)	$e^{-\alpha\|t\|}$	$\frac{2\alpha}{\Omega^2+\alpha^2}$	$\frac{2\alpha}{(2\pi f)^2+\alpha^2}$
f)	$\frac{1}{\beta-\alpha}\left[e^{-\alpha t}-e^{-\beta t}\right]u(t)$	$\frac{1}{(j\Omega+\alpha)(j\Omega+\beta)}$	$\frac{1}{(j2\pi f+\alpha)(j2\pi f+\beta)}$
g)	$te^{-\alpha t}u(t)$	$\frac{1}{(j\Omega+\alpha)^2}$	$\frac{1}{(j2\pi f+\alpha)^2}$
h)	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(j\Omega+\alpha)^n}$	$\frac{1}{(j2\pi f+\alpha)^n}$
i)	$e^{-(\alpha t)^2}$	$\frac{\sqrt{\pi}}{\alpha}e^{-(\Omega/2\alpha)^2}$	$\frac{\sqrt{\pi}}{\alpha}e^{-(\pi f/\alpha)^2}$
j)	$e^{-\alpha t}\sin(\Omega_o t)u(t)$	$\frac{\Omega_o}{(j\Omega+\alpha)^2+\Omega_o^2}$	$\frac{\Omega_o}{(j2\pi f+\alpha)^2+\Omega_o^2}$
	$e^{\alpha t}\sin(\Omega_o t)u(-t)$	$\frac{-\Omega_o}{(\alpha-j\Omega)^2+\Omega_o^2}$	$\frac{-\Omega_o}{(\alpha-j2\pi f)^2+\Omega_o^2}$
k)	$e^{-\alpha t}\cos(\Omega_o t)u(t)$	$\frac{\alpha+j\Omega}{(j\Omega+\alpha)^2+\Omega_o^2}$	$\frac{\alpha+j2\pi f}{(j2\pi f+\alpha)^2+\Omega_o^2}$
	$e^{\alpha t}\cos(\Omega_o t)u(-t)$	$\frac{\alpha-j\Omega}{(\alpha-j\Omega)^2+\Omega_o^2}$	$\frac{\alpha-j2\pi f}{(\alpha-j2\pi f)^2+\Omega_o^2}$
l)	$(\cos\Omega_o t)\left[u\left(t+\frac{T}{2}\right)-u\left(t-\frac{T}{2}\right)\right]$	$\frac{T}{2}\left[\frac{\sin(\Omega-\Omega_o)\frac{T}{2}}{(\Omega-\Omega_o)\frac{T}{2}}+\frac{\sin(\Omega+\Omega_o)\frac{T}{2})}{(\Omega+\Omega_o)\frac{T}{2})}\right]$	$\frac{T}{2}\left[\frac{\sin\pi T(f-f_o)}{\pi T(f-f_o)}+\frac{\sin\pi T(f+f_o)}{\pi T(f+f_o)}\right]$

Generalized Fourier transform (power signals)

