Skolkovo Institute of Science and Technology

National Research University Higher School of Economics

as a manuscript

Ilya Vilkoviskiy

Integrable structures of the affine Yangian

Summary of the PhD thesis for the purpose of obtaining academic degree
Doctor of Philosophy in Mathematics at HSE
and Doctor of Philosophy in Mathematics and Mechanics at Skoltech

Academic supervisors:
Candidate of Physical and Mathematical Sciences
Associate Professor
Litvinov Alexey Viktorovich
Doctor of Physical and Mathematical Sciences
Professor
Marshakov Andrey Vladimirovich

Moscow – 2022

Introduction

0.1 Integrable field theories, integrable structures of CFT

As was pointed out by Zamolodchikov [Zam89] there is a natural relation between integrable and conformal field theories. Namely having an integrable field theory it is always possible to consider its ultraviolet (UV) limit which is controlled by conformal field theory (CFT). The infinite tower of Integrals of Motion $\mathbf{I_s}(\lambda)$ in this limit splits into two independent family of Integrals of Motion defined in a purely CFT terms.

$$\mathbf{I}_s(\lambda) = \mathbf{I}_s + O(\lambda), \qquad \mathbf{I}_{-s}(\lambda) = \bar{\mathbf{I}}_s + O(\lambda),$$

here λ is a scale parameter and turns to zero in a UV limit, $\mathbf{I}_s$ and $\bar{\mathbf{I}}_s$ are two decoupled integrable systems acting in the space of holomorphic and antiholomorphic fields correspondingly. More importantly, as explained in [Zam89] it is often possible to recover the massive integrable field theory out of integrable structure of CFT. The integrable systems in CFTs is much more simple than the ones in massive integrable field theories, and so, the study of integrable structures in CFT serves as a good playground to understand the space of Integrable quantum field theories (IQFT). In particular the integrable structures of CFTs plays an important role in [Lit19], [LV20] and allows to guess new integrable Toda field theories, and provide a duality between them and Integrable sigma models.

Despite the great simplifications complete diagonalization of chiral Integrals of Motion (IOMs) is yet a nontrivial problem. The study of integrable structure of conformal field theory began with the seminal series of papers of Bazhanov, Lukyanov and Zamolodchikov (BLZ) [BLZ96, BLZ97, BLZ99] devoted to study of quantum KDV integrable system, which appears in the UV limit of sine-Gordon theory. In particular, the set of generating functions for local and non-local Integrals of Motion has been explicitly constructed. Unfortunately the construction of [BLZ96, BLZ97, BLZ99] does not known to provide by itself any equations for the spectrum of the Integrals of Motion.

New ideas appear since the discovery of Ordinary Differential Equation/Integrable Model (ODE/IM) correspondence [DT99a, BLZ01, DT99b]. Using this approach and bunch of analytic intuition, Bazhanov, Lukyanov and Zamolodchikov [BLZ04] were able to express the spectrum of the local IOMs in terms of solutions of certain algebraic system of equations. Later these equations were generalized for some other integrable structures, such as Fateev models or quantum AKNS model (see [KL20] for the list of all known cases). Despite the obvious success of BLZ program, it is still unclear where the algebraic equations of [BLZ04] come from, and whether they can be easily generalized for other models of CFT.

In this thesis we develop a parallel approach based on the affine Yangian symmetry. The advantage of this approach is that it fits in general framework of the quantum inverse scattering method, provides Bethe ansatz equations for the spectrum and allows to treat a lot of integrable structures in a unified way. Being originally formulated geometrically [Var00, Nak01, MO19], it can be rephrased entirely algebraically in CFT terms[1]. In [LV20], using this algebraic approach, we studied the integrable structures in CFT related to $\mathrm{Y}\big(\widehat{\mathfrak{gl}}(1)\big)$, the affine Yangian of $\mathfrak{gl}(1)$ [Tsy17]. These integrable structures describe W algebras of A_n type and its super-algebra generalizations and can be viewed as twist

[1] For the modern review of the geometric approach and more advanced topics see Andrei Okounkov's summer lecture course `sites.google.com/view/andrei-okounkov-lecture-course/home`.

deformations of the quantum Gelfand-Dikii hierarchies (quantum ILWtype integrable systems). We also were able to study integrable structures of W algebras of BCD type, by realising corresponding integrable systems as an affine Yangian "spin chain" with boundaries [LV21].

Affine Yangian of $\mathfrak{gl}(1)$ admits two different descriptions: the current realisation which is useful in studying the spectrum and Bethe eigenfunctions, and the so called Chevalley description in terms of generators of $W_{1+\infty}$ algebra. The second description is more useful in study of the local Integrals of Motion. In order to clarify the structure of W algebra, it may be useful to study its q-deformation. The q-deformations of W algebras have been provided in [AKOS96] for type A, and in [FR97] for simple Lie algebras. The deformations of the local IOMs associated to W algebras of type A were constructed in [KOJ06], [FJM17]. In the third chapter we review the q-deformation of W algebras defined as a commutant of screenings and provide a construction for a q-deformation of local integrals of motion of arbitrary high spin for W algebras of type $\mathrm{B}, \mathrm{C}, \mathrm{D}$.

Chiral Integrals of Motion, example. In order to clarify the ideas above, let us consider an example of classical Sinh-Gordon model living on a cylinder of length $L = 2\pi$ and defined by the action:

$$S = \int \left(\frac{1}{\pi} (\partial_z \varphi \partial_{\bar{z}} \varphi) + \lambda \cosh\left(2b\varphi\right) \right) d^2 z, \tag{0.1.1}$$

where $z = x + iy$, $\bar{z} = x - iy$ are the complex coordinates.
The theory is known to contain an infinite tower of Integrals of Motions:

$$\partial_{\bar{z}} T_{s+1} = \lambda \partial_z \Theta_{s-1}, \quad \partial_z T_{-s-1} = \lambda \partial_{\bar{z}} \Theta_{-s+1}, \quad s \geq 1$$

$$\mathbf{I}_s(\lambda) = \int \frac{dx}{2\pi} \Big(T_{s+1} - \lambda \Theta_{s-1} \Big).$$

In the classical limit the first few IOMs are given by the following formulas:

$$T_2 = (\partial_z \varphi)^2, \quad \Theta_0 = 2\pi\lambda \cosh(2b\varphi) \tag{0.1.2}$$

$$T_4 = (\partial_z \varphi)^4 + b^{-2} (\partial_z^2 \varphi)^2, \quad \Theta_2 = 4\pi\lambda \, (\partial_z \varphi)^2 \cosh(2b\varphi), \tag{0.1.3}$$

which should be corrected at the quantum level. One may already see that in the UV limit $(\lambda \to 0)$ Θ_s vanishes, and we are left with the chiral mutually commuting Integrals of Motion $\mathbf{I}_s \stackrel{\text{def}}{=} \mathbf{I}_s(0)$. It may also be shown (see for eg [FF95]) that that the classical chiral Integrals of Motion may be selected by the condition of the Poisson commutativity with the screenings

$$\{\mathbf{I}_s, \mathcal{S}_i\} = 0,$$

where

$$\mathcal{S}_1 = \oint e^{2b\varphi(z)} \frac{dz}{2\pi}, \quad \mathcal{S}_2 = \oint e^{-2b\varphi(z)} \frac{dz}{2\pi}.$$

It turns out that the quantization of chiral integrable system may be defined very directly. Namely, following the ideas of Zamolodchikov [Zam89] developed in [LF91] and also [FF96] we will postulate the following formula for the chiral Integrals of Motion in the quantum case[2]:

$$[\mathbf{I}_s, \mathcal{S}_i] = 0, \tag{0.1.4}$$

and

$$\mathcal{S}_1 = \oint e^{2b\varphi(z)} \frac{dz}{2\pi}, \quad \mathcal{S}_2 = \oint e^{-2b\varphi(z)} \frac{dz}{2\pi}.$$

[2]Strictly speaking, the commutator in the LHS is not well defined, as the contour of integration is not closed for the general values of the zero mode of the field $\varphi(z)$. We, nevertheless, make this inaccuracy, the proper definition is explained below (see (0.1.6), (0.1.7)).

