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Painlevé equations and representation theory

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1 Summary of the thesis

This research is devoted to studying solutions of Painlevé equations and their q-deformations, using methods of representation theory and geometry of instanton moduli space.

1.1 Historical survey

Painlevé equations were introduced more than 100 years ago, initially as second-order differential equations with no movable singular points except poles in the works of P. Painlevé and B. Gambier [P1900], [P1900], [G1910].

A bit later in works of R. Fuchs [F1905], [F1907], it was found that Painlevé equations naturally arise as simplest non-trivial cases of isomonodromic deformations of linear systems on $\mathbb{CP}^1$. Isomonodromic deformation theory was subsequently developed in papers of Schlesinger [S1912] and Garnier [G1912], [G1917]. It was also found that Painlevé equations admit non-autonomous Hamiltonian formulation [M1922]. Also, at that time by the works [B1911], [B1913] of G.D.Birkhoff started study of the Riemann-Hilbert problem for the difference and q-difference linear equations.

Starting from the early seventies, Painlevé equations and their solutions (called Painlevé transcendents) have been playing an increasingly important role in mathematical physics, especially in the applications to classical and quantum integrable systems and random matrix theory. For example, this includes such problems as

- Spin-spin correlation function in 2D Ising model [WMTB76]
- Density matrix of an impenetrable Bose gas [JMMS80]
- KPZ scaling of one-dimensional ASEP on integer lattice [TW09]
- Scaling functions in 2D polymers [FS92], [Z94] (see also [L11] for generalizations)
- Fredholm determinants of integrable kernels [TW92], [TW93], see ref. in [GIL13]
- Sine–Gordon model at the free-fermion point [BL94], [SMJ78]
- Matrix models (see book [F10Book] and references therein)

In parallel, several fundamental notions for the Painlevé theory were introduced. In particular, in the work [JMU81] there was introduced notion of isomonodromic tau function. These tau functions shows up in different applications of Painlevé equations, probably, the most important is its intimate connection with quantum field theory, which was discovered in the first two papers of the series [SMJ78].

Another fundamental notion, important for this thesis is space of initial conditions of Painlevé equations, introduced by K. Okamoto in [O79]. This space is certain compactification of $\mathbb{C}^2$, s.t. unique solution of corresponding Painlevé equation passes through each point of this space, its symmetries are symmetries of initial Painlevé equation.

This survey do not pretend to provide full overview of Painlevé theory developments and applications, so see also reviews and books [CDL], [C99], [FIKN06], as well as reviews cited below.

At this time also investigation of discrete integrable systems started and this study, in particular, attract attention to discretisation of Painlevé equations. Systematical approach to discrete Painlevé equations started from the work [GRP91] where it was introduced the concept of singularity confinement as a discrete counterpart of the Painlevé property and applied (in [RGH91]) to obtain non-autonomous version of QRT mappings (see [QRT88], [QRT89]). For these and further developments in study of discrete Painlevé equations see reviews [GR04], [TTGR04]. Also, in those years, q-Painlevé VI equation was constructed in [JS96] from q-isomonodromic problem.

Other approach to discrete Painlevé equations was presented by H. Sakai in the celebrated work [S01], which generalize Okamoto approach. In this work he classified certain rational surfaces and related each surface to differential, difference or q-difference Painlevé equations, thereby classifying discrete Painlevé equations. Notion of tau function for discrete Painlevé equations also appear in this approach [T06]. This approach appeared to be very powerful, one see recent review [KNY15].

The celebrated work by Gamayun, Iorgov and Lisovyy [GIL12] introduced connection between Painlevé equations (and, more generally, isomonodromic problems on $\mathbb{CP}^1$) and Conformal Field Theory. Namely, they proposed formula for general (2-parametric) solution of Painlevé VI equation: tau function is a Fourier series of Virasoro $c = 1$ four-point conformal blocks of Conformal Field Theory. This result gave a new impetus for studying of Painlevé equations and, especially, their solutions. This resulted in the series of new results (including the results of the thesis)

- Initially proposed (conjectured) formula was proved by several different approaches [ILT14], [BS14] (see also [BS16b]), [GL16], [NTalk].
- Initial conjecture was extended (and proved) to Painlevé V and III's equations [GIL13], Garnier systems [ILT14], higher rank isomonodromic problems [G15], general Fuchsian systems [GIL18], [GIL18FST], isomonodromic problem on torus [BMGT19], [BMGT19W].
- Expansions of tau functions in irregular singularities were studied [ItsLTy14], [N15], [BLMST16], [N18].
- Different Fredholm determinant representations of tau function were constructed and studied [GL16], [GL17], [CGL17], [GIL18FST].
- Connection problem for tau function expansions in different points was studied for different Painlevé equations [ILTy13], [ItsLTy14], [ItsLP16], results were applied for construction of crossing invariant correlation functions in $c = 1$ CFT [GS18].
- q-deformation of initial formula was built and proved [BS16q], [JNS17], [BGM18], [MN18], [BS18].
- Correspondence between deautonomized Goncharov-Kenyon integrable systems and Sakai classification of q-deformed Painlevé equations was constructed, using cluster nature of latter integrable systems [BGM17]. Under this approach in loc.cit., quantization of isomonodromic tau function was introduced.
- Initial formula and q-deformed formula for resonant values of parameters was interpreted through matrix models [MM17], [MM17q], [MMZ19]
- $c = -2$ analog of Painlevé tau function was introduced and studied [BS18]
- Tau functions of Painlevé equations appear to be connected to Topological Strings/Spectral Theory duality [BGT16], [BGT17]. Particularly, in loc. cit. discussed formula for tau function used to prove certain limit version of TS/ST duality.
- Developing ideas of [BE11], [EM08], in paper [I19] there was constructed 2-parametric tau function of Painlevé I equation as a Fourier series of topological recursion partition function for a family of elliptic curves. This could be viewed as a certain analog of discussed formula for the tau function.

1.2 Painlevé equations and their solutions

1.2.1 Painlevé equations hierarchies

The highest equation in the hierarchy of Painlevé equations, Painlevé VI equation has the form

$$\frac{d^2w}{dz^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{dw}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\frac{dw}{dz} + \\ + \frac{2w(w-1)(w-z)}{z^2(z-1)^2}\left(\left(\theta_\infty - \frac{1}{2}\right)^2 - \frac{\theta_0^2 z}{w^2} + \frac{\theta_1^2(z-1)}{(w-1)^2} - \frac{\left(\theta_z^2 - \frac{1}{4}\right)z(z-1)}{(w-z)^2}\right),$$

where $\theta_0, \theta_z, \theta_1, \theta_\infty$ are the parameters of the equation. All other Painlevé equations can be obtained from Painlevé VI by a coalescence cascade Fig. 1, where arrows correspond to certain limits and

$$\begin{array}{ccccccccc} \mathrm{PVI} & \longrightarrow & \mathrm{PV} & \longrightarrow & \mathrm{PIII}_1 & \longrightarrow & \mathrm{PIII}_2 & \longrightarrow & \mathrm{PIII}_3 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathrm{PIV} & \longrightarrow & \mathrm{PII} & \longrightarrow & \mathrm{PI} & & \end{array}$$

Figure 1: Coalescence diagram for Painlevé equations

each limitation decreases number of parameters by 1. Our goal is to study general solutions of these equations. The thesis focuses on Painlevé VI equation, as the highest equation of the hierarchy, from which it is known, how to degenerate solution along the first line of the diagram. The thesis also focuses on Painlevé III_3 equation — the parameterless endpoint of this degeneration procedure which appear to be more suitable to illustrate our approaches.