	$x(t)$	$X(j\Omega)$	$X(f)$
a)	$\delta(t)$	1	1
b)	$\delta(t-t_0)$	$e^{-j\Omega t_0}$	$e^{-j2\pi f t_0}$
c)	$\sum_{n=-\infty}^{\infty} \delta(t-nT)$	$\frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\Omega-\frac{2\pi}{T}n)$	$\frac{1}{T}\sum_{n=-\infty}^{\infty} \delta(f-\frac{n}{T})$
d)	$u(t)$	$\pi\delta(\Omega)+\frac{1}{j\Omega}$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
e)	$\mathrm{sgn}(t)=\frac{t}{\mid t \mid}$	$\frac{2}{j\Omega}$	$\frac{1}{j\pi f}$
f)	$\frac{1}{\pi t}$	$-j\mathrm{sgn}(\Omega)$	$-j\mathrm{sign}(f)$
g)	K	$2\pi K\delta(\Omega)$	$K\delta(f)$
h)	$tu(t)$	$j\pi\delta'(\Omega)-\frac{1}{\Omega^2}$	$\frac{j}{4\pi}\delta'(f)-\frac{1}{4\pi^2 f^2}$
i)	t^n	$2\pi(j)^n\delta^{(n)}(\Omega)$	$\left(\frac{j}{2\pi}\right)^n\delta^{(n)}(f)$
j)	$\cos\Omega_o t$	$\pi[\delta(\Omega-\Omega_o)+\delta(\Omega+\Omega_o)]$	$\frac{1}{2}[\delta(f-f_o)+\delta(f+f_o)]$
k)	$\sin\Omega_o t$	$\frac{\pi}{j}[\delta(\Omega-\Omega_o)-\delta(\Omega+\Omega_o)]$	$\frac{1}{j2}[\delta(f-f_o)-\delta(f+f_o)]$
l)	$\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}$	$2\pi\sum_{n=-\infty}^{\infty} c_n\delta\left(\Omega-\frac{2\pi n}{T}\right)$	$\sum_{n=-\infty}^{\infty} c_n\delta\left(f-\frac{n}{T}\right)$
m)	$e^{j\Omega_o t}$	$2\pi\delta(\Omega-\Omega_o)$	$\delta(f-f_o)$
n)	Periodic square wave $\begin{cases} 1, & \mid t \mid\leq T_1 \\ 0, & T_1 <\mid t \mid\leq \frac{T_o}{2}\end{cases}$ periodT_o	$\sum_{k=-\infty}^{\infty} A_k(\Omega)\delta(\Omega-k\Omega_o)$ $A_k(\Omega)=\frac{2\sin k\Omega_o T_1}{k}$	$\sum_{k=-\infty}^{\infty} A_k(f)\delta(f-kf_o)$ $A_k(f)=\frac{\sin k2\pi f_o T_1}{k\pi}$

III. Discrete-time Fourier series

A. Properties of discrete-time Fourier series

Periodic signal	*Fourier serie coefficient*
$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$	$a_k \triangleq \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$
$\left.\begin{array}{r} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N	$\left.\begin{array}{r} a_k \\ b_k \end{array}\right\}$ Periodic with period N
$Ax[n] + By[n]$	$Aa_k + Bb_k$
$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
$x^*[n]$	a^*_{-k}
$x[-n]$	a_{-k}
$x_{(m)}[n] = \begin{cases} x[n/m], & \text{If } n \text{ is a multiple av } m \\ 0, & \text{otherwise} \end{cases}$	$\frac{1}{m} a_k, \quad$ period mN
$\sum_{r=<N>} x[r] y[n-r]$	$N a_k b_k$
$x[n] y[n]$	$\sum_{l=<N>} a_l b_{k-l}$
$x[n] - x[n-1]$	$\left(1 - e^{-j2\pi/Nk}\right) a_k$
$\sum_{k=-\infty}^{n} x[k]$ Bounded and periodic only if $a_0 = 0$	$\frac{1}{1 - e^{-jk2\pi/N}} a_k$
If $x[n]$ is real valued then	
$x[n]$	$\begin{cases} a_k = a^*_{-k} \\ \Re\{a_k\} = \Re\{a_{-k}\} \\ \Im\{a_k\} = -\Im\{a_{-k}]\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \arg\{a_k\} = -\arg\{a_{-k}\} \end{cases}$
$x_e[n] = \mathcal{E}\{x[n]\}$	$\Re\{a_k\}$
$x_o[n] = \mathcal{O}\{x[n]\}$	$j\Im\{a_k\}$

