NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS
FACULTY OF MATHEMATICS

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Leonid Rybnikov

Geometry and Combinatorics of Gaudin algebras

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Introduction

The main subject of the present thesis is the Bethe ansatz problem for the Gaudin quantum magnet chain. The main results of the thesis fall into the following three directions:

1. (joint with Boris Feigin and Edward Frenkel) Proof of the Bethe Ansatz Conjecture which states the completeness of the algebraic Bethe ansatz for the Gaudin model in the Feigin–Frenkel form.

2. (joint with Alexander Chervov and Gregorio Falqui) Classification of degenerations of the Gaudin model. Presenting Deligne-Mumford compactification of the moduli space of rational curves with marked points as the parameter space for Gaudin models.

3. (joint with Iva Halacheva, Joel Kamnitzer and Alex Weekes) Indexing of solutions of algebraic Bethe ansatz for the Gaudin model by Kashiwara crystals and description of the monodromy of the solutions in combinatorial terms.

This summary is a brief overview of the research. In the Introduction part of the thesis we give a more detailed description of the main results.

Acknowledgements

The most of the ideas of the present thesis grew from various problems posed by my advisors Ernest Borisovich Vinberg and Boris Feigin when I was a graduate student. I first learned about the Gaudin model from Alexander Chervov and in fact he was the person who made me think particularly about this subject. I am grateful to him for his enthusiasm and for numerous stimulating discussions. The relation between the Gaudin Bethe ansatz problem and Kashiwara crystals was conjectured by Pavel Etingof during our 10-minute discussion at MIT. I believe the significant part of the progress in the monodromy problem for the Gaudin model was made in these 10 minutes! I would like to thank all coauthors of the papers contained in this thesis, especially Joel Kamnitzer for explaining me everything I know now about Kashiwara crystals.

1 Gaudin model and related integrable systems

The Gaudin model was introduced in [G76] as a spin model being a generate version of the XXX Heisenberg chain related to the Lie algebra $\mathfrak{sl}_2$. It generalized to the case of arbitrary semisimple Lie algebra $\mathfrak{g}$ in [G83], 13.2.2 and can be described algebraically as follows. Let V_λ be an irreducible representation of $\mathfrak{g}$ with the highest weight λ. For any collection of integral dominant weights $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$, let $V_{\underline{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$. For any $x \in \mathfrak{g}$, consider the operator $x^{(i)} = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$ (x stands at the i-th place), acting on the space $V_{\underline{\lambda}}$. Let $\{x_a\},\ a = 1, \dots, \dim \mathfrak{g}$, be an orthonormal basis of $\mathfrak{g}$ with respect to Killing form, and let $z_1, \dots, z_n$ be pairwise distinct complex numbers. The Hamiltonians of Gaudin model are the following commuting operators acting in the space $V_{\underline{\lambda}}$:

$$H_i = \sum_{k \neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{x_a^{(i)} x_a^{(k)}}{z_i - z_k}. \tag{1}$$

We can regard the H_i as $\mathfrak{g}$-invariant elements of $U(\mathfrak{g})^{\otimes n}$. In [FFR] Feigin, Frenkel and Reshetikhin constructed a maximal commutative subalgebra $\mathcal{A}(\underline{z})$ in the invariants in the tensor power of the universal enveloping algebra $[U(\mathfrak{g})^{\otimes n}]^{\mathfrak{g}}$ which depends on the collection of points $\underline{z} = (z_1, \dots z_n)$ on the complex line and contains the quadratic elements H_i.

We call $\mathcal{A}(\underline{z})$ the *Gaudin algebra*. One can regard the algebra $U(\mathfrak{g})$ as the algebra of global (twisted) differential operators of the flag variety G/B, then $[U(\mathfrak{g})^{\otimes n}]^{\mathfrak{g}}$ is the algebra of differential operators on the moduli space $\mathrm{Bun}_G(\mathbb{P}^1)_{\underline{z}}$ of principal G-bundles on the projective line with a B-structure at the points $z_1, \dots, z_n, \infty$. The Gaudin model can be regarded as the corresponding Hitchin system. Respectively, the commutative subalgebra $\mathcal{A}(\underline{z})$ is generated by all its quantum integrals.