The Painlevé III_3' equation has the form

$$\frac{d^2w}{dz^2} = \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{2w^2}{z^2} - \frac{2}{z}, \tag{1.1}$$

where prime in the notation means certain simple change of variables. Painlevé III_3' equation comes out from Painlevé VI equation by limit $R \to 0$ under rescaling

$$w \mapsto R^2 w,\ z \mapsto R^4 w,\ \theta_{z,1} \mapsto \frac{1}{2}R^{-2},\ \theta_{0,\infty} \mapsto -\frac{1}{2}R^{-2}. \tag{1.2}$$

This equation can be rewritten as system of two bilinear Toda-like equations on two tau functions [BS16b]

$$\tau(z)\frac{d^2}{d\log z^2}\tau(z) - \left(\frac{d}{d\log z}\tau(z)\right)^2 = -z^{1/2}\tau_1(z)\tau_1(z), \tag{1.3}$$

$$\tau_1(z)\frac{d^2}{d\log z^2}\tau_1(z) - \left(\frac{d}{d\log z}\tau_1(z)\right)^2 = -z^{1/2}\tau(z)\tau(z), \tag{1.4}$$

which has obvious symmetry (Bäcklund transformation) $\pi : \tau \leftrightarrow \tau_1$. The function $w(z)$ is equal to $z^{\frac{1}{2}}\frac{\tau(z)^2}{\tau_1(z)^2}$, Bäcklund transformation acts on w as $w \mapsto z/w$. Stationary point of this transformation $\pm\sqrt{z}$ is the only algebraic solution of Painlevé III'$_3$ [OKSO06], [Gr84], corresponding tau functions are $z^{1/16}e^{\mp 4\sqrt{z}}$.

As it was mentioned before, there exists natural geometric construction, which relates Painlevé equations to spaces of initial conditions [O79]. This space appear to be certain blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in 8 points (or, equivalently, $\mathbb{P}^2$ in 9 points), which is characterized by affine root system R. More precisely, each Painlevé equation corresponds to two orthogonal affine root sublattices of affine root lattice $E_8^{(1)}$. First of them, R corresponds to the space of initial conditions and second, $R^\perp$, characterizes symmetry of this space, as well as symmetry of corresponding Painlevé equation. This symmetry is given by extended affine Weyl group $\widetilde{W}(R^\perp)$. For example, Painlevé VI corresponds to sublattices $R^\perp/R = D_4^{(1)}/D_4^{(1)}$ and Painlevé III'$_3$ equation correspond to $A_0^{(1)}/D_8^{(1)}$.

In the work of Sakai [S01] this approach was generalized significantly and cover also discrete Painlevé equations, namely, difference and q-difference. Namely, he classified all $\mathbb{P}^2$ surfaces blowed up in 9 points up to automorphismes. These surfaces could be composed in the Sakai degeneracy scheme (Fig. 2, where only surface type matchings are written). Spaces of initial conditions of continuous Painlevé equations lay into this classification as surfaces of types D and E (cf. Fig. 1). In Sakai's approach, surfaces of A type in the row, starting from $A_0^{(1)*}$ to $A_8^{(1)}$, as well as $A_7^{(1)'}$ correspond to q-difference Painlevé equations as follows. Corresponding extended affine Weyl group $W(R^\perp)$ contains a subgroup of free translations. Certain discrete flow in this subgroup correspond to certain q-Painlevé equations. Note that there is two different surfaces which correspond to the same affine root system $A_7^{(1)}$, they are distinguished by prime. Note that vertical arrows from surfaces, corresponding to q-deformed equations to surfaces, corresponding to continuous equations correspond also to continuous limit.

Besides continuous and q-difference equations, Sakai classification contains also elliptic and difference equations, one surface could correspond to several equations (because it could be more than one flow in the subgroup of free translations). More details about this one can found in original paper [S01] and also in further paper [KNY15].

From now on Painlevé equations are denoted by their surface type, for example, Painlevé III'$_3$ is $D_8^{(1)}$.

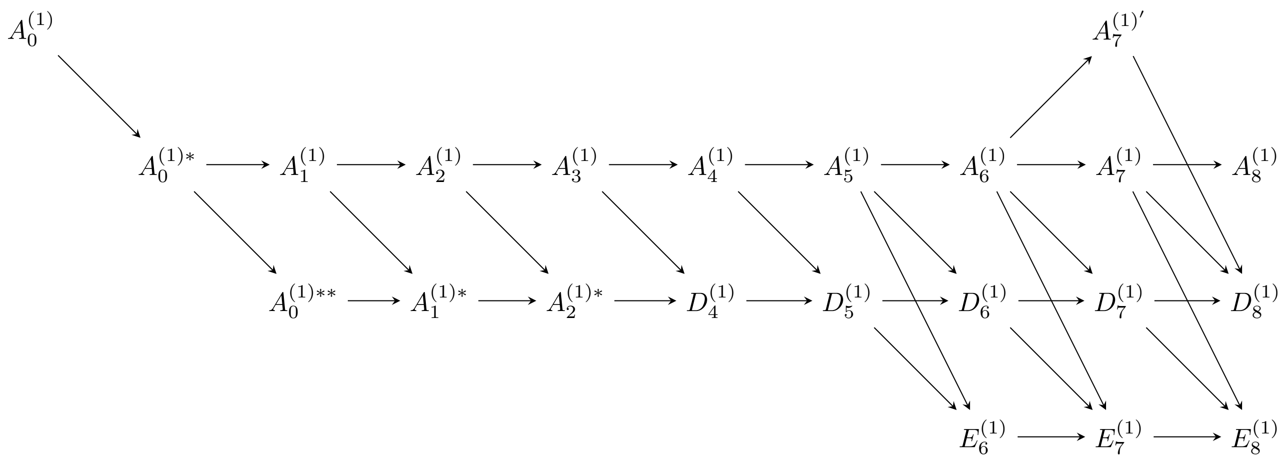

Figure 2: Sakai's classification scheme for discrete Painlevé equations: the surface type.

The thesis is focused on two q-deformations of differential Painlevé $D_8^{(1)}$ equation, namely on Painlevé $A_7^{(1)'}$ and Painlevé $A_7^{(1)}$ equations. Painlevé $A_7^{(1)'}$ has the form

$$G(qz)G(q^{-1}z) = \frac{(G(z)-z)^2}{(G(z)-1)^2} \tag{1.5}$$

and Painlevé $A_7^{(1)}$ has the form

$$G(qz)G(q^{-1}z) = \frac{1-G(z)}{z^2G(z)^2}, \tag{1.6}$$

where G is direct analog of w.

They also could be written as bilinear Toda-like equations. In this form Painlevé $A_7^{(1)'}$ reads

$$\begin{aligned} \tau(qz)\tau(q^{-1}z) &= \tau^2 - z^{1/2}\tau_1^2 \\ \tau_1(qz)\tau_1(q^{-1}z) &= \tau_1^2 - z^{1/2}\tau^2 \end{aligned} \tag{1.7}$$

and Painlevé $A_7^{(1)}$ reads as a single equation

$$\tau(q^2z)\tau(q^{-2}z) = \tau^2 - z^{1/2}\tau(qz)\tau(q^{-1}z). \tag{1.8}$$

Some aspects for the Painlevé $A_3^{(1)}$ equation, which is q-deformation of differential Painlevé VI $D_4^{(1)}$ equation, is also studied, see Subsection 6.2 in the main text.

1.2.2 Fourier series for Painlevé tau function

In celebrated work [GIL12] Gamayun,Iorgov and Lisovyy proposed formula (also informally known as "Kiev formula") for the general solution of Painlevé VI equation in terms of Virasoro algebra of central charge $c = 1$ four-point conformal blocks. The latter are special functions, appearing in context of Conformal Field Theory and having explicit representation-theoretic definition (see Sec. 2 in the main text). Namely, their conjecture is

Theorem 1.1. *Generic Painlevé VI tau function equals to Fourier series of four-point Virasoro $c = 1$ conformal blocks $\mathcal{F}(\{\theta_\kappa^2\};\sigma^2|z)$, $\{\theta_\kappa\} = \{\theta_0, \theta_z, \theta_1, \theta_\infty\}$*

$$\tau(\{\theta_\kappa\};\sigma,s|z) = \sum_{n\in\mathbb{Z}} s^n \mathcal{C}_{VI}(\{\theta_\kappa\};\sigma+n)\mathcal{F}(\{\theta_\kappa^2\};(\sigma+n)^2|z), \tag{1.9}$$

where parameters $\sigma, s \neq 0$ play role of the integration constants of Painlevé VI equation and $\mathcal{C}_{VI}(\{\theta_\kappa\};\sigma)$ is certain simple function (see (3.2)).