We would like to stress out that the rigorous quantization and definition of the IOMs for the full massive integrable model with non zero λ is far more non trivial problem, which we don't even touch in this thesis.
Let us be more precise, we are working in a second quantisation picture, $\varphi(z)$ is the free bosonic field:

$$\partial\varphi(z) = u + \sum_{n\neq 0} a_n e^{inz}, \quad [a_n, a_m] = \frac{m}{2}\delta_{m,-n}. \tag{0.1.5}$$

The field $\varphi(z)$ acts in the standard Fock space $\mathcal{F}_u$:

$$F_u = \{\mathbf{C}[a_{-1}, a_{-2}, \dots]|\varnothing\rangle\},$$
$$a_n|\varnothing\rangle = 0, \text{ for } n > 0,$$
$$a_0|\varnothing\rangle = u.$$

We will search for the Integrals of Motion of fixed spin s, as an integrals of local densities $\mathbf{I}_s = \int_0^{2\pi} G_{s+1}(\partial\varphi(z), \partial^2\varphi(z), \dots)\frac{dz}{2\pi}$, which are polynomials in $\partial\varphi$ and its derivatives. We further introduce two vertex operators $V_\pm(z) = e^{\pm 2b\varphi(z)}$, the equations (0.1.4) then reads as a conditions on the coefficients in the operator product expansion:

$$V_\pm(w)G_{s+1}(z) = \text{reg} + \frac{\partial X_s^{(1)}(z)}{z-w} + \sum_{k\geq 2}\frac{X_s^{(k)}(z)}{(z-w)^k}, \tag{0.1.6}$$

or equivalently:

$$\oint_z V_\pm(w)G_{s+1}(z)\frac{dw}{2\pi} = \partial X_s^{(1)}(z), \tag{0.1.7}$$

where $X_s^{(k)}(z)$ are some local fields. Equations (0.1.4) then is nothing but a system of a linear equations on a coefficients of density G_{s+1}. Direct computation provides for the first few Integrals of Motion:

$$G_2 =: (\partial_z\varphi(z))^2 :$$
$$G_4 =: (\partial_z\varphi)^4 : +(Q^2+1) : (\partial_z^2\varphi)^2 :$$
$$G_6 =: (\partial_z\varphi)^6 : -\frac{5}{8} : (\partial\varphi)^4 : +5(Q^2+2)\Big(: (\partial_z^2\varphi)^2\partial_z\varphi^2 : -\frac{1}{24} : (\partial^2\varphi)^2 : \Big) + \Big(Q^4 + \frac{8}{3}Q^2 + \frac{19}{12}\Big) : (\partial^2\varphi)^2 :$$

$$G_8 = \Big(: (\partial_z\varphi)^8 : + \dots\Big),$$

here $Q = b + \frac{1}{b}$, and " : : " denotes the Wick normal ordering.

While I_3 and I_5 obviously commute with the I_1 which plays the role of grading operator, the commutativity of I_3 and I_5 is not obvious but straightforward to check. Note that this densities coincide with densities T_s (0.1.2),(0.1.3) in the semiclassical limit $b \to \infty$ as it should be.

More generally, one can consider the tensor product of n Fock spaces $\mathcal{F}_{u_1} \otimes \dots \otimes \mathcal{F}_{u_n}$ and the corresponding n-component bosonic field $\boldsymbol{\varphi}(z) = (\varphi_1, \dots, \varphi_n)$[3]:

$$\partial\varphi_j(z) = u_j + \sum_{n\neq 0} a_n^{(j)} e^{inz}, \quad [a_n^i, a_m^j] = m\delta_{i,j}\delta_{m,-n},$$

and affine set of screenings corresponding to an affine Lie algebra $\hat{\mathfrak{g}}$.

$$\mathcal{S}_r = \oint e^{b(\boldsymbol{\alpha}_r\cdot\boldsymbol{\varphi}(z))}\frac{dz}{2\pi},$$

[3]Note that commutation relation of bosonic modes $a_n^{(i)}$ are different from ones defined previously in case of a single field (0.1.5). This is because the Sinh-Gordon model is an $\hat{\text{A}}_1$ Toda and in (0.1.1) we already decouple the $U(1)$ center of mass $U = \frac{\varphi_1+\varphi_2}{2}$, and left with a single bosonic field $\varphi = \frac{\varphi_1-\varphi_2}{2}$.

where $\boldsymbol{\alpha}_r$ have scalar products corresponding to the Dynkin diagram of an affine Lie algebra $\hat{\mathfrak{g}}$: $(\alpha_r \cdot \alpha_s) = c_{r,s}$. The Integrals of Motion can be again found as the intersection of kernels of all the screenings [LF91], [FF96]:

$$[\mathbf{I}_s, \mathcal{S}_r] = 0.$$

In this thesis we will consider in details the cases of $\hat{\mathfrak{g}} = \hat{\mathbf{A}}_n$ and $\hat{\mathfrak{g}} = \hat{\mathbf{B}}_{\mathbf{n}}, \hat{\mathbf{C}}_{\mathbf{n}}, \hat{\mathbf{D}}_{\mathbf{n}}$. The existence of a grading operator $\mathbf{I}_1$ among the Integrals of Motion allows to restrict the IOMs on a finite dimensional space $\mathbf{I}_1 = N$, such that they becomes a finite dimensional matrices. Nonetheless their exact diagonalization is by far a non trivial problem. Our strategy in analysing this problem is to identify corresponding integrable systems with integrable "spin chains" with the symmetry of affine Yangian, and then apply to them a machinery of algebraic Bethe ansatz.

0.2 Thesis results

The main results of chapters 1 and 2 are the Bethe ansatz equations and the Bethe eigenvectors, which provide a diagonalization of the chiral integrals of motion obtained as a UV limit of the Toda integrable system.

- For the A_n case we derive the Bethe ansatz equations for the spectrum of the local (1.1.13) and KZ (1.4.2) Integrals of Motion:

$$q\prod_{j\neq i}\prod_{\alpha=1}^{3}\frac{x_i-x_j-\epsilon_\alpha}{x_i-x_j+\epsilon_\alpha}\prod_{k=1}^{n}\frac{x_i-u_k+\epsilon_3}{x_i-u_k}=1 \quad \text{for all} \quad i=1,\dots,N, \tag{0.2.1}$$

  here we used Nekrasov epsilon notations $\epsilon_1 \sim b^{-1}, \epsilon_2 \sim b, \epsilon_3 \sim -Q$, see formula (1.3.7) for details. Corresponding Bethe vectors are given by the formula (1.4.19).

- For the BCD case we derive the boundary Bethe ansatz equations for the spectrum of the local (2.2.1),(2.2.5) and KZ (2.3.8) Integrals of Motion:

$$\begin{gathered} r^\alpha(x_i)r^\beta(x_i)A(x_i)A^{-1}(-x_i)\prod_{j\neq i}G(x_i-x_j)G^{-1}(-x_i-x_j)=1, \\ G(x)=\frac{(x-\epsilon_1)(x-\epsilon_2)(x-\epsilon_3)}{(x+\epsilon_1)(x+\epsilon_2)(x+\epsilon_3)}, \qquad A(x)=\prod_{k=1}^{n}\frac{x-u_k+\frac{\epsilon_3}{2}}{x-u_k-\frac{\epsilon_3}{2}}, \quad r^\alpha(x)=-\frac{x+\epsilon_\alpha/2}{x-\epsilon_\alpha/2}. \end{gathered} \tag{0.2.2}$$

  And the Bethe vectors are defined in (2.4.5).