The classical analog of this commutative subalgebra is generated by the classical integrals of the Hitchin system on the rational curve with marked points $z_1, \dots, z_n, \infty$. This subalgebra can be obtained by the Lenard-Magri scheme from some pair of compatible Poisson brackets. In the paper [R08] contained in this thesis we explain that the quantization of these classical integrals is unique (in any reasonable sense). It was noticed in [R06] that this pair of compatible brackets can be degenerated in such a way that the corresponding Poisson-commutative subalgebra becomes the *shift of argument* subalgebra generated by the integrals of the Euler-Manakov top [Ma]. The shift of argument subalgebras were introduced by Fomenko and Mishchenko in [MF] and studied by Vinberg [Vin], Shuvalov [Sh], Tarasov [Tar], Panyushev and Yakimova [PY].

The observation that shift of argument subalgebras are limit cases of Gaudin algebras is important by two reasons: on the one hand, it solves the quantization problem problem for shift of argument subalgebras posed by Vinberg in [Vin] (this is explained in [R06]), and on the other hand, this reduces the problem of enumerating solutions of the algebraic Bethe ansatz for Gaudin model (this is explained in [HKRW] which is a part of the present thesis). Moreover, this leads to a more general family of commutative subalgebras $\mathcal{A}_\chi(\underline{z}) \subset U(\mathfrak{g})^{\otimes n}$ (so called inhomogeneous Gaudin subalgebras, see [R06, FFT, FMTV]) describing the Gaudin magnet chain with an outer magnet field which depend on an additional parameter $\chi \in \mathfrak{g}^*$ which contain both the usual Gaudin algebras and shift of argument algebras as special cases. This general family of algebras plays crucial role in proving completeness of Bethe ansatz [FFR10] and in relating Bethe ansatz to Kashiwara crystals [HKRW].

2 Bethe ansatz

The main problem in Gaudin model is the problem of simultaneous diagonalization of (higher) Gaudin Hamiltonians. It is very much studied (cf. [Fr95, Fr05, Fr05', Fr04, FFR, MTV, MTV']). One can look for the eigenvectors in some prescribed form depending on auxiliary parameters, and write algebraic relations on these parameters guaranteeing the vector to be an eigenvector for Gaudin Hamiltonians. This (very general) method is called *algebraic Bethe ansatz.* Bethe ansatz is said to be *complete* if all the eigenvectors can be obtained this way. For $\mathfrak{g} = \mathfrak{sl}_2$, the Bethe ansatz form of the solutions of the corresponding eigenproblem was already known to Gaudin [G83]. Namely, introduce Bethe vectors

$$v(w_1, \ldots, w_m) = f(w_1) \ldots f(w_m) v_\lambda \in \otimes_{i=1}^n V_{\lambda_i},$$

where

$$f(w) = \sum_{i=1}^n \frac{f^{(i)}}{w - z_i}.$$

The vector $v = v(w_1, \ldots, w_m)$ is an eigenvector for H_i if and only if the following *Bethe ansatz equations* are satisfied:

$$\sum_{i=1}^N \frac{\lambda_i}{w_j - z_i} - \sum_{s \neq j} \frac{2}{w_j - w_s} = 0, \qquad j = 1, \ldots, m.$$

For arbitrary $\mathfrak{g}$, a similar form of Bethe ansatz was first written by Babujian and Flume in [BF] and interpreted by Feigin, Frenkel and Reshetikhin [FFR] in the following way. According to Beilinson and Drinfeld [BD], the quantum integrals of the Hitchin system come from the center of the universal enveloping algebra of the corresponding affine Kac-Moody algebra at the critical level. This center is naturally identified with the classical W-algebra attached to the Langlands dual Lie algebra ${}^L\mathfrak{g}$, hence the spectrum of this center is naturally the space of ${}^L\mathfrak{g}$-opers (i.e. connections in a principal ${}^LG$-bundle with certain transversality condition) on the formal punctured disc. Respectively, the spectrum of the quantum integrals of the Hitchin system is given by global ${}^L\mathfrak{g}$-opers on the corresponding curve with prescribed singularities at the marked points. The Bethe ansatz equations are interpreted as the condition of existence of a Miura structure on the oper while the auxiliary parameters in the equations are the coordinates of the points where the Miura structure degenerates. Equivalently, the joint eigenvalues for Gaudin Hamiltonians correspond to monodromy-free opers on the projective line with prescribed singularities and with trivial monodromy representation, and the completeness of Bethe ansatz means that this is one-to-one. The precise formulation and the proof of this result for the Gaudin model is given in the papers [FFR10, R20] which are contained in the present thesis. This result is known as *Bethe ansatz conjecture* in the Feigin–Frenkel form.