This theorem was proved in [ILT14], [BS14] (see also [BS16b]), [GL16], using different approaches. In this thesis proof from [BS14], [BS16b] is presented in Section 3. Word "generic" means that complex dimension of dropped solutions equals 1.

Formula (1.9) generalizes asymptotic behaviour of Painlevé VI tau function, obtained in the work [J82]. Analogous formula, corresponding to special case $\sigma = 1/4$, was written in [NO03], but, unfortunately, connection to Painlevé equations was missed.

Approach to write tau function of Painlevé equations (and, more generally, isomonodromic tau functions) as a Fourier series of some "partition functions" $\mathcal{Z}$ (including different conformal blocks)

$$\tau(\sigma,s|z) = \sum_{n\in\mathbb{Z}} s^n \mathcal{Z}(\sigma+n|z) \tag{1.10}$$

appeared to be very efficient. After work [GIL12] a huge amount of degenerations and extensions for different differential and q-difference isomonodromic problems appeared. Such connection between isomonodromic problems and conformal blocks have a common name Isomonodromy/CFT correspondence, or, sometimes, Painlevé/gauge theory correspondence. The latter name is because conformal blocks are equal to instanton partition functions of SUSY gauge theories according to AGT correspondence ([AGT09], [AY09] and further developments). From this point of view, such tau function is called dual partition function, in this form it appeared in [NO03].

These developments are already listed in the Historical survey, now in more details will be described those, that are presented in this thesis

- In [GIL13] the solution (1.9) was degenerated along the first row of coalescence cascade Fig. 1. Tau functions for these Painlevé equations are given by Fourier series of certain limits of four-point Virasoro $c = 1$ conformal block.

  For example, for Painlevé III($D_8^{(1)}$) equation, tau function is given by Fourier series (1.10) of conformal blocks given by so-called irregular (or Whittaker) limit $\mathcal{Z}(\sigma|z) = C_{III_3'}(\sigma)\mathcal{F}(\sigma^2|z)$ (see Thm. 1.4 below). These blocks besides z depend only on intermediate Verma module highest weight σ^2. This formula by her own was proved in [BS14] and [BS16b]. For details see Sec. 3.

- In [BS16q] q-deformation of the formula for tau function of Painlevé III($D_8^{(1)}$) equation was built, which proposed to be a general solution of Painlevé $A_7^{(1)'}$ equation. Similar formula for the solution of Painlevé $A_7^{(1)}$ equation was proposed in [BGM18]. Tau functions for these q-Painlevé equations are given by Fourier series (1.10) of irregular limits of conformal blocks of q-Virasoro algebra with $c = 1$. For the case of $A_7^{(1)'}$ these conformal blocks are equal to 5d pure gauge SUSY $SU(2)$ instanton partition functions according to the AGT relation [AY09]. Case $A_7^{(1)}$ differ from the case $A_7^{(1)'}$ by additional Chern-Simons term in partition function. These formulas were proved in [BS18], [MN18], using different approaches. For details see Sec. 5, 6.

  The formula for the solution of Painlevé $A_3^{(1)}$ equation (but not for tau functions), which is q-deformation of Painlevé VI ($D_4^{(1)}$) equation was proposed and proved in paper [JNS17], using q-deformation of [ILT14] approach.

- Isomonodromy/CFT correspondence for rank 2 isomonodromic problems states, that isomonodromic tau function is given by Fourier series of $c = 1$ Virasoro conformal blocks [ILT14]. Avoiding this central charge constraint leads to quantum tau function and corresponding quantization of the space of monodromy parameters (they are no longer commutative) [BGM17]. However, there are reasons to believe that generalizations of Isomonodromy/CFT correspondence exist for the case of Virasoro algebra central charge $c = 1 - 6\frac{(n-1)^2}{n}, n \in \mathbb{Z}_{\geq 1}$. (which is central charge of "minimal models" $M(1,n)$). That is because, following approach of [ILT14], for these central charges there is a commutativity of the operator-valued monodromies, so it is possible to build a solution of a certain isomonodromic problem, using conformal blocks with such central charges. For other reasons see Sec. 6.

  In the paper [BS18] there was studied tau functions, given by Fourier series (1.10) of Virasoro $c = -2$ irregular conformal blocks, which is the very first case of such generalization. They are natural from the various point of views. Namely, $c = 1$ tau function decomposes into product of two such tau functions. Analogous decomposition arose from the point of view of ABJM theory ([BGT17]). These $c = -2$ tau functions appear to be $c = 1$ tau functions of special case of Painlevé $A_3^{(1)}$ equation. For details see Sec. 6.

1.2.3 Main results

The main results of my PhD studies are:

1. The original Gamayun–Iorgov–Lisovyy formula for the tau function of Painlevé VI, V and III's equations is proven ([1]=[BS14], [3]=[BS16b], Sec. 3).

2. The q-deformation of the Isomonodromy/CFT correspondence is built. Namely, it was proposed ([2]=[BS16q]) and proved ([4]=[BS18], Sec. 5,6) formulas for the tau function of Painlevé $A_7^{(1)'}$ and $A_7^{(1)}$ equations, which are Fourier series of 5d pure gauge SUSY $SU(2)$ partition functions.

3. The Isomonodromy/CFT correspondence was extended to the central charge $c=-2$. The $c=-2$ analog of Gamayun–Iorgov–Lisovyy formula was introduced and related to Painlevé theory ([4]=[BS18], Sec. 6).

This PhD thesis is based on 4 papers

1. M. Bershtein, A. Shchechkin, *Bilinear equations on Painlevé τ functions from CFT*, [`arXiv:1406.3008`], Communications in Mathematical Physics 339 (3), (2015), 1021-1061.

2. M. Bershtein, A. Shchechkin, *q-deformed Painlevé tau function and q-deformed conformal blocks*, [`arXiv:1608.02566`], Journal of Physics A: Mathematical and Theoretical 50 (8), (2017), 085202.

3. M. Bershtein, A. Shchechkin, *Bäcklund transformation of Painlevé III(D_8) tau function*, [`arXiv:1608.02568`] Journal of Physics A: Mathematical and Theoretical 50 (11), (2017), 115205.

4. M. Bershtein, A. Shchechkin, *Painlevé equations from Nakajima-Yoshioka blowup relations*, [`arXiv:1811.04050`], Letters in Mathematical Physics 109 (11), (2019), 2359-2402

See also preprint A. Shchechkin, *Blowup relations on $\mathbb{C}^2/\mathbb{Z}_2$ from Nakajima-Yoshioka blowup relations*, [`arXiv:2006.08582`]
and review N. Iorgov, O. Lisovyy, A. Shchechkin, Yu. Tykhyy, *Painlevé functions and conformal blocks*, Constructive Approximation 39 (1), (2014), 255-272.

1.2.4 Methods

It turns out that Painlevé equations (and other isomonodromic problems) could be written as certain bilinear equations on tau functions. Then, looking for the solutions given by Fourier series (1.10) of some "partition functions" is equivalent to looking for the appropriate bilinear relations on these functions

$$\sum_{n\in\mathbb{Z}} \mathrm{D}(\mathcal{Z}((\sigma+n)^2|z), \mathcal{Z}((\sigma-n)^2|z)) = 0, \tag{1.11}$$

where D is certain bilinear differential (or q-difference) operator.

In the thesis two "sources" of bilinear relations are used:

- Bilinear relations on Virasoro algebra conformal blocks from the representation theory of Super Virasoro algebra. These relations are based on embedding of two Virasoro algebras into Super Virasoro algebra, extended by Majorana fermion algebra [CPSS90]. It is briefly described below in the Overview of the thesis, see also Sections 2,3. This approach appear for us to be more suitable for the differential case.

- Other approach to obtain bilinear relations comes from blowup relations on Nekrasov partition functions of SUSY gauge theory. This approach appears to be more suitable for q-deformed case, but it is also suitable for continuous case. As it was already mentioned above, 4d and 5d partition functions are equal via the AGT correspondence [AGT09], [AY09] to Virasoro or q-Virasoro conformal blocks. Blowup relation is based on relation between partition functions on initial manifold and the same manifold, blowed up in one point. Such relations were found by Nakajima-Yoshioka in [NY03] for partition functions on $\mathbb{C}^2$, they are called "Nakajima-Yoshioka blowup relations". Unlike the representation-theoretic issues, this geometry is not studied in this thesis, but only its results are used and simple consequences of them are studied. These issues are briefly describe below in the Overview of the thesis, more details could be found in Sections 4,6.