Another important results include:

- explicit computation of the current realisation (1.3.9) of the RLL algebra with Maulik-Okounkov [MO19] $R-$matrix.

- three different solutions $\mathcal{K}^i$ (2.3.5)-(2.3.6) of the Sklyanin's KRKR relation with the Maulik-Okounkov [MO19] $R-$matrix.

In chapter 3 we studied Integrals of Motion for the q-deformed W algebras.

- We provide explicit formulas for the Integrals of Motion of the q-deformed W algebras of BCD type (3.4.23).

- We construct the q-deformed versions of the reflection R and K operators (3.5.1).

0.3 Thesis review

This section is a short guide through the thesis, which contains main statements and ideas. The thesis consists of three chapters. The chapter 1 of the thesis is devoted to the study of $\hat{\mathfrak{gl}}_1$ affine Yangian and related integrable systems. We studied in details the connection between the RLL algebra and its current realisation. We derive the local Integrals of Motion for W algebras of type A (1.1.13) and corresponding Bethe ansatz equations (1.4.17) for their spectrum. In the chapter 2 we introduce the Integrals of Motion of BCD type (2.2.1),(2.2.5), and studied their spectrum by means of the boundary Bethe ansatz of the affine Yangian. We provide three different solutions $K^{1,2,3}$ of the Sklyanin's KRKR equation (2.3.5)-(2.3.6), and the Bethe ansatz equations (2.4.7) for the spectrum of the local Integrals of Motion. In the chapter 3 we studied the q-deformation of the Local and KZ Integrals of Motion. We provide explicit formulas for the q-deformed versions of the local Integrals of Motion of arbitrary high spin (3.4.23) for the q-deformed W algebras of type BCD.

***W* algebras and Maulik-Okounkov *$R-$*matrix.** In section 1.2 we recall the definition of our main tool the Maulik-Okounkov $R-$matrix [MO19] as a unique (up to a normalisation factor) solution of the intertwining relation:

$$\mathcal{R}_{i,j}\big(Q\partial-\partial\varphi_i\big)\big(Q\partial-\partial\varphi_j\big)=\big(Q\partial-\partial\varphi_j\big)\big(Q\partial-\partial\varphi_i\big)\mathcal{R}_{i,j}, \tag{0.3.1}$$

where the product of two brackets is a Miura-Gelfand-Dikii transformation [FL88, Luk88] which defines generators of W algebra. Multiplying the brackets in different orders we obtain two isomorphic but not identical W algebras

$$\begin{aligned}\big(Q\partial-\partial\varphi_j\big)\big(Q\partial-\partial\varphi_i\big)&=(Q\partial)^2+W^{(1)}(z)(Q\partial)+W^{(2)}(z),\\ \big(Q\partial-\partial\varphi_i\big)\big(Q\partial-\partial\varphi_j\big)&=(Q\partial)^2+\tilde{W}^{(1)}(z)(Q\partial)+\tilde{W}^{(2)}(z)\end{aligned}$$

The operator $\mathcal{R}_{i,j}$ then intertwines the two W algebras and acts in the tensor product of two Fock representations of Heisenberg algebra with the highest weight parameters u_i and u_j

$$\mathcal{F}_{u_i}\otimes\mathcal{F}_{u_j}\xrightarrow{\mathcal{R}_{i,j}}\mathcal{F}_{u_i}\otimes\mathcal{F}_{u_j}$$

and its matrix depends on the difference u_i-u_j. Then, by considering W_3 algebra generated by the product of three terms $\big(Q\partial-\partial\varphi_1\big)\big(Q\partial-\partial\varphi_2\big)\big(Q\partial-\partial\varphi_3\big)$, we immediately obtain from the definition (0.3.1) that the $\mathcal{R}_{i,j}(u_i-u_j)$ matrix satisfies the Yang-Baxter equation

$$\mathcal{R}_{1,2}(u_1-u_2)\mathcal{R}_{1,3}(u_1-u_3)\mathcal{R}_{2,3}(u_2-u_3)=\mathcal{R}_{2,3}(u_2-u_3)\mathcal{R}_{1,3}(u_1-u_3)\mathcal{R}_{1,2}(u_1-u_2),$$

and hence the whole machinery of quantum inverse scattering method can be applied.

RLL algebra and its current realisation. In section 1.3 we introduce an RLL algebra:

$$\mathcal{R}_{ij}(u-v)\mathcal{L}_i(u)\mathcal{L}_j(v)=\mathcal{L}_j(v)\mathcal{L}_i(u)\mathcal{R}_{ij}(u-v). \tag{0.3.2}$$

The left and right hand sides of this equation both act in the tensor product of three Fock spaces $\mathcal{F}_{u_i}\otimes\mathcal{F}_{u_j}\otimes\mathcal{F}_q$. The $\mathcal{R}_{ij}(u_i-u_j)$ matrix acts in the product of two Fock spaces $\mathcal{F}_{u_i}\otimes\mathcal{F}_{u_j}$, and $\mathcal{L}_i(u_i)$ operator acts in $\mathcal{F}_{u_i}\otimes\mathcal{F}_q$. Hence the RLL algebra (0.3.2) may be considered as an infinite set of quadratic relations between the matrix elements of $L-$operator, labeled by two partitions

$$\mathcal{L}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(u)\overset{\text{def}}{=}\langle u|a_{\boldsymbol{\lambda}}\mathcal{L}(u)a_{-\boldsymbol{\mu}}|u\rangle\quad\text{where}\quad a_{-\boldsymbol{\mu}}|u\rangle=a_{-\mu_1}a_{-\mu_2}\dots|u\rangle.$$

It is well known that the commutation relations of RLL algebras could be rewritten in an equivalent current form, see [DF93] where such an analysis was performed for $U_q(gl(n))$. In this thesis we provide similar analysis for the case of Maulik-Okounkov $R-$matrix. We conjecture that the *RLL* algebra (0.3.2) factorized over its center is related to the Yangian of $\widehat{\mathfrak{gl}}(1)$ considered by Tsymbaliuk in [Tsy17]. This is similar to the well known fact that the Yangians of $\mathfrak{gl}(n)$ and of $\mathfrak{sl}(n)$ are differ by central elements [KS82]. We will usually refer to the RLL algebra as Yang-Baxter algebra and denote as $\mathrm{YB}\big(\widehat{\mathfrak{gl}}(1)\big)$, reserving the notation $\mathrm{Y}\big(\widehat{\mathfrak{gl}}(1)\big)$ for Tsymbaliuk's algebra.