Another geometric interpretation of Bethe ansatz developed by Mukhin, Tarasov and Varchenko deals with Schubert calculus on Grassmannians, cf.

[MTV, MTV'], and also gives a proof of Bethe ansatz conjecture in type A. The relation of this approach to that of [FFR] is explained in [Fr04].

3 Parameter space for Gaudin models

The interpretation of the Gaudin model as a Hitchin system suggests the generalization of the model to nodal rational curves. In the papers [CFR09, CFR10, R18] contained in this thesis we describe the family of Gaudin models depending on a point in the Deligne-Mumford moduli space $\overline{M_{0,n+1}}$ of stable rational curves with $n+1$ marked points (which are $z_1, \ldots, z_n, \infty$ for the usual Gaudin model). This generalizes the result of Aguirre, Felder and Veselov [AFV] extending the family of quadratic Gaudin Hamiltonians to the Deligne-Mumford moduli space. Bethe ansatz conjecture proved in [FFR10, R20] guarantees that when the parameter $\underline{z} \in \overline{M_{0,n+1}}$ is real, then the Gaudin algebra acts with simple spectrum on the tensor product multiplicity space and gives us a basis of eigenvectors.

Similarly, the parameter space for shift of argument subalgebras compactifies as the De Concini – Procesi wonderful closure [DCP] of the complement of the root hyperplane arrangement in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We show this in [HKRW].

4 Kashiwara crystals and commutors

The Gaudin algebras depend on a parameter which is a point in the Deligne-Mumford moduli space $\overline{M_{0,n+1}}$ of marked stable genus 0 curves, and the eigenvectors of the Gaudin Hamiltonians are given by some multivalued functions of the parameter in $\overline{M_{0,n+1}}$. Moreover, these multivalued functions are unramified over $\overline{M_{0,n+1}}(\mathbb{R})$, the real locus of $\overline{M_{0,n+1}}$. So it is natural to study the monodromy of these eigenvectors as the parameter varies within the real locus; this gives an action of the fundamental group of $\overline{M_{0,n+1}}(\mathbb{R})$ which is called the cactus group J_n (see [D, DJS]), on the finite set of the eigenlines for the Gaudin Hamiltonians in the space of states $V_{\underline{\lambda}}$.

The group J_n arises naturally as an analog of the braid group in coboundary monoidal categories, cf. [HK1, Sa]. A *coboundary category* is a monoidal category $\mathcal{C}$ along with a natural isomorphism called *commutor*

$$\sigma_{A,B} : A \otimes B \to B \otimes A$$

satisfying the following two axioms.

1. For all $A, B \in \mathcal{C}$, we have $\sigma_{B,A} \circ \sigma_{A,B} = id_{A\otimes B}$
2. For all $A, B, C \in \mathcal{C}$ we have $\sigma_{A,B\otimes C} \circ (id \otimes \sigma_{B,C}) = \sigma_{A\otimes B,C} \circ (\sigma_{A,B} \otimes id)$.

All possible commutors on the n-fold tensor product $X_1 \otimes \ldots \otimes X_n$ in a coboundary category $\mathcal{C}$ generate an action of the cactus group J_n.