1.3 Overview of the thesis

This Subsection is devoted to overview the thesis, namely, to describe its organization, as well as to formulate main theorems and crucial points, describing them in some details.

In Sec. 1 basic facts about Painlevé equations, necessary for the thesis, are reviewed. Some important details are already overviewed in above Subsubsection 1.2.1.

Section starts from explaining, how Painlevé VI equation appears from the isomonodromic deformation problem of a linear system with 4 singularities on $\mathbb{CP}^1$ (Subsection 1.1).

Then Painlevé VI and Painlevé III($D_8^{(1)}$) equations are described as non-autonomous Hamiltonian systems, tau forms of these equations are introduced (Subsubsection 1.2.1). Here we also review Bäcklund transformation and algebraic solutions for Painlevé III($D_8^{(1)}$). In Subsubsection 1.2.2 solutions of Painlevé VI and III($D_8^{(1)}$) are parametrized by their asymptotic behaviour, namely by two parameters σ, s. Using this asymptotic behavior, it is studied, how Bäcklund transformation generator act on these parameters σ, s of solutions of Painlevé III($D_8^{(1)}$) equation

Proposition 1.1. *(Prop. 1.3 in the main text) Bäcklund transformation π of Painlevé III($D_8^{(1)}$) equation acts on parameters σ, s by $\pi(\sigma, s) = (1/2 - \sigma, s^{-1})$.*

In Subsubsection 1.2.3 additionally Toda-like (1.4) tau form of Painlevé III($D_8^{(1)}$) equation is constructed, as well as Okamoto-like tau form

$$\begin{aligned} D^2_{[\log z]}(\tau, \tau_1) - \frac{1}{2}\left(z\frac{d}{dz} - \frac{1}{8}\right)(\tau\tau_1) = 0, \\ D^3_{[\log z]}(\tau, \tau_1) - \frac{1}{2}\left(z\frac{d}{dz} - \frac{1}{8}\right)D^1_{[\log z]}(\tau, \tau_1) = 0. \end{aligned} \tag{1.12}$$

More precisely, it was proved that these tau forms are equivalent to Painlevé III($D_8^{(1)}$) equation (Prop. 1.4, Prop. 1.5).

Subsection 1.3 start from describing Sakai's approach to Painlevé equations for the case of Painlevé $A_7^{(1)'}$ equation in details (Subsubsection 1.3.1). Then in Subsubsection 1.3.2 tau functions for this case are introduced (Prop. 1.6), this was done in [BS16q] similarly to analogs, known for higher Painlevé equations. Here we also discuss continuous limit to Painlevé III($D_8^{(1)}$) equation, as well as Bäcklund transformation and algebraic solutions for Painlevé $A_7^{(1)'}$. In Subsubsection 1.3.3 Painlevé $A_7^{(1)}$ equation is briefly introduced.

After this Section there starts description of two main approaches to the solutions of Painlevé equations, already stated in Subsubsection 1.2.4. These approaches are briefly overwieved below.

1.3.1 Conformal blocks of Super Virasoro algebra

Section 3 reviews and prove various facts about Verma modules, vertex operators and conformal blocks of Virasoro and Super Virasoro algebra.

Virasoro and Super Virasoro algebras. Review starts at Subsection 2.1 from reviewing definitions of Virasoro and Super Virasoro algebras and their Verma modules as well as free-field representation of Super Virasoro algebra.

Virasoro algebra is defined by OPE

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg \tag{1.13}$$

for Virasoro current $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$.

Super Virasoro or Neweu-Schwarz-Ramond (NSR) algebra is $\mathcal{N}=1$ superextension of Virasoro algebra by odd current $G(z)=\sum_{r\in\mathbb{Z}+1/2}G_r z^{-r-3/2}$ with OPE

$$G(z)G(w)=\frac{c_{\mathsf{NSR}}}{(z-w)^3}+\frac{2T(w)}{z-w}+reg \tag{1.14}$$

$$L(z)G(w)=\frac{3/2G(w)}{(z-w)^2}+\frac{\partial G(w)}{z-w}+reg, \tag{1.15}$$

where $c_{\mathsf{NSR}}=3/2c$. Half-integer indices of G_r mean that so-called NS sector of this algebra is considered in above formulas. Then this algebra is extended by Majorana fermion F in the same sector to the direct sum $\mathsf{F}\oplus\mathsf{NSR}$.

Subsection 2.2 starts from the embedding of $\mathsf{Vir}\oplus\mathsf{Vir}\subset\overline{U(\mathsf{F}\oplus\mathsf{NSR})}$ [CPSS90], which for the case $c_{\mathsf{NSR}}=1, c^{(1)}=c^{(2)}=1$ have the form

$$T^{(\eta)}(z)=\frac{1}{2}T(z)+\frac{1}{2}T_f(z)+\frac{(-1)^\eta}{2i}f(z)G(z),\quad \eta=1,2, \tag{1.16}$$

where $T_f(z)=\frac{1}{2}:f'(z)f(z):$ is standard fermionic energy-momentum tensor.

Below in this overview, all results will be stated for the case $c_{\mathsf{NSR}}=1, c^{(1)}=c^{(2)}=1$ for simplicity. This is the case, were given embedding is applied to obtain bilinear relations on $c=1$ Virasoro conformal blocks necessary for studying solutions of Painlevé equations. However, all representation-theoretic results exist for arbitrary value of c_{NSR}, they could be found in the main text.

Verma module and Whittaker vector decomposition. Subsection 2.2 continues by discussion of decomposition of Verma modules in Neweu-Schwarz and Ramond sectors.

In NS sector this decomposition is as follows ([BBFLT11], Prop. 2.1 in the main text)

$$\pi_{\mathsf{F}}\otimes\pi_{\mathsf{NSR}}^{2\sigma^2}\cong\bigoplus_{2n\in\mathbb{Z}}\pi_{\mathsf{Vir}}^{(\sigma+n)^2}\otimes\pi_{\mathsf{Vir}}^{(\sigma-n)^2}. \tag{1.17}$$

Here in l.h.s. one has tensor product of fermion algebra Verma module π^{F}, freely (which means with respect to universal enveloping algebra) generated by negative fermionic modes with Super Virasoro Verma module $\pi_{\mathsf{NSR}}^{2\sigma^2}$ of highest weight $2\sigma^2$ (with respect to L_0), freely generated by negative modes of $T(z)$ and $G(z)$. This tensor product decomposes into direct sum of tensor products of two Virasoro Verma modules $\pi_{\mathsf{Vir}}^{(\sigma\pm n)^2}$ of highest weights $(\sigma\pm n)^2$ (with respect to $L_0^{(1,2)}$), freely generated by negative modes of $T^{(1,2)}(z)$. Verma module decomposition for Ramond sector was proved in [BS16b, Thm. 4.2.]. In this case Verma module π_{F} is freely generated by non-positive fermionic generators, so it has two highest weight vectors, which differ by action of f_0. Analogously, Verma module $\pi_{\mathsf{NSR}}^{2\sigma^2+\frac{1}{16}}$ is freely generated by NSR algebra negative modes and G_0, so it also has two highest weight vectors of highest weight $2\sigma^2+\frac{1}{16}$, which differ by action of G_0. Module $\pi_{\mathsf{F}}\otimes\pi_{\mathsf{NSR}}^{2\sigma^2+\frac{1}{16}}$ could be divided into even and odd parts, using parity operator. Decomposition in the case $c_{\mathsf{NSR}}=1$ is as follows

Theorem 1.2. *(for the case of arbitrary c_{NSR} see Thm. 2.2 in the main text) Verma module of $c_{\mathsf{NSR}}=1$ Ramond sector $\mathsf{F}\oplus\mathsf{NSR}$ algebra of highest weight $2\sigma^2+\frac{1}{16}$ decomposes into sum of two isomorphic eigenspaces, distinguished by parity (denoted by $0,1$ in superscript)*

$$\pi_{\mathsf{F}\oplus\mathsf{NSR}}^{2\sigma^2+\frac{1}{16}}\cong\pi_{\mathsf{F}\oplus\mathsf{NSR}}^{2\sigma^2+\frac{1}{16},0}\oplus\pi_{\mathsf{F}\oplus\mathsf{NSR}}^{2\sigma^2+\frac{1}{16},1} \tag{1.18}$$

each ot them decomposes into the sum

$$\bigoplus_{2n+1/2\in\mathbb{Z}}\pi_{\mathsf{Vir}}^{(\sigma+n)^2}\otimes\pi_{\mathsf{Vir}}^{(\sigma-n)^2}. \tag{1.19}$$

Then, Subsection 2.3 reviews notions of Virasoro and Super Virasoro vertex operators, chain and Whittaker vectors in Verma modules of these algebras and corresponding conformal blocks.