Our methods are similar to the analysis performed in [DF93]. We introduce three basic currents of degree 0, 1 and -1 (see appendix A.2 for more details)

$$h(u) \stackrel{\text{def}}{=} \mathcal{L}_{\varnothing,\varnothing}(u), \qquad e(u) \stackrel{\text{def}}{=} h^{-1}(u)\cdot\mathcal{L}_{\varnothing,\square}(u) \quad \text{and} \quad f(u) \stackrel{\text{def}}{=} \mathcal{L}_{\square,\varnothing}(u)\cdot h^{-1}(u),$$

as well as an auxiliary current (as we will see (1.3.9a) it also belongs to the Cartan subalgebra of $\mathrm{YB}\big(\widehat{\mathfrak{gl}}(1)\big)$)

$$\psi(u) \stackrel{\text{def}}{=} \Big(\mathcal{L}_{\square,\square}(u-Q) - \mathcal{L}_{\varnothing,\square}(u-Q)h^{-1}(u-Q)\mathcal{L}_{\square,\varnothing}(u-Q)\Big)h^{-1}(u-Q). \tag{0.3.3}$$

The direct computation (provided in the appendix A.2) allows to rewrite the RLL commutation relations (0.3.2) in terms of e, f, h currents. The results are presented at the beginning of section 1.3.1. There also exists an inverse mapping which allows to express $\mathcal{L}_{\lambda,\mu}(u)$ operators in terms of e, h, f, ψ currents. In particular there is an important for us formula

$$\mathcal{L}_{\boldsymbol{\lambda},\varnothing}(u) = \frac{1}{(2\pi i)^{|\boldsymbol{\lambda}|}} \oint \cdots \oint F_{\lambda}(\boldsymbol{z}|u)\, h(u) f(z_{|\boldsymbol{\lambda}|}) \dots f(z_1) dz_1 \dots dz_{|\boldsymbol{\lambda}|} \tag{0.3.4}$$

where $F_\lambda(\boldsymbol{z})$ is a concrete function and contours go clockwise around ∞ and all poles of $F_\lambda(\boldsymbol{z})$. This formula and recurrent definition of function $F_\lambda(\boldsymbol{z})$ is explained in the appendix A.3, see formulas (A.3.8),(A.3.10).

ϵ- notations. It is easy to note that quantum Integrals of Motion depends only on combination $Q = b + \frac{1}{b}$ and not b, b^{-1} themselves. Which results in a very well known symmetry $b \to b^{-1}$. As can be seen for example in [Tsy14], defining relations of affine Yangian algebra are symmetric under all three parameters b, b^{-1} and Q parameters[4]. For this reason it will be more convenient to use Nekrasov epsilon notations rather than Liouville notations. Formally, they are obtained by replacing central charge parameter

$$b \to \frac{\epsilon_2}{\sqrt{\epsilon_1\epsilon_2}}, \quad b^{-1} \to \frac{\epsilon_1}{\sqrt{\epsilon_1\epsilon_2}}, \quad Q \to -\frac{\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}} \implies \epsilon_1+\epsilon_2+\epsilon_3 = 0.$$

Note that without loss of generality it is always possible to put $\epsilon_1\epsilon_2 = 1$.

Center of $\mathbf{YB}(\widehat{\mathfrak{gl}}(1))$ The section 1.3.2 is insufficient for the understanding of the main results of the thesis. In this section we show that the algebra $\mathrm{YB}(\widehat{\mathfrak{gl}}(1))$ contains an infinite dimensional center. Namely for any singular vector $|s\rangle$ of W_n algebra acting in the space of n bosons we assign a central element D_s (1.3.21). First element of this series is related to the operator $\psi(u)$ (0.3.3) as

$$D_{1,1}(u) = \psi(u)\frac{h(u)h(u+\epsilon_3)}{h(u-\epsilon_1)h(u-\epsilon_2)}.$$

$$\psi(u) = \frac{\langle s_{1,1}|\mathcal{L}^1(u)\mathcal{L}^2(u+\epsilon_3)|s_{1,1}\rangle}{h(u)h(u+\epsilon_3)},$$

[4] For the case of KDV and ILW integrable systems this symmetry is broken by a particular choice of Fock representation.

where

$$|s_{1,1}\rangle_u \stackrel{\text{def}}{=} \big(a^{(1)}_{-1} - a^{(2)}_{-1}\big)|\varnothing\rangle_u \otimes |\varnothing\rangle_{u+\epsilon_3}$$

is a singular vector of a W algebra which appears in the tensor product of two Fock spaces $\mathcal{F}_{u_1} \otimes \mathcal{F}_{u_2}$ at special value of spectral parameters $u_2 = u_1 + \epsilon_3$.
In general, for the singular vector $|s\rangle$ of W_n algebra acting in the space of n Fock spaces $\mathcal{F}_1(u_1)\dots\mathcal{F}_n(u_n)$[5] we may define a Cartan current acting on a quantum space as

$$h_s = \langle s|\mathcal{L}^1(u-u_1)\dots\mathcal{L}^n(u-u_n)|s\rangle.$$

And the operator:

$$D_s = \frac{h_s(u)}{\prod\limits_{i=1}^{n} h(u-v_i)} \tag{0.3.5}$$

is central.

Zero twist integrable system. In section 1.3.3 we considered the integrable system with zero twist $q = 0$. In this case twist deformed transfer matrix $\mathbf{T}_q$ turns to the $h(u)$ current introduced in previous section. The spectrum and eigenbasis of $h(u)$ is very simple and may be written explicitly. For example for a representation in the tensor product of n Fock spaces: $\mathcal{F}_{x_1}\otimes\dots\otimes\mathcal{F}_{x_n}$ the eigenbasis is enumerated by the collection of n Young diagrams $\vec{\boldsymbol{\lambda}} = \{\lambda^{(1)},\dots\lambda^{(n)}\}$ and known as a basis of generalised Jack polynomials. The eigenvalues may be conveniently written in terms of contents of the Young diagrams

$$h(u)|\vec{\boldsymbol{\lambda}}\rangle = \prod_{\square\in\vec{\boldsymbol{\lambda}}} \frac{(u-c_\square)}{(u-c_\square-\epsilon_3)}|\vec{\boldsymbol{\lambda}}\rangle.$$

For a cell $\square = (i,j)$ the content $c_\square$ is defined as

$$c_\square = x_k - (i-1)\epsilon_1 - (j-1)\epsilon_2.$$

We proof that explicit formulas for the action of e, f generators in the eigenbasis of h are given by the formulas (1.3.29):

$$\begin{aligned} e(u)|\vec{\boldsymbol{\lambda}}\rangle &= \sum_{\square\in\text{addable}(\vec{\boldsymbol{\lambda}})} \frac{E(\vec{\boldsymbol{\lambda}},\vec{\boldsymbol{\lambda}}+\square)}{u-c_\square}|\vec{\boldsymbol{\lambda}}+\square\rangle, \\ f(u)|\vec{\boldsymbol{\lambda}}\rangle &= \sum_{\square\in\text{removable}(\vec{\boldsymbol{\lambda}})} \frac{F(\vec{\boldsymbol{\lambda}},\vec{\boldsymbol{\lambda}}-\square)}{u-c_\square}|\vec{\boldsymbol{\lambda}}-\square\rangle, \end{aligned} \tag{0.3.6}$$

where the amplitudes $E(\vec{\boldsymbol{\lambda}},\vec{\boldsymbol{\lambda}}+\square)$ and $F(\vec{\boldsymbol{\lambda}},\vec{\boldsymbol{\lambda}}-\square)$ are given by the formulas

$$\begin{aligned} E(\vec{\boldsymbol{\lambda}},\vec{\boldsymbol{\lambda}}+\square) &= \frac{\epsilon_1\epsilon_2}{\epsilon_3}\prod_{\square'\in\vec{\boldsymbol{\lambda}}+\square} S^{-1}(c_{\square'}-c_\square)\prod_{k=1}^{n}\frac{(c_\square - x_k+\epsilon_3)}{(c_\square-x_k)}, \\ F(\vec{\boldsymbol{\lambda}},\vec{\boldsymbol{\lambda}}-\square) &= \prod_{\square'\in\vec{\boldsymbol{\lambda}}-\square} S(c_\square - c_{\square'}), \end{aligned} \tag{0.3.7}$$

with

$$S(x) = \frac{(x+\epsilon_1)(x+\epsilon_2)}{x(x-\epsilon_3)}.$$

This formulas plays the crucial role in definition of Bethe vector, study of its matrix elements.