Parsevals relation for periodic signals

$$\frac{1}{N} \sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$$

B. Fourier series table

$x[n]$	a_k
$\sum_{k=-\infty}^{\infty} \delta(n-kN)$	$a_k = \frac{1}{N}$,for all k
1	$a_k = \begin{cases} 1, & k=0,\pm N,\pm 2N,\ldots \\ 0, & \text{otherwise} \end{cases}$
$e^{j\omega_o n}$	$\begin{cases} \omega_o = \frac{2\pi m}{N} \\ a_k = \begin{cases} 1, & k=m,m\pm N,m\pm 2N,\ldots \\ 0, & \text{otherwise} \end{cases} \end{cases}$ $\frac{\omega_o}{2\pi} = \text{irrational}:$ The signal is non-periodic
$\cos \omega_o n$	$\begin{cases} \omega_o = \frac{2\pi m}{N} \\ a_k = \begin{cases} \frac{1}{2}, & k=\pm m,\pm m\pm N,\pm m\pm 2N,\ldots \\ 0, & \text{otherwise} \end{cases} \end{cases}$ $\frac{\omega_o}{2\pi} = \text{irrational}:$ The signal is non-periodic
$\sin \omega_o n$	$\begin{cases} \omega_o = \frac{2\pi m}{N} \\ a_k = \begin{cases} \frac{1}{2j}, & k=m,m\pm N,m\pm 2N,\ldots \\ -\frac{1}{2j}, & k=-m,-m\pm N,-m\pm 2N,\ldots \end{cases} \end{cases}$ $\frac{\omega_o}{2\pi} = \text{irrational}:$ The signalen is non-periodic
$\begin{cases} 1, & \mid n \mid \le N_1 \\ 0, & N_1 < \mid n \mid \le \frac{N}{2} \end{cases}$ period N	$a_k = \begin{cases} \frac{\sin \frac{2\pi k}{N}\left(N_1+\frac{1}{2}\right)}{N \sin \frac{\pi k}{N}}, & k \neq 0,\pm N,\pm 2N,\ldots \\ \frac{2N_1+1}{N}, & k=0,\pm N,\pm 2N,\ldots \end{cases}$

IV. Discrete-time Fourier transform

A. Properties of the discrete-time Fourier transform

Non-periodic signal	*Fourier transform*
$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$	$X(e^{j\omega}) \stackrel{\triangle}{=} \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ Periodic with period 2π
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
$x^*[n]$ $x[-n]$	$X^*(e^{j(-\omega)})$ $X(e^{j(-\omega)})$
$x_{(m)}[n] = \begin{cases} x[n/m], & n \text{ multiple of } m \\ 0, & n \text{ not multiple av } m \end{cases}$	$X(e^{j(m\omega)})$
$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
$x[n] - x[n-1]$	$\left(1 - e^{j\omega}\right)X(e^{j\omega})$
$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{j}\omega}X(e^{j\omega}) + \pi X(0)\sum_{k=-\infty}^{\infty}\delta(\omega - 2\pi k)$
$nx[n]$	$j\frac{d}{d\omega}X(e^{j\omega})$
If $x[n]$ is real valued then	
$x[n]$	$\begin{cases} X(e^{j\omega}) = X^*(e^{j(-\omega)}) \\ \Re\{X(e^{j\omega})\} = \Re\{X(e^{j(-\omega)})\} \\ \Im\{X(e^{j\omega})\} = -\Im\{X(e^{j(-\omega)})\} \\ \left\|X(e^{j\omega})\right\| = \left\|X(e^{j(-\omega)})\right\| \\ \arg\{X(e^{j\omega})\} = -\arg\{X(e^{j(-\omega)})\} \end{cases}$
$x_e[n] = \mathcal{E}\{x[n]\}$ $x_o[n] = \mathcal{O}\{x[n]\}$	$\Re\{X(e^{j\omega})\}$ $j\Im\{X(e^{j\omega})\}$

Parsevals relation for non-periodic signals

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

B. *Discrete-time Fourier transform table*

$x[n]$	$X(e^{j\omega})$
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$\sum_{k=-\infty}^{\infty} \delta(n-kN)$	$\frac{2\pi}{N}\sum_{k=-\infty}^{\infty} \delta\left(\omega-\frac{2\pi k}{N}\right)$
1	$2\pi\sum_{k=-\infty}^{\infty} \delta\left(\omega-2\pi k\right)$
$e^{j\omega_o n}$	$2\pi\sum_{k=-\infty}^{\infty} \delta\left(\omega-\omega_o-2\pi k\right)$
$\cos\omega_o n$	$\pi\sum_{k=-\infty}^{\infty} [\delta(\omega-\omega_o-2\pi k)+\delta(\omega+\omega_o-2\pi k)]$
$\sin\omega_o n$	$\frac{\pi}{j}\sum_{k=-\infty}^{\infty} [\delta(\omega-\omega_o-2\pi k)-\delta(\omega+\omega_o-2\pi k)]$
$u[n]$	$\frac{1}{1-e^{-j\omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$
$a^n u(n), \quad \|a\|<1$	$\frac{1}{1-ae^{-j\omega}}$
$(n+1)a^n u[n], \quad \|a\|<1$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{(n+m-1)!}{n!(m-1)!}a^n u[n], \quad \|a\|<1$	$\frac{1}{(1-ae^{-j\omega})^m}$
$\frac{1}{1-a^2}a^{\|n\|}, \quad \|a\|<1$	$\frac{1}{1+a^2-2acos\omega}$
$\begin{cases} 1, & \|n\|\le N_1 \\ 0, & N_1<\|n\|\le \frac{N}{2} \end{cases}$ period N	$2\pi\sum_{k=-\infty}^{\infty} a_k\delta\left(\omega\frac{2\pi k}{N}\right)$
$\begin{cases} 1, & \|n\|\le N_1 \\ 0, & \|n\|>N_1 \end{cases}$	$\frac{\sin\omega\left(N_1+\frac{1}{2}\right)}{\sin\frac{\omega}{2}}$
$\begin{cases} \frac{\sin Wn}{\pi n}=\frac{W}{\pi}\text{sinc}\frac{Wn}{\pi} \\ 0<W<\pi \end{cases}$	$\begin{cases} 1, & \|\omega\|\le W \\ 0, & W<\|\omega\|\le\pi \end{cases}$ period 2π