Virasoro Whittaker vector $|W(z)\rangle$ from $\pi^{\Delta}_{\mathsf{Vir}}$ is defined by

$$L_0|W(z)\rangle = z\frac{d}{dz}|W(z)\rangle, \quad L_1|W(z)\rangle = z|W(z)\rangle, \quad L_2|W(z)\rangle = 0 \tag{1.20}$$

and Super Virasoro Whittaker vector $|W_{\mathsf{NS}}(z)\rangle$ from $\pi^{\Delta^{\mathsf{NS}}}_{\mathsf{NSR}}$ defined by

$$L_0|W_{\mathsf{NS}}(z)\rangle = z\frac{d}{dz}|W_{\mathsf{NS}}(z)\rangle, \quad G_{1/2}|W_{\mathsf{NS}}(z)\rangle = z^{1/2}|W_{\mathsf{NS}}(z)\rangle, \quad G_{3/2}|W_{\mathsf{NS}}(z)\rangle = 0 \tag{1.21}$$

Irregular limits of Super Virasoro and Virasoro conformal blocks are given by Shapovalov form scalar products of correponding Whittaker vectors

$$\mathcal{F}^{\mathsf{NS}}(\Delta^{\mathsf{NS}}|z) = \langle W_{NS}(1)|W_{NS}(z)\rangle, \quad \mathcal{F}(\Delta|z) = \langle W(1)|W(z)\rangle \tag{1.22}$$

Subsection 2.4 describes decomposition of Super Virasoro Whittaker and chain vectors into sum of $\mathsf{Vir}\oplus\mathsf{Vir}$ Whittaker and chain vectors respectively, which follow from decomposition of $\pi_{\mathsf{F}}\otimes\pi^{\Delta^{\mathsf{NS}}}_{\mathsf{NSR}}$.

Proposition 1.2. *(for the case of arbitrary c_{NSR} see Prop. 2.1 in the main text) Super Virasoro Whittaker vectors decomposes into the sum of $\mathsf{Vir}\oplus\mathsf{Vir}$ Whittaker vectors*

$$|1\otimes W_{\mathsf{NS}}(z)\rangle = \sum_{2n\in\mathbb{Z}}\left(l_n\left(|W^{(1)}_n(z)\rangle\otimes|W^{(2)}_n(z)\rangle\right)\right), \tag{1.23}$$

with certain independent of z decomposition coefficcients l_n

Also more general result about decomposition of Super Virasoro vertex operator into product of two Virasoro vertex operators is considered there (Thm. 2.3 in the main text).

Finally, in Subsection 2.5 we present calculation of coefficcients l_n. Result is as follows

Theorem 1.3. *(see more general Thm. 2.4 in the main text) Coefficients l_n are given by the formula*

$$l_n^2(\sigma) = (-1)^{2n}\frac{2^{4\sigma^2}}{\prod\limits_{k=1}^{2|n|-1}(k^2-4\sigma^2)^{2(2|n|-k)}(4\sigma^2)^{2|n|}} \tag{1.24}$$

Thm. 2.4 is for much more general case of decomposition coefficcients of above mentioned vertex operator decomposition, it is an important technical result.

Proof of the bilinear relations. In Section 3 it is presented representation-theoretical approach to finding solutions of to the Painlevé III($D_8^{(1)}$) equation and more general Painlevé VI($D_4^{(1)}$) equation. Using this approach, Theorem 1.1 is proved, as well as formulas for Painlevé V and III's tau functions, which could be obtained by certain limitation procedure, as we briefly explained above. However, we start from the proof of formula for the tau function of Painlevé III($D_8^{(1)}$) equation, which is more illustrative. The analog of Theorem 1.1 in this case is as follows

Theorem 1.4. *(Thm. 3.1. in the main text) Generic Painlevé III($D_8^{(1)}$) tau function equals to Fourier series of irregular Virasoro $c=1$ conformal blocks $\mathcal{F}(\sigma^2|z)=\mathcal{F}_{c=1}(\sigma^2|z)$ given by* (1.22)

$$\tau(\sigma,s|z) = \sum_{n\in\mathbb{Z}}\mathcal{C}_{III_3'}(\sigma+n)s^n\mathcal{F}((\sigma+n)^2|z), \quad \operatorname{Re}\sigma\in\mathbb{R}\setminus\left\{\frac{1}{2}\mathbb{Z}\right\}, \quad s\in\mathbb{C}\setminus\{0\}, \tag{1.25}$$

where coefficients $\mathcal{C}(\sigma)$ are expressed in terms of Barnes G-function

$$\mathcal{C}_{III_3'}(\sigma) = \frac{1}{\mathsf{G}(1-2\sigma)\mathsf{G}(1+2\sigma)}, \tag{1.26}$$

The parameters s and σ play role of the integration constants of Painlevé III($D_8^{(1)}$) equation.

Section starts from Subsection 3.1, which give all related formulations.

In Subsection 3.2 formula (1.25) is proved. We start from simple case of algebraic tau function $z^{1/16}e^{\mp 4\sqrt{z}}$ corresponding to the case $\sigma = 1/4$, $s = \pm 1$ (according to Prop. 1.1), using decomposition of odd boson Fock module. Then it is proved that supposed tau function (1.25) satisfy Toda-like equations (1.4) as well as certain bilinear tau form of order 4, using NS sector of $\mathsf{F}\oplus\mathsf{NSR}$ algebra, namely, proved Thm 0.4. Finally, we also demonstrate from the Ramond sector of $\mathsf{F}\oplus\mathsf{NSR}$ algebra, that supposed tau function satisfy Okamoto-like tau form of Painlevé III($D_8^{(1)}$) equation.

Main idea is to calculate matrix element

$$\langle 1\otimes W_{\mathsf{NS}}(1)|\left(\sum_{r\in\mathbb{Z}+1/2} f_{-r}G_r\right)^k|1\otimes W_{\mathsf{NS}}(z)\rangle = i^k\langle W_n^{(1)}(1)\rangle\otimes|W_n^{(2)}(1)|(L_0^{(1)}-L_0^{(2)})^k|W_n^{(1)}(z)\rangle\otimes|W_n^{(2)}(z)\rangle \tag{1.27}$$

from two sides: $\mathsf{Vir}\oplus\mathsf{Vir}$ and $\mathsf{F}\oplus\mathsf{NSR}$ side.