[5]Note that a singular vector may exist only if evaluation parameters u_i are not arbitrary, but they are restricted by some resonance conditions.

Transfer matrix and ILW Integrals of Motion. At the beginning of section 1.4 we recall that the transfer matrix defined by

$$\mathbf{T}_q(u) = \mathrm{Tr}\Big(q^{L_0^{(0)}}\mathcal{R}_{0,1}(u-u_1)\mathcal{R}_{0,2}(u-u_2)\dots\mathcal{R}_{0,n-1}(u-u_{n-1})\mathcal{R}_{0,n}(u-u_n)\Big)\Big|_{\mathcal{F}_u},$$

admits the following large u expansion

$$\mathbf{T}_q(u) = \Lambda(u,q)\exp\left(\frac{1}{u}\mathbf{I}_1(q) + \frac{1}{u^2}\mathbf{I}_2(q) + \dots\right),$$

where $\Lambda(u,q)$ is a normalization factor and $\mathbf{I}_1$ and $\mathbf{I}_2$ are the first ILW$_n$ Integrals of Motion.

$$\begin{aligned}
\mathbf{I}_1(q) &= \frac{iQ}{2\pi}\int\left[\frac{1}{2}\sum_{k=1}^{n}(\partial\varphi_k)^2\right]dx,\\
\mathbf{I}_2(q) &= \frac{iQ}{2\pi}\int\left[\frac{1}{3}\sum_{k=1}^{n}(\partial\varphi_k)^3 + Q\left(\frac{i}{2}\sum_{i,j}\partial\varphi_i D\partial\varphi_j + \sum_{i<j}\partial\varphi_i\partial^2\varphi_j\right)\right]dx,\\
\mathbf{I}_3(q) &= \frac{iQ}{2\pi}\int\left[\frac{1}{4}\sum_{k=1}^{n}(\partial\varphi_k)^4 + \dots\right]dx,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

where ($Q = -\frac{\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}}$), and D is the non-local operator whose Fourier image is

$$D(k) = k\frac{1+q^k}{1-q^k}.$$

Now let us define KZ Integral of Motion as $T_q(u)$ operator at the special value of the parameter $u = u_1$. Using the fact that $\mathcal{R}_{0,1}(0) = \mathcal{P}_{0,1}$ is a permutation operator, one finds for the KZ IOM:

$$\mathcal{I}_1^{\mathrm{KZ}}(q) \stackrel{def}{=} T_q(u_1) = q^{L_0^{(1)}}\mathcal{R}_{1,2}(u_1-u_2)\mathcal{R}_{1,3}(u_1-u_3)\dots\mathcal{R}_{1,n}(u_1-u_n).$$

The rest of this section is aimed to show that the simultaneous spectrum of KZ and first few local Integrals of Motion is governed by Bethe ansatz equations (0.2.1).

Special vector $|\chi\rangle$, definition of Bethe vector. In section 1.4.1 we define the Bethe vector $B(\boldsymbol{x})$ which turns to the eigenvector of corresponding integrable system after imposing the Bethe equations. In order to define Bethe vector we introduce the tensor product of $n+N$ Fock spaces, with n "quantum" and N "auxiliary" spaces

$$\underbrace{\mathcal{F}_{u_1}\otimes\dots\otimes\mathcal{F}_{u_n}}_{\text{quantum space}}\otimes\underbrace{\mathcal{F}_{x_1}\otimes\dots\otimes\mathcal{F}_{x_N}}_{\text{auxiliary space}}$$

generated from the vacuum state

$$|\varnothing\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}} = |x_1\rangle\otimes\dots\otimes|x_N\rangle\otimes|u_1\rangle\otimes\dots\otimes|u_n\rangle.$$

We then searched for the Bethe vector in the form[6]:

$$|B(\boldsymbol{x})\rangle_{\boldsymbol{u}} \stackrel{\text{def}}{=} {}_{\boldsymbol{x}}\langle\varnothing|\mathcal{R}(\boldsymbol{x},\boldsymbol{u})|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}} \quad \text{where} \quad \mathcal{R}(\boldsymbol{x},\boldsymbol{u}) = \mathcal{R}_{x_1u_1}\dots\mathcal{R}_{x_Nu_1}\dots\mathcal{R}_{x_1u_n}\dots\mathcal{R}_{x_Nu_n},$$

[6] This definition is similar to the very general approach investigated in [AO17] (in particular this construction is explained in section 1.3.3 of [AO17]).

here $|\chi\rangle_{\boldsymbol{x}}$ is some vector in auxiliary space. The convenient choice for the vector $|\chi\rangle$ is to choose it equal to an eigenvector of zero twist integral of motion $h(u)$ acting on auxiliary Fock space. Among the various eigenvectors the simplest one is (see (1.4.4) for details)

$$|\chi\rangle_{\boldsymbol{x}} \stackrel{\text{def}}{=} |\underbrace{\square,\dots,\square}_{N}\rangle \sim \oint\limits_{x_N} dz_N \cdots \oint\limits_{x_1} dz_1\, e(z_N)\dots e(z_1)|\varnothing\rangle_{\boldsymbol{x}}.$$

Alternatively vector $|\chi\rangle_{\boldsymbol{x}}$ is fixed (up to proportionality factor) as an eigenvector of zero twist integrable system with concrete eigenvalue

$$h(u)|\chi\rangle_{\boldsymbol{x}} = \prod_{k=1}^{N} \frac{u-x_k}{u-x_k-\epsilon_3}|\chi\rangle_{\boldsymbol{x}}.$$

Explicit computation of Bethe vector and its properties. Here we continue to describe the results of section 1.4.1. A direct consequence of (0.3.6),(0.3.7) implies a convenient formula :

$${}_{\boldsymbol{x}}\langle\varnothing|f(z_N)\dots f(z_1)|\chi\rangle_{\boldsymbol{x}} = \mathrm{Sym}_{\boldsymbol{x}}\left(\prod_{a=1}^{N}\frac{1}{z_a-x_a}\prod_{a<b}S(x_a-x_b)\right),$$

where $\mathrm{Sym}_{\boldsymbol{x}}$ means the symmetrization over the x_i variables. Together with the formula (0.3.4) for an $\mathcal{L}$-operators in terms of f and h currents it allows to explicitly compute the matrix elements of Bethe vector - the so called weight functions:

$$\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) \stackrel{\text{def}}{=} {}_{\boldsymbol{u}}\langle\varnothing|a^{(1)}_{\boldsymbol{\lambda}^{(1)}}\dots a^{(n)}_{\boldsymbol{\lambda}^{(n)}}|B(\boldsymbol{x})\rangle_{\boldsymbol{u}} = {}_{\boldsymbol{x}}\langle\varnothing|\mathcal{L}_{\boldsymbol{\lambda}^{(1)},\varnothing}(u_1)\dots\mathcal{L}_{\boldsymbol{\lambda}^{(n)},\varnothing}(u_n)|\chi\rangle_{\boldsymbol{x}}.$$