The main example of a coboundary monoidal category is the category of Kashiwara $\mathfrak{g}$-crystals (i.e. combinatorial version of the category of finite-dimensional $\mathfrak{g}$-modules). More precisely, to any irreducible representation V_λ one assigns an oriented graph B_λ (called normal crystal of the highest weight λ) whose vertices correspond to basis vectors of V_λ and are labelled by the weights of V_λ while the edges correspond to the action of the Chevalley generators and are labelled by the simple roots of $\mathfrak{g}$, with a purely combinatorial rule of tensor multiplication. In [HKRW] we prove the conjecture of Pavel Etingof on the existence of a bijection between the set of eigenvectors for the Gaudin Hamiltonians in $V_{\underline{\lambda}}$ and the tensor product of $\mathfrak{g}$-crystals $B_{\lambda_1}\otimes\ldots\otimes B_{\lambda_n}$, compatible with the J_n-action (for $\mathfrak{g}=\mathfrak{sl}_2$ it was done earlier in [R18]). In fact, we prove that the coboundary category of normal $\mathfrak{g}$-crystals can be reconstructed using the coverings of the moduli spaces given by the eigevectors of the Gaudin model. Our main tool is the construction of a crystal structure on the set of eigenvectors for shift of argument algebras, another family of commutative algebras which act on any irreducible $\mathfrak{g}$-representation. We also prove that the monodromy of such eigenvectors is given by the internal cactus group action on $\mathfrak{g}$-crystals (in type A this recovers the cactus group action on the Gelfand-Tsetlin polytope described by Berenstein and Kirillov in [BK]). This gives (as complete as possible) combinatorial indexing of solutions of Bethe ansatz for the Gaudin model. This also generalizes the results of White [W] (see also Brochier, Gordon and White [BGW]) and also should be related to the construction of Calogero-Moser cells due to Bonnafe and Rouquier [BR]. The possible generalization of this result to the XXX Heisenberg magnet chain should give a new explanation of the Kirillov–Reshetikhin indexing of solutions of Bethe ansatz equations [KR].

5 Gaudin model and KZ connection

The quadratic Gaudin Hamiltonians naturally arise in the Knizhnik-Zamolodchikov connection, which is a flat connection in the trivial bundle over the configuration space $\mathbb{C}^n\setminus\{z_i=z_j\}$ with the fiber $V_{\underline{\lambda}}$:

$$\nabla = d+\frac{1}{\varkappa}\sum_{i=1}^{n} H_i dz_i.$$

More precisely, the Gaudin model can be regarded as the limit of the KZ connection as $\varkappa\to 0$. The Bethe vectors are the leading terms of asymptotic solutions considered by Varchenko [Var]. The asyptotic zones for KZ connection are open strata on $\overline{M_{0,n+1}}(\mathbb{R})$, and the action of the cactus group J_n on Bethe vectors comes from the reordering of asymptotics when crossing the walls between the open strata.

Drinfeld-Kohno correspondence provides a natural isomorphism between the space of flat sections with respect to the KZ connection and the tensor product of the corresponding representations $V^q_{\underline{\lambda}}=V^q_{\lambda_1}\otimes\cdots\otimes V^q_{\lambda_n}$ of the quantum group

$U_q(\mathfrak{g})$ with $q = \exp(\frac{\pi i}{\varkappa})$ which is compatible with the braid group action (by monodromy on the KZ side and by R-matrices on the U_q side).

The tensor product of crystals $B_{\underline{\lambda}}$ arise as the $q \to \infty$ limit of crystal bases in $V^q_{\underline{\lambda}}$, and the action of the unitarized R-matrices on the crystal base give rise to the J_n action on the tensor product crystal coming from the coboundary monoidal structure. Moreover, according to [Var], for $\mathfrak{g} = \mathfrak{sl}_2$, asymptotic flat sections correspond to some crystal base under the Drinfeld-Kohno correspondence. So Etingof's conjecture proved in [HKRW] can be regarded as a $\varkappa \to 0$, $q \to \infty$ limit of the Drinfeld-Kohno correspondence. In [R18] we give a detailed explanation of this approach and prove Etingof's conjecture for $\mathfrak{g} = \mathfrak{sl}_2$ by taking the limit $\varkappa \to 0$, $q \to \infty$ and using the results of Varchenko [Var].

6 The list of main results of the thesis

The main results of the thesis are as follows:

1. In [R08] we prove that there is a unique commutative subalgebra $\mathcal{A} \subset U(t^{-1}\mathfrak{g}[t^{-1}])$ whose image under the homomorphism of evaluation at the points $z_1, \ldots, z_n$ is the Gaudin subalgebra $\mathcal{A}(\underline{z})$. In particular, this means, that explicit formulas for higher Gaudin Hamiltonians known in classical types (cf. [Tal, CM]) give the same commutative subalgebra as the general recipe of [FFR].