From the $\mathsf{Vir}\oplus\mathsf{Vir}$ side calculation gives

$$\langle W_n^{(1)}(1)\rangle\otimes|W_n^{(2)}(1)|(L_0^{(1)}-L_0^{(2)})^k|W_n^{(1)}(z)\rangle\otimes|W_n^{(2)}(z)\rangle = \sum_{2n\in\mathbb{Z}} l_n^2\cdot D^k_{[\log z]}(\mathcal{F}((\sigma+n)^2|z),\mathcal{F}((\sigma-n)^2|z)), \tag{1.28}$$

where Hirota differential operator $D^k_{[\log z]}$ is given by

$$f(e^{\alpha}z)g(e^{-\alpha}z) = \sum_{k=0}^{\infty} D^k_{[\log z]}(f(z),g(z))\frac{\alpha^k}{k!}. \tag{1.29}$$

From the other, $\mathsf{F}\oplus\mathsf{NSR}$ side calculation gives ($k=0$)

$$\mathcal{F}_{\mathsf{NS}} = \sum_{2n\in\mathbb{Z}} l_n^2\cdot\mathcal{F}((\sigma+n)^2|z)\mathcal{F}((\sigma-n)^2|z) \tag{1.30}$$

and next non-trivial relation is obtained for $k=2$

$$-z^{1/2}\mathcal{F}_{\mathsf{NS}} = \sum_{2n\in\mathbb{Z}} l_n^2\cdot D^2_{[\log z]}(\mathcal{F}((\sigma+n)^2|z),\mathcal{F}((\sigma-n)^2|z)). \tag{1.31}$$

Eliminating $\mathcal{F}_{\mathsf{NS}}$ from these two relations, bilinear relation is obtained

$$\sum_{2n\in\mathbb{Z}} l_n^2\cdot D^k_{[\log z]}(\mathcal{F}((\sigma+n)^2|z),\mathcal{F}((\sigma-n)^2|z)) = -z^{1/2}\sum_{2n\in\mathbb{Z}} l_n^2\cdot\mathcal{F}((\sigma+n)^2|z)\mathcal{F}((\sigma-n)^2|z). \tag{1.32}$$

And this is desired bilinear relation to prove Theorem 1.4(after comparing coefficcients l_n with those that appear from the function $\mathcal{C}_{III_3'}$) namely, prove that tau function (1.25) satisfy Toda-like form (1.4) of Painlevé III($D_8^{(1)}$) equation.

In Subsection 3.3 formula for tau function of Painlevé VI($D_4^{(1)}$) (Theorem 1.1) is proved in analogous way.

1.3.2 Nakajima-Yoshioka blowup relations and $c=-2$ tau functions

In Section 4 we review and prove various facts about Nekrasov partition functions and blowup equations.

Nekrasov partition functions. Subsection 4.1 reviews formulas and properties for instanton and perturbative parts of 5d SUSY $SU(2)$ pure gauge partition function, convergence of instanton part of the partition function in the case $-\epsilon_1/\epsilon_2 \in \mathbb{Q} > 0$ is proved. Then 4d limit of this partition function is studied.

It is well known that (pure) gauge SUSY $U(r)$ Yang-Mills theory on $\mathbb{C}^2$ has special solutions, called instantons, their type are decribed by so-called instanton number $n \in \mathbb{Z}_{\geq 0}$, which runs from 0 (no instanton) to $+\infty$. Denote by $M(r,n)$ a (Gieseker) compactification of moduli space of instantons with instanton number n. Nekrasov instanton partition function for given theory is defined as an equivariant volume of the spaces $M(r,n)$

$$\mathcal{Z}_{inst}(\epsilon_1, \epsilon_2, \{a_i\}; z) = \sum_{n=0}^{+\infty} z^n \int_{M(r,n)} 1, \tag{1.33}$$

where $a_i, i = 1, r$ are coordinates on Lie algebra of torus $(\mathbb{C}^*)^r \subset U(r)$ and ϵ_1, ϵ_2 are coordinates on Lie algebra of torus $(\mathbb{C}^*)^2$. Space $M(r,n)$ have ADHM description [ADHM78], in which these two tori act simply by matrix multiplications. From the physical point of view, a_i are vacuum expectation values and ϵ_1, ϵ_2 are so-called Ω-background parameters.

This integral localizes on points of $M(r,n)$, fixed by action of above mentioned $r+2$-dimensional torus. Fixed points are labeled by r-tuples of the Young diagrams $(\lambda^{(1)}, \ldots \lambda^{(r)})$, $n = \sum_{i=1}^r |\lambda^{(i)}|$. For $SU(2)$ case such calculation gives (in this case $a_1 + a_2 = 0$, we denote $a = a_1 - a_2$)

$$\begin{gathered}\mathcal{Z}_{inst}(a_1, a_2; \epsilon_1, \epsilon_2 | z) = \sum_{\lambda^{(1)}, \lambda^{(2)}} \frac{z^{|\lambda^{(1)}| + |\lambda^{(2)}|}}{\prod_{i,j=1}^2 \mathsf{N}_{\lambda^{(i)}, \lambda^{(j)}}(a_i - a_j; \epsilon_1, \epsilon_2)}, \\ \mathsf{N}_{\lambda,\mu}(a; \epsilon_1, \epsilon_2) = \\ = \prod_{s\in\lambda} (a - \epsilon_2(a_\mu(s) + 1) + \epsilon_1 l_\lambda(s)) \prod_{s\in\mu} (a + \epsilon_2 a_\lambda(s) - \epsilon_1(l_\mu(s) + 1)),\end{gathered} \tag{1.34}$$

where $a_\lambda(s), l_\lambda(s)$ denote the lengths of arms and legs for the box s in the Young diagram λ. As it was mentioned above, due to AGT relation [AGT09], this Nekrasov instanton partition function is equal to irregular limit of Virasoro four-point conformal block $\mathcal{F}(\Delta|z)$ (normalized on $1 + O(z)$) with central charge and highest weight given by

$$c = 1 + 6\frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2}, \quad \Delta = \frac{(\epsilon_1 + \epsilon_2)^2 - a^2}{4\epsilon_1 \epsilon_2}. \tag{1.35}$$

Case $c = 1$ corresponds to $\epsilon_1 + \epsilon_2 = 0$, in this case parametrization $\Delta = \sigma^2$ is used. Instead of instanton partition function $\mathcal{Z}_{inst}$ one can take full partition function $\mathcal{Z}$ of pure gauge SUSY $U(r)$ Yang-Mills theory, which contain also additional multipliers, called classical and 1-loop part. They are given by certain simple functions, which are analogs of prefactor of z^{σ^2} of conformal block $\mathcal{F}(\sigma^2|z)$ and function $\mathcal{C}$ from (1.25) respectively. Thereby, this full partition function $\mathcal{Z}$ is equal to the function $\mathcal{C}(\sigma)\mathcal{F}(\sigma^2|z)$ on the CFT side. So (1.25) is (1.10) with these full partition functions $\mathcal{Z}$.

To construct tau functions of q-Painlevé equations, one needs q-analog of this partition function $\mathcal{Z}(u; q_1, q_2|z)$, and this is partition function on $\mathbb{C}^2$, extended by the compact 5th dimension of radius $R = -\log q$, more precisely, on certain factor $(\mathbb{C}^2 \times \mathbb{R})/\mathbb{Z}$. Here $q_i = e^{R\epsilon_i}$, $u_i = e^{R\epsilon_i}$, $u = u_1/u_2, u_1 u_2 = 1$. Formulas for instanton partition function for 5d case differ from 4d case roughly by q-deformation of all multipliers $(\ldots) \mapsto 1 - q^{\cdots}$.

Nakajima-Yoshioka blowup relations and its dual formulation. Subsection 4.2 reviews relations on introduced above partition functions, known as blowup relations, starting from the most usual, initially found by H. Nakajima and K. Yoshioka

These are relations on 4d and 5d instanton partition functions [NY03], [NY05] (see also developments [GNY06] and [NY09]). They express instanton partition function on $\widehat{\mathbb{C}^2}$ = ($\mathbb{C}^2$ blowed up in the point) as a bilinear expression on $\mathbb{C}^2$ instanton partition function

$$\mathcal{Z}_{\widehat{\mathbb{C}^2}}(a;\epsilon_1,\epsilon_2|z) = \sum_{n\in\mathbb{Z}} \mathcal{Z}_{\mathbb{C}^2}(a+2\epsilon_1 n;\epsilon_1,\epsilon_2-\epsilon_1|z)\mathcal{Z}_{\mathbb{C}^2}(a+2\epsilon_2 n;\epsilon_1-\epsilon_2,\epsilon_2|z). \tag{1.36}$$

From the other hand partition function on $\widehat{\mathbb{C}^2}$ equals

$$\mathcal{Z}_{\widehat{\mathbb{C}^2}}(a;\epsilon_1,\epsilon_2|z) = \mathcal{Z}_{\mathbb{C}^2}(a;\epsilon_1,\epsilon_2|z). \tag{1.37}$$

Eliminating partition function on $\widehat{\mathbb{C}^2}$, one obtains Nakajima-Yoshioka blowup relations

$$\mathcal{Z}_{\mathbb{C}^2}(a;\epsilon_1,\epsilon_2|z) = \sum_{n\in\mathbb{Z}} \mathcal{Z}_{\mathbb{C}^2}(a+2\epsilon_1 n;\epsilon_1,\epsilon_2-\epsilon_1|z)\mathcal{Z}_{\mathbb{C}^2}(a+2\epsilon_2 n;\epsilon_1-\epsilon_2,\epsilon_2|z). \tag{1.38}$$

There are also differential (for 4d) and q-difference (for 5d) Nakajima-Yoshioka blowup relations.