After the straightforward computation we get

$$\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) = \frac{1}{(2\pi i)^N}\oint\cdots\oint \Omega_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\ \mathrm{Sym}_{\boldsymbol{x}}\left(\prod_{a=1}^{N}\frac{1}{z_a-x_a}\prod_{a<b}S(x_a-x_b)\right)d\vec{\boldsymbol{z}},$$

where function

$$\Omega_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u}) = F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\left(\prod_{j=1}^{|\boldsymbol{\lambda}^{(1)}|}\frac{u_2-z_j^{(1)}}{u_2-z_j^{(1)}-\epsilon_3}\right)\left(\prod_{j=1}^{|\boldsymbol{\lambda}^{(2)}|}\frac{u_3-z_j^{(2)}}{u_3-z_j^{(2)}-\epsilon_3}\prod_{j=1}^{|\boldsymbol{\lambda}^{(1)}|}\frac{u_3-z_j^{(1)}}{u_3-z_j^{(1)}-\epsilon_3}\right)\dots$$
$$\dots\left(\prod_{j=1}^{|\boldsymbol{\lambda}^{(n-1)}|}\frac{u_n-z_j^{(n-1)}}{u_n-z_j^{(n-1)}-\epsilon_3}\prod_{j=1}^{|\boldsymbol{\lambda}^{(n-2)}|}\frac{u_n-z_j^{(n-2)}}{u_n-z_j^{(n-2)}-\epsilon_3}\cdots\prod_{j=1}^{|\boldsymbol{\lambda}^{(1)}|}\frac{u_n-z_j^{(1)}}{u_n-z_j^{(1)}-\epsilon_3}\right)$$

The integral shrinks to the points $\boldsymbol{x}$ and one obtains explicit formula (see (1.4.14) for details)

$$\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) = \mathrm{Sym}_{\boldsymbol{x}}\left(\Omega_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{x}}|\boldsymbol{u})\prod_{a<b}S(x_a-x_b)\right).$$

The simplicity of this formula explains our choice of vector $|\chi\rangle$.

Diagonalization of local and KZ Integrals of Motion. Using the computation methods described above, in sections 1.4.2,1.4.4 we were able to compute the action of local and KZ Integrals of Motion on a Bethe vector. Namely we were able to prove that upon the Bethe equations:

$$q\prod_{j\neq i}\prod_{\alpha=1}^{3}\frac{x_i-x_j-\epsilon_\alpha}{x_i-x_j+\epsilon_\alpha}\prod_{k=1}^{n}\frac{x_i-u_k+\epsilon_3}{x_i-u_k} = 1 \quad \text{for all} \quad i=1,\dots,N,$$

the Bethe vector becomes an eigenvector of KZ integral of motion $\mathcal{I}_1^{\mathrm{KZ}} = q^{L_0^{(1)}}\mathcal{R}_{1,2}\mathcal{R}_{1,3}\dots\mathcal{R}_{1,n-1}\mathcal{R}_{1,n}$, with eigenvalue:

$$t_q^1(\boldsymbol{u}) = \prod_{k=1}^{N}\frac{x_k - u_1}{x_k - u_1 + \epsilon_3}.$$

And also becomes an eigenvector of Local integral of motion $\mathbf{I}_2$:

$$-\epsilon_3\int\left[\frac{1}{3}\sqrt{\epsilon_1\epsilon_2}\sum_i(\partial\phi_i)^3 - \epsilon_3\left(\frac{1}{2}\sum_{i,j}\partial\phi_i D(q)\partial\phi_j + \sum_{i<j}\partial\phi_i\partial^2\phi_j\right)\right]\frac{dx}{2\pi} - \frac{\epsilon_3\mathbf{I}_1(q)}{2} - \frac{\epsilon_3}{3}\sqrt{\epsilon_1\epsilon_2}\sum_i u_i^3,$$

with eigenvalue $\left(\sum_1^N x_k\right)$. We were also able to write explicitly the solution (1.4.19) of a difference Knizhnik-Zamolodchikov (KZ) (1.4.20) and Okounkov-Pandharipande (OP) (1.4.32) equation in terms of Bethe vector. This finishes the review of the first chapter.

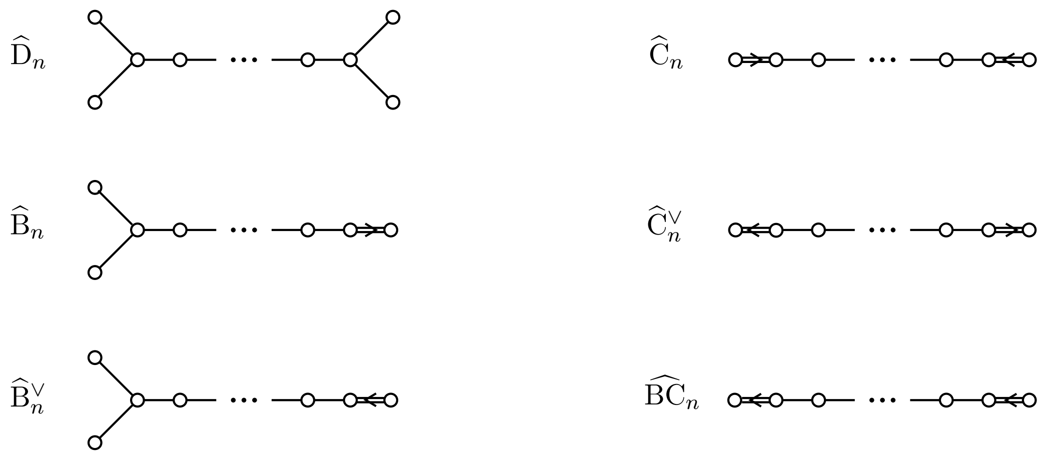
Integrable structure of $\mathrm{B}, \mathrm{C}, \mathrm{D}$ **conformal field theory.** The second chapter is devoted to the study of integrable structure of $\mathrm{B}, \mathrm{C}, \mathrm{D}$ conformal field theory and its relation to boundary Bethe ansatz of affine Yangian.

In section 2.2 we introduce the affine Toda QFT associated to an affine Lie algebra $\mathfrak{g}$ of BCD type. We recall that Integrals of Motion can be found as a commutant of the affine set of screenings:

$$\mathcal{S}_r = \oint e^{b(\boldsymbol{\alpha_r}\cdot\boldsymbol{\varphi(z)})}\frac{dz}{2\pi}, \tag{0.3.8}$$

$$[\mathbf{I}_s, \mathcal{S}_r] = 0,$$

where vectors $\boldsymbol{\alpha_r}$ have the Gram matrix of BCD type affine Lie algebra and $b = \frac{\epsilon_2}{\sqrt{\epsilon_1\epsilon_2}}$ is the coupling constant.



Using the standard parametrization for the roots one can express the scalar products in the exponents in (0.3.8) as

$$(\boldsymbol{\alpha}_0\cdot\boldsymbol{\varphi}) = \begin{cases}-\varphi_1\\ -2\varphi_1\\ -\varphi_1-\varphi_2\end{cases} \qquad (\boldsymbol{\alpha}_r\cdot\boldsymbol{\varphi}) = \varphi_r - \varphi_{r+1} \quad \text{for} \quad 0<r<n, \quad (\boldsymbol{\alpha}_n\cdot\boldsymbol{\varphi}) = \begin{cases}\varphi_n\\ 2\varphi_n\\ \varphi_{n-1}+\varphi_n\end{cases}$$

That is each of the affine diagrams can be interpreted as non-affine A_{n-1} diagram with two boundary conditions which can be of three types B, C or D corresponding to the short root, the long root or the root of the length $\sqrt{2}$ correspondingly.