2. In [FFR10, R20] we prove that the joint eigenvalues of (higher) Gaudin Hamiltonians on $V_{\underline{\lambda}}$ are in one-to one correspondence with monodromy-free ${}^L\mathfrak{g}$-opers on the projective line with prescribed singularities at the points $z_1, \ldots, z_n, \infty$. More precisely, in [FFR10] we show this for inhomogeous version of Gaudin model (i.e. Gaudin model with a non-degenerate outer magnet field) which turns to be easier and in [R20] deduce this for the original Gaudin model. Moreover, for real values of the parameters z_i (and hence for generic z_i) these joint eigenvalues have no multiplicities. This is the Feigin-Frenkel version of the Bethe ansatz conjecture.

3. In [CFR09, CFR10, R18] we show that the family of Gaudin algebras extends to the Deligne-Mumford compactification $\overline{M_{0,n+1}}$ and show that the results of [FFR10, R20] on simplicity of spectrum still hold true for the Gaudin algebras corresponding to the boundary points of $\overline{M_{0,n+1}}$. This allows to consider the set of Bethe vectors as a covering over $\overline{M_{0,n+1}}$ which is unramified over the real points.

4. In [HKRW] we solve the same compactification problem for the parameter space of shift of argument subalgebras obtaining the De Concini-Procesi wonderful compactification as the answer.

5. In [R18, HKRW] we prove the conjecture of Etingof on the monodromy of Bethe vectors. Namely, we constuct a natural structure of tensor product of Kashiwara crystal on the set of joint eigenvectors for (inhomogeneous)

Gaudin algebra on the space $V_{\underline{\lambda}}$. This implies that the monodromy of Bethe vectors along $\overline{M_{0,n+1}(\mathbb{R})}$ is given by crystal commutors.

7 Organization of the thesis

The thesis consists of the Introduction, the list of references, and the following papers:

1. L. Rybnikov, *Uniqueness of higher Gaudin hamiltonians*, Reports on Mathematical Physics **61** (2008) No 2, pp. 247-252. [R08]
2. A. Chervov, G. Falqui, L. Rybnikov, *Limits of Gaudin Systems: Classical and Quantum Cases*, SIGMA. Symmetry, Integrability and Geometry: Methods and Applications **5** (2009), 029 [CFR09]
3. B. Feigin, E. Frenkel, L. Rybnikov, *Opers with irregular singularity and spectra of the shift of argument subalgebra*, Duke Mathematical Journal **155** (2010), No 2, pp. 337-363. [FFR10]
4. A. Chervov, G. Falqui, L. Rybnikov, *Limits of Gaudin algebras, quantization of bending flows, Jucys–Murphy elements and Gelfand–Tsetlin bases*, Letters in mathematical physics **91** (2010), No 2, pp. 129-150. [CFR10]
5. L. Rybnikov, *Cactus group and monodromy of Bethe vectors*, International Mathematics Research Notices **2018** No 1, pp. 202-235. [R18]
6. L. Rybnikov, *A proof of the Gaudin Bethe Ansatz conjecture*, International Mathematics Research Notices **2020** No 22, pp. 8766-8785. [R20]
7. I Halacheva, J Kamnitzer, L Rybnikov, A Weekes, *Crystals and monodromy of Bethe vectors*, Duke Mathematical Journal **169** (2020) No 12, pp. 2337-2419. [HKRW]

The Introduction part of the thesis and is organized as follows. In Section 1 we introduce the Gaudin algebras and show how they help to construct quantum integrals of Hamiltonian systems. In Section 2 we state the Bethe ansatz conjecture for Gaudin model in the Feigin-Frenkel form and explain its proof following [FFR10, R20]. In Section 3 we explain the compactification procedure for the parameter space of Gaudin algebras and extend Bethe ansatz conjecture to this compactification, following [CFR09, CFR10, R18]. In Section 4 we describe the monodromy of solutions of Bethe ansatz along the compactified parameter space, following [HKRW]. In Section 5 we outline some directions of further development of the results of [HKRW].

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