However, bilinear relations on $c=1$ conformal blocks we studied above correspond to other blowup relations, namely to blowup relations on the space $\mathbb{C}^2/\mathbb{Z}_2$ (such relations were studied in 4d case in papers $\mathbb{C}^2$ [BMT11I], [BPSS13], [O18])

$$\tilde{\beta}_j^D \mathcal{Z}_{X_2}(a,\epsilon_1,\epsilon_2|z) = \sum_{n\in\mathbb{Z}+j/2} \mathrm{D}\Big(\mathcal{Z}(a+2n\epsilon_1;2\epsilon_1,-\epsilon_1+\epsilon_2|z),\mathcal{Z}(a+2n\epsilon_2;\epsilon_1-\epsilon_2,2\epsilon_2|z)\Big), \quad j\in\mathbb{Z}/2\mathbb{Z}, \tag{1.39}$$

where X_2 is a minimal resolution of $\mathbb{C}^2/\mathbb{Z}_2$, D is some differential operator, $\tilde{\beta}_j^D$ is some simple coefficcient. Eliminating $\mathcal{Z}_{X_2}$ from two such relations one obtains bilinear relations on $\mathcal{Z}$, which in the case $\epsilon_1+\epsilon_2=0$ become bilinear relations on $c=1$ conformal blocks studied above. Looking for a tau functions (1.10), satisfying (1.7), one can use 5d analogs of such relations.

In Subsubsections 4.2.2, 4.2.3 we present, how we could obtain blowup relations on $\mathbb{C}^2/\mathbb{Z}_2$ from Nakajima-Yoshioka blowup relations (see also [S20]). The most important relation we proved is (formula (4.60) in the main text)

$$\begin{aligned}&\sum_{n\in\mathbb{Z}} \mathcal{Z}(uq_1^{2n};q_1^2,q_2q_1^{-1}|q_1^2z)\mathcal{Z}(uq_1^{2n};q_1q_2^{-1},q_2^2|q_2^2z)\\ &= (1-(q_1q_2)^{1/2}z^{1/2})\sum_{n\in\mathbb{Z}} \mathcal{Z}(uq_1^{2n};q_1^2,q_2q_1^{-1}|z)\mathcal{Z}(uq_1^{2n};q_1q_2^{-1},q_2^2|z)\end{aligned} \tag{1.40}$$

In Subsubsection 4.2.4 we prove the connection between partition functions $\mathcal{Z}^{[2]}$ and $\mathcal{Z}^{[0]}$ which differ by the level of additional Chern-Simons theory, using Nakajima-Yoshioka blowup relations for these partition functions.

Proposition 1.3. *(Prop. 4.4 in the main text) Nekrasov function $\mathcal{Z}^{[2]}$ is equal to $\mathcal{Z}^{[0]}$ up to double q-Pochhammer symbol*

$$\mathcal{Z}^{[2]}(u;q_1,q_2|z) = (z;q_1,q_2)_\infty \mathcal{Z}^{[0]}_{inst}(u;q_1,q_2|z). \tag{1.41}$$

In the same way we also prove invariance of $\mathcal{Z}^{[1]}(u;q_1,q_2|z)$ under $q_1,q_2\mapsto q_1^{-1},q_2^{-1}$ (Prop. 4.5 in the main text).

From Nakajima-Yoshioka blowup relations to Painlevé $A_7^{(1)'}$ equation. In short Section 5 we prove formula for the tau function of Painlevé $A_7^{(1)'}$ (proposed in [BS16q]) and Painlevé $A_7^{(1)}$ equations, using previously obtained $\mathbb{C}^2/\mathbb{Z}_2$ blowup relations (Subsection 5.1). For example, for Painlevé $A_7^{(1)'}$ the Theorem we proved is as follows.

Theorem 1.5. *(Thm. 5.1. in the main text) Introduce functions $\tau_j(u,s;q|z)$, $j\in\mathbb{Z}/2\mathbb{Z}$ by the formula*

$$\tau_j(u,s;q|z)=\sum_{n\in\mathbb{Z}+j/2} s^n\mathcal{Z}(uq^{2n};q^{-1},q|z), \tag{1.42}$$

where $\mathcal{Z}$ is 5d pure SUSY $SU(2)$ gauge partition function. These functions are tau functions of Painlevé III($A_7^{(1)'}$) equation, i.e., they satisfy (1.7).

For instance, proof for the $A_7^{(1)'}$ case is based on formula (1.40). Other proof of this Theorem, based on $c=-2$ tau functions, is presented below.

Then convergence and continuous limit of these tau functions are studied (Subsection 5.2), as well as tau function for the algebraic solution $G(z)=\pm\sqrt{z}$ of Painlevé $A_7^{(1)'}$ equation (1.5) (Subsection 5.3). The latter equals

$$\tau=\tau_1=z^{1/16}(\pm q^{1/2}z^{1/2};q^{1/2},q^{1/2})_\infty. \tag{1.43}$$

Section 6 starts from introducing $c=-2$ tau functions of Painlevé $A_7^{(1)'}$ equation (Subsubsection 6.1.1).

Take particular case $\epsilon_1+\epsilon_2=0$ in Nakajima-Yoshioka blowup relations (4d or 5d)

$$\mathcal{Z}(a;\epsilon_1,-\epsilon_1|z)=\sum_{n\in\mathbb{Z}}\mathcal{Z}(a+2n\epsilon_1;\epsilon_1,-2\epsilon_1|z)\mathcal{Z}(a+2n\epsilon_1;2\epsilon_1,-\epsilon_1|z). \tag{1.44}$$

According to (1.35) partition function in l.h.s. corresponds to central charge $c=1$ and partition functions in r.h.s. both correspond to central charge $c=-2$. Partition functions in r.h.s. differ on some simple function, corresponding tau functions are distinguished by a superscript $\pm$.

Then it is natural to compose Fourier series from l.h.s. and r.h.s.

$$\tau(\sigma,s|z)=\mathrm{\tau}^-(\sigma,s|z)\mathrm{\tau}^+(\sigma,s|z) \tag{1.45}$$

where in l.h.s. one has standard $c=1$ tau function (1.10) and in r.h.s. there are so-called $c=-2$ tau functions defined by

$$\mathrm{\tau}^{\pm}(a,s|z)=\sum_{n\in\mathbb{Z}}s^{n/2}\mathcal{Z}(a+2n\epsilon_1;\pm\epsilon_1,\mp2\epsilon_1|z), \tag{1.46}$$

which is a Fourier series (1.10) of $c=-2$ conformal blocks unlike standard case $c=1$. Relation (1.45) could be viewed as Nakajima-Yoshioka blowup relation (1.38) in dual form.

Subsubsection 6.1.2 relates $c=-2$ tau functions to Painlevé $A_7^{(1)'}$ and $A_7^{(1)}$ equations, it give us another prove of the formulas for ordinary tau functions of these equations (in particular, Theorem 1.5 for $A_7^{(1)'}$ case).

Besides Nakajima-Yoshioka blowup relations (1.38) (or (1.45) in dual formulation), there also exist q-difference (in 5d case) Nakajima-Yoshioka blowup relations, which in dual formulation have the form

$$\begin{aligned}\overline{\mathrm{\tau}^+}\underline{\mathrm{\tau}^-}-\underline{\mathrm{\tau}^+}\overline{\mathrm{\tau}^-}&=-2z^{1/4}\tau_1,\\ \overline{\mathrm{\tau}^+}\underline{\mathrm{\tau}^-}+\underline{\mathrm{\tau}^+}\overline{\mathrm{\tau}^-}&=2\tau,\end{aligned} \tag{1.47}$$

where notations $\overline{\mathrm{\tau}}=\mathrm{\tau}(qz)$, $\underline{\mathrm{\tau}}=\mathrm{\tau}(q^{-1}z)$ are introduced.