As in the first chapter, we will search local Integrals of Motion in terms of integrals of local densities $\mathbf{I}_s = \int\limits_0^{2\pi} G_{s+1}(z)\frac{dz}{2\pi}$. First few local Integrals of Motion can be computed explicitly by solving the equation

$$\frac{1}{2\pi i}\oint_z e^{b(\boldsymbol{\alpha}_r\cdot\boldsymbol{\varphi}(\xi))}G_{s+1}(z)d\xi = \partial X_s(z),$$

where $X_s(z)$ is some local field. The first non trivial density has the form

$$\begin{aligned}
G_4(z) = \left(\partial\boldsymbol{\varphi}\cdot\partial\boldsymbol{\varphi}\right)^2 - \frac{1}{3}\left(2n - \frac{\epsilon_\alpha+\epsilon_\beta}{\epsilon_3}\right)\sum_{k=1}^{n}\left(\partial\varphi_k\right)^4 + \\
+\frac{4\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}}\sum_{k=1}^{n}\partial\varphi_k^2\left(\sum_{j<k}\left(j-1+\frac{\epsilon_3-\epsilon_\alpha}{2\epsilon_3}\right)\partial^2\varphi_j - \sum_{j>k}\left(n-j+\frac{\epsilon_3-\epsilon_\beta}{2\epsilon_3}\right)\partial^2\varphi_j\right) + \\
+\left(2n+\frac{4(n-1)(\epsilon_1^2+\epsilon_2^2)}{3\epsilon_1\epsilon_2}+\frac{(\epsilon_1\epsilon_2-2\epsilon_3^2)(\epsilon_\alpha+\epsilon_\beta-2\epsilon_3)}{3\epsilon_1\epsilon_2\epsilon_3}\right)\left(\partial^2\boldsymbol{\varphi}\cdot\partial^2\boldsymbol{\varphi}\right) - \\
-\frac{4\epsilon_3^2}{\epsilon_1\epsilon_2}\sum_{i\leq j}\left(i-1+\frac{\epsilon_3-\epsilon_\alpha}{2\epsilon_3}\right)\left(n-j+\frac{\epsilon_3-\epsilon_\beta}{2\epsilon_3}\right)(2-\delta_{ij})\partial^2\varphi_i\partial^2\varphi_j, \quad (0.3.9)
\end{aligned}$$

here $\alpha,\beta=\{1,2,3\}$ for the B,C or D type of endings correspondingly.

Sklyanin's $K-$matrix of affine Yangian. The crucial step in understanding the relation of this integrable structure to the boundary affine Yangian is to introduce reflection $K-$matrix. This is done in section 2.3. The idea is to consider reflection operator K as an intertwining operator of W algebra, analogically to how it was done for the $R-$matrix (0.3.1).

Let us introduce two currents of W_4 algebra acting in the space of two bosonic Fock modules $\mathcal{F}_{u_1}\otimes\mathcal{F}_{u_2}$:

$$W^{(2)} = (\partial\varphi_1)^2+(\partial\varphi_2)^2+\frac{2\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}}\partial^2\varphi_1+\frac{\epsilon_3-\epsilon_\alpha}{\sqrt{\epsilon_1\epsilon_2}}(\partial^2\varphi_2+\partial^2\varphi_1)$$

and

$$\begin{aligned}
W^{(4)} = (\partial\varphi_1)^2(\partial\varphi_2)^2+\frac{2\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}}\partial\varphi_1\partial\varphi_2\partial^2\varphi_2+\frac{\epsilon_3-\epsilon_\alpha}{\sqrt{\epsilon_1\epsilon_2}}\left((\partial\varphi_1)^2\partial^2\varphi_2+(\partial\varphi_2)^2\partial^2\varphi_1\right) - \\
-\frac{\epsilon_3\epsilon_\alpha}{\epsilon_1\epsilon_2}(\partial^2\varphi_1)^2+\frac{(\epsilon_3-\epsilon_\alpha)^2}{\epsilon_1\epsilon_2}\partial^2\varphi_1\partial^2\varphi_2-\frac{(\epsilon_1-\epsilon_\alpha)(\epsilon_2-\epsilon_\alpha)}{2\epsilon_1\epsilon_2}\left(\partial\varphi_1\partial^3\varphi_1+\partial\varphi_2\partial^3\varphi_2\right) - \\
-\frac{\epsilon_3(\epsilon_3-\epsilon_\alpha)}{\epsilon_1\epsilon_2}\left(\partial\varphi_1\partial^3\varphi_1-\partial\varphi_1\partial^3\varphi_2\right)+\frac{\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}}\left(\frac{\epsilon_\alpha(\epsilon_3-\epsilon_\alpha)}{2\epsilon_1\epsilon_2}-\frac{\epsilon_3^2}{\epsilon_1\epsilon_2}-\frac{1}{3}\right)\partial^4\varphi_1
\end{aligned}$$

where $\alpha=1,2,3$ correspond to the W algebras of types B, C or D correspondingly.

By definition the R and K operators are defined by the following intertwining relations:

$$\mathcal{R}_{1,2}W^{(s)} = W^{(s)}\Big|_{\varphi_1\leftrightarrow\varphi_2}\mathcal{R}_{1,2}, \qquad \mathcal{K}_2W^{(s)} = W^{(s)}\Big|_{\varphi_2\to-\varphi_2}\mathcal{K}_2, \qquad (0.3.10)$$

for $s=2,4$. The $R_{1,2}$ operator is identified with the Maulik-Okounkov $R-$matrix defined earlier (0.3.1) $\mathcal{R}_{1,2}=\mathcal{R}[\partial\varphi_1-\partial\varphi_2]$, while $\mathcal{K}_2$ is also equal to the MO $R-$matrix of the re-scaled argument

$$\begin{aligned}
\mathcal{K}_2^1 &= \mathcal{R}[\sqrt{2}\partial\varphi_2]\Big|_{\epsilon_1\to\sqrt{2}\epsilon_1,\epsilon_2\to\epsilon_2/\sqrt{2}} && \text{for B series}\\
\mathcal{K}_2^2 &= \mathcal{R}[\sqrt{2}\partial\varphi_2]\Big|_{\epsilon_1\to\epsilon_1/\sqrt{2},\epsilon_2\to\sqrt{2}\epsilon_2} && \text{for C series}\\
\mathcal{K}_2^3 &= \mathrm{Id} && \text{for D series}
\end{aligned}$$

Note that the simplest K operator is very explicit $\mathcal{K}_2^3 = \mathrm{Id}$ and it does not depend on spectral parameter.

Now, similar to the argument of Maulik and Okounkov, the $K-$operator obeys Sklyanin's KRKR equation[7]

$$\mathcal{R}[\partial\varphi_1 - \partial\varphi_2]\mathcal{K}_1^\alpha \mathcal{R}[\partial\varphi_1 + \partial\varphi_2]\mathcal{K}_2^\alpha = \mathcal{K}_2^\alpha \mathcal{R}[\partial\varphi_1 + \partial\varphi_2]\mathcal{K}_1^\alpha \mathcal{R}[\partial\varphi_1 - \partial\varphi_2]. \tag{0.3.11}$$

KZ Integrals of Motion. In section 2.3.1 we have defined KZ Integrals of Motion:

$$\begin{aligned} \mathcal{T}_i^+ &= \mathcal{R}_{i,\overline{i+1}} \dots \mathcal{R}_{i,\bar{n}} \mathcal{K}_i^\alpha \mathcal{R}_{i,n} \dots \mathcal{R}_{i,i+1}, \\ \mathcal{T}_i^- &= \mathcal{R}_{i,1} \dots \mathcal{R}_{i,i-1} \mathcal{K}_i^\beta \mathcal{R}_{1,\bar{i}} \dots \mathcal{R}_{i-1,\bar{i}}, \\ \mathcal{I}_i^{\mathrm{KZ}} &= \mathcal{T}_i^- \mathcal{T}_i^+ \end{aligned} \tag{0.3.12}$$

where the barred index $\bar{i}$ means the conjugation by the operator of sign reflection D_i

$$\begin{aligned} D_i f(\boldsymbol{\varphi}) &= f(\boldsymbol{\varphi})\Big|_{\varphi_i \to -\varphi_i} D_i, \\ \mathcal{R}_{i,\bar{j}} &= D_j \mathcal{R}_{i,j} D_j = \mathcal{R}[\partial\varphi_i + \partial\varphi_j], \\ \mathcal{R}_{\bar{i},j} &= D_i \mathcal{R}_{i,j} D_i = \mathcal{R}[-\partial\varphi_i - \partial\varphi_j], \end{aligned}$$