V. Sampling and reconstruction

The sampling theorem:

Let $x(t)$ with transform $X_c(j\Omega)$ be a bandlimited signal such that $X_c(j\Omega) = 0, \ \ |\Omega| > \Omega_M$. Then $x(t)$ is uniquely described by the samples $x(nT), \ \ n = 0, \pm 1 \pm 2 \ldots$ if

$$\Omega_s > 2\Omega_M$$

where

$$\Omega_s = \frac{2\pi}{T} = 2\pi f_s$$

Given $x(nT)$, if the sampling theorem is satisfied, it is possible with an ideal reconstruction filter to exactly reconstruct $x(t)$.

Discrete-time processing of continuous-time signals

Sampling:

$$x_d(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \longleftrightarrow X_d(j\Omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\Omega nT}$$

"Normalization in time" gives

$$x[n] = x(nT) \longleftrightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\Omega nT}$$

where

$$\Omega T = \omega = \frac{\Omega}{f_s} \ \text{ or } \ fT = q = \frac{f}{f_s}$$

Poissons summation formula:

$$X_d(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\frac{\Omega}{T} - \frac{2\pi k}{T}))$$

If the sampling theorem is satisfied then

$$X_d(j\Omega) = \frac{1}{T}X(j\Omega), \quad -\frac{\pi}{T} < \Omega < \frac{\pi}{T}$$

or

$$X_d(f) = \frac{1}{T}X(f), \quad -\frac{1}{2T} < f < \frac{1}{2T}$$

Ideal reconstruction:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)h(t - nT)$$

where

$$h(t) = T\frac{\Omega_c}{\pi}\text{sinc}\frac{\Omega_c t}{\pi} \longleftrightarrow H(j\Omega) = \begin{cases} T, & \mid \omega \mid \leq \Omega_c \\ 0, & \text{otherwise} \end{cases}$$

VI. Z-transform

A. Properties of the Z-transform

signal	*Z-transform*	*ROC*
$x[n]$	$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n}$	R_x
$ax[n] + by[n]$	$aX(z) + bY(z)$	Contains $R_x \cap R_y$
$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except possible addition or deletion of the origin or ∞
$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
$x^*[n]$	$X^*(z^*)$	R_x
$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
$x[n] * y[n]$	$X(z)Y(z)$	Contains $R_x \cap R_y$
$nx[n]$	$-z\frac{d}{dz}X(z)$	R_x, except possible addition or deletion of the origin or ∞
$\Re\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
$\Im\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x

Initial value theorem

$x[n] = 0,\ n < 0 \quad \lim_{z\to\infty} X(z) = x[0]$

B. Z-transform table

$x[n]$	$X(z)$	*ROC*
$\delta[n]$	1	All z
$\delta[n-n_0]$	z^{-n_0}	All z, except $0 (n_0 > 0)$ or $\infty (n_0 < 0)$
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > a$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < a$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > a$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < a$
$[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
$[\sin\omega_0 n]u[n]$	$\frac{1-[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
$[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
$[r^n\sin\omega_0 n]u[n]$	$\frac{1-[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
$\begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$