Proposition 1.4. *(Prop. 6.1 in the main text) Take* (1.47) *and* (1.45). *Then τ and τ_1 satisfy Toda-like equation*

$$\overline{\tau}\underline{\tau}=\tau^2-z^{1/2}\tau_1^2. \tag{1.48}$$

Proof. Proof is extremely elementary. We substitute τ_1 and τ in different ways

$$\overline{\mathrm{\tau}^+\mathrm{\tau}^-}\,\underline{\mathrm{\tau}^+\mathrm{\tau}^-}=\frac{1}{4}(\overline{\mathrm{\tau}^+}\underline{\mathrm{\tau}^-}+\underline{\mathrm{\tau}^+}\overline{\mathrm{\tau}^-})^2-\frac{1}{4}(\overline{\mathrm{\tau}^+}\underline{\mathrm{\tau}^-}-\underline{\mathrm{\tau}^+}\overline{\mathrm{\tau}^-})^2 \tag{1.49}$$

□

Nakajima-Yoshioka blowup relations are proven, so we obtain the proof of the formula (1.10) for the tau function of Painlevé $A_7^{(1)'}$ equation.

In Subsubsection 6.1.3 $c=-2$ tau functions, which correspond to the algebraic solution of Painlevé $A_7^{(1)'}$ equation are found.

Properties of $c=-2$ tau function. As it already mentioned, these $c=-2$ tau functions have natural meaning on their own. Thesis continues by studying properties of $c=-2$ tau functions.

In Subsubsection 6.1.4 decomposition of $c=1$ tau function to a product of $c=-2$ tau functions (1.45) is related to the decomposition of spectral determinant of operator, inverse to relativistic Toda Hamiltonian. Namely, such factorization of $c=1$ tau function is known in the framework of ABJ theory [BGT17], where authors initially consider Painlevé $A_7^{(1)'}$ $c=1$ tau function from the point of view of Topological strings/Spectral Theory duality. Namely, they found that this tau function for $|q|=1$, $s=1$ up to some simple factor is equal to ABJ grand canonical partition function Ξ. Partition function Ξ, which equals Fredholm determinant of above mentioned operator, naturally factorizes according to the parity of eigenvalues of this operator as $\Xi=\Xi^+\Xi^-$.

Additionally, these factors satisfy so-called "Quantum Wronskian" relations, observed in [GHM14']. They appear to be equivalent (up to some simple factors) to Nakajima-Yoshioka blowup relations (1.47) on functions Ξ^+ and Ξ^-.

Then (in Subsubsection 6.1.5) we consider continuous limit of $c=-2$ tau functions and study Knizhnik-Zamolodchikov equations on $c=-2$ tau functions. Namely, $c=-2$ tau functions in 4d case satisfy

$$z\frac{d}{dz}\mathsf{\tau}^{\pm}=\frac{1}{2}(\zeta\mp i\sqrt{\zeta'})\mathsf{\tau}^{\pm}, \tag{1.50}$$

where ζ is Painlevé III$(D_8^{(1)})$ equation Hamiltonian, connected to $c=1$ tau function by $z\frac{d}{dz}\tau=\zeta\tau$. "Eigenvalues" in (1.50) apparently are Painlevé III$(D_6^{(1)})$ Hamiltonians.

Other interesting phenomenon related to $c=-2$ tau functions, we separate into disctinct Subsection 6.2.

As we mentioned above, $c=-2$ tau functions, corresponding to Painlevé $A_7^{(1)'}$, appear to be $c=1$ tau function of special case of Painlevé $A_3^{(1)}$ equation.

q-Painlevé VI equation (or $A_3^{(1)}$) is a system of eight first order q-difference bilinear equations on 8 tau functions $\tau_1\dots\tau_8$ depending on 4 (except q) parameters $\theta_0,\theta_z,\theta_1,\theta_\infty$. In the special case when $q^{\theta_\kappa}=i$ for all κ, these dynamics is equivalent (under substitution $z\mapsto z^{1/2}$) to the dynamics of four $c=-2$ tau functions given by four second-order q-difference equations ($j\in\mathbb{Z}/2\mathbb{Z}$)

$$\overline{\mathsf{\tau}_j^+}=\frac{\mathsf{\tau}_j^+\mathsf{\tau}_j^- - z^{1/4}\mathsf{\tau}_{j+1}^+\mathsf{\tau}_{j-1}^-}{\underline{\mathsf{\tau}_j^-}},\quad \overline{\mathsf{\tau}_j^-}=\frac{\mathsf{\tau}_j^+\mathsf{\tau}_j^- + z^{1/4}\mathsf{\tau}_{j+1}^+\mathsf{\tau}_{j-1}^-}{\underline{\mathsf{\tau}_j^+}}, \tag{1.51}$$

where $\mathsf{\tau}_0^{\pm}(u;q|z)=\mathsf{\tau}^{\pm}(u;q|z)$, $\mathsf{\tau}_1^{\pm}(u;q|z)=s^{1/2}\mathsf{\tau}^{\pm}(uq;q|z)$, $\tau(u;q|z)$ is 5d analog of (1.46). These equations follow from (1.47) and from the same relations for $u\mapsto uq$, when we exclude $c=1$ tau functions, using (1.45).

Namely,

Proposition 1.5. *(Prop. 6.4. in the main text) Consider the tuple* $(\tau_1,\tau_2,\tau_3,\tau_4,\tau_5,\tau_6,\tau_7,\tau_8)=(\mathsf{\tau}_0^+,\mathsf{\tau}_0^-,\mathsf{\tau}_1^+,\mathsf{\tau}_1^-,\underline{\mathsf{\tau}_0^-},\underline{\mathsf{\tau}_0^+},\underline{\mathsf{\tau}_1^-},\underline{\mathsf{\tau}_1^+})$*, where the functions* $\mathsf{\tau}_0^{\pm},\mathsf{\tau}_1^{\pm}$ *satisfy above dynamics. This tuple is a solution of tau form of Painlevé* $A_3^{(1)}$ *equation in the case* $q^{\theta_0}=q^{\theta_t}=q^{\theta_1}=q^{\theta_\infty}=i$ *under substitution* $z\mapsto z^{1/2}$.

According to the formula for the tau function of the q-Painlevé VI equation, which is (1.10) where $\mathcal{Z}$ are q-deformed four-point conformal blocks [JNS17], we obtain that such relation should entail relation on Nekrasov instanton partition functions

$$(-qz^{1/2};q,q)_\infty^2 \mathcal{Z}_{inst}(i,i,i,iq^{\pm 1/2},u|z^{1/2}) = \mathcal{Z}_{inst}(u;q^{-1},q^2|z) \qquad (1.52)$$

In Subsubsection 6.2.2 we make continuous limit of the obtained relation (1.52), which are apparently related to well known folding of Painlevé $D_6^{(1)}$ to $D_8^{(1)}$ equations [TOS05].

Appendix A collects some necessary facts about certain q-special functions (App. A.1), multiple gamma functions (App. A.2) and connects them by continuous limits (App. A.3).

The results of this dissertation are published in four articles:

1. M. Bershtein, A. Shchechkin, *Bilinear equations on Painlevé τ functions from CFT*, Communications in Mathematical Physics 339 (3), (2015), 1021-1061.

2. M. Bershtein, A. Shchechkin, *q-deformed Painlevé tau function and q-deformed conformal blocks*, Journal of Physics A: Mathematical and Theoretical 50 (8), (2017), 085202.

3. M. Bershtein, A. Shchechkin, *Bäcklund transformation of Painlevé III(D_8) tau function*, Journal of Physics A: Mathematical and Theoretical 50 (11), (2017), 115205.

4. M. Bershtein, A. Shchechkin, *Painlevé equations from Nakajima-Yoshioka blowup relations*, Letters in Mathematical Physics 109 (11), (2019), 2359-2402

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