Their mutual commutativity is provided by KRKR equation (0.3.11)

$$[\mathcal{I}_i^{\mathrm{KZ}}, \mathcal{I}_j^{\mathrm{KZ}}] = 0.$$

We also proved a commutativity between KZ and local Integrals of Motion $[\mathbf{I}_s, \mathcal{I}_i^{\mathrm{KZ}}] = 0$ which follows from the intertwining relations (0.3.10) (see (2.3.9) for details)

$$\mathcal{T}_i^+ \mathbf{I}_s = \mathbf{I}_s\Big|_{\varphi_i \to -\varphi_i} \mathcal{T}_i^+, \qquad \mathcal{T}_i^- \ \mathbf{I}_s\Big|_{\varphi_i \to -\varphi_i} = \mathbf{I}_s \ \mathcal{T}_i^-.$$

Of-shell Bethe vector. The section 2.4 goes in parallel to the section 1.4.1 where we considered the type A integrable structures. We introduce a product of $n+N$ Fock spaces where the first n products is a quantum $\mathcal{F}_{\boldsymbol{u}}$ Fock space and the second N products is an auxiliary Fock space $\mathcal{F}_{\boldsymbol{x}}$

$$\underbrace{\mathcal{F}_{u_n} \otimes \dots \otimes \mathcal{F}_{u_1}}_{\text{quantum space}} \otimes \underbrace{\mathcal{F}_{x_1} \otimes \dots \otimes \mathcal{F}_{x_N}}_{\text{auxiliary space}} = \mathcal{F}_{\boldsymbol{u}} \otimes F_{\boldsymbol{x}}.$$

We then define two types of $L-$ operators (2.4.1),(2.4.2), and $\mathcal{K}_{\boldsymbol{u}|\boldsymbol{x}}$ operator fixed by the recurrent relations (2.4.4).

Finally we define an of-shell Bethe vector by the formula (see (2.4.5) for details)

$$|B(\boldsymbol{x})\rangle = {}_x\langle\varnothing|\bar{\mathcal{L}}_{\boldsymbol{v}} \mathcal{K}_{\boldsymbol{x}} L_{\boldsymbol{v}}|\varnothing\rangle_v |\chi\rangle_x = {}_x\langle\varnothing|\mathcal{K}_{\boldsymbol{v}|\boldsymbol{x}}|\varnothing\rangle_v|\chi\rangle_x, \tag{0.3.13}$$

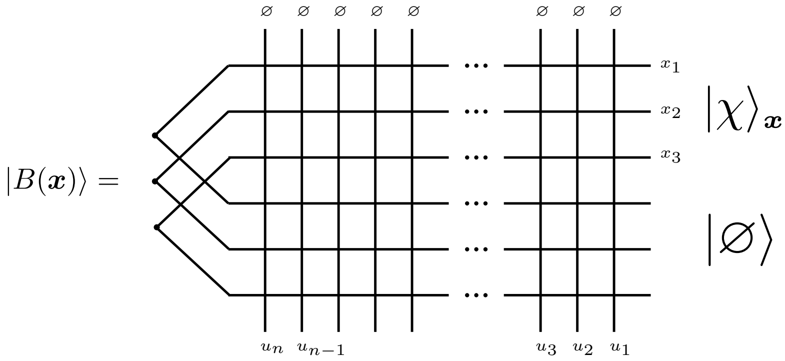
where $|\chi\rangle_{\boldsymbol{x}}$ is the same vector as in the first chapter (1.4.4). The definition of Bethe vector may be

[7] Let us note that originally [Skl88] the KRKR equation was written in a quite different form:

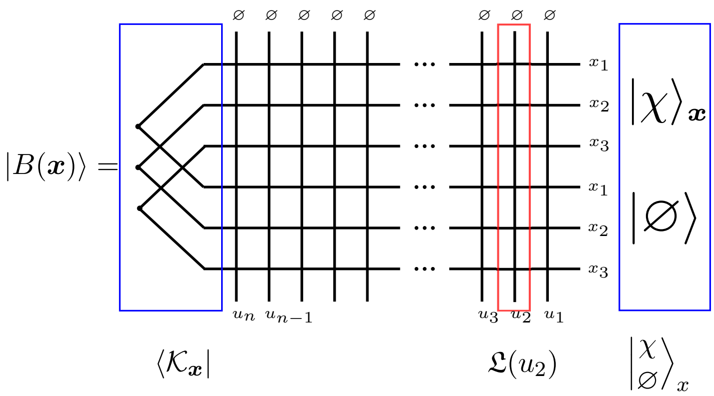
$$\mathcal{R}_{1,2}(u_1 - u_2)\tilde{\mathcal{K}}_1(u_1)\mathcal{R}_{2,1}(u_2 + u_1)\tilde{\mathcal{K}}_2(u_2) = \tilde{\mathcal{K}}_2(u_2)\mathcal{R}_{1,2}(u_1 + u_2)\tilde{\mathcal{K}}_1(u_1)\mathcal{R}_{2,1}(u_1 - u_2).$$

The difference is actually insufficient as the two equations are differ by the redefinition of $K-$operator and overall conjugation by the reflection of bosonic modes $a_n^{1,2} \to -a_n^{1,2}$, $n \neq 0$

illustrated by a picture:



In the beginning of section (2.5) we suggest to interpret the Bethe vector $|B(\boldsymbol{x})\rangle$ as a product of some $L-$operators $\mathfrak{L}(u_n)\ldots\mathfrak{L}(u_1)$ sandwiched between bra and ket states $\langle\mathcal{K}_{\boldsymbol{x}}|$ and $\left|\begin{matrix}\chi\\ \varnothing\end{matrix}\right\rangle_x$, see the picture below. This bra and ket vectors then should live in the tensor product of the Fock space and its dual $\mathcal{F}_{\boldsymbol{x}}\otimes\mathcal{F}^\star_{\boldsymbol{x}}$.



Strange module. In section 2.5 we observe that modified operators $\mathfrak{L}$ obeys the same RLL commutation relations:

$$\mathcal{R}_{ij}(u-v)\mathfrak{L}_i(u)\mathfrak{L}_j(v)=\mathfrak{L}_j(v)\mathfrak{L}_i(u)\mathcal{R}_{ij}(u-v).$$

And we still can define $\mathfrak{h}\overset{def}{=}\mathfrak{L}_{\varnothing,\varnothing},\mathfrak{e}\overset{def}{=}\mathfrak{h}^{-1}\mathfrak{L}_{\varnothing,\square},\mathfrak{f}\overset{def}{=}\mathfrak{L}_{\square,\varnothing}\mathfrak{h}^{-1}$ operators.

The difference is that $\mathfrak{L}-$operators act in the tensor product of Fock module and its dual $\mathcal{F}_{\boldsymbol{x}}\otimes\mathcal{F}^\star_{\boldsymbol{x}}$. This representation for the $\mathfrak{L}-$operator doesn't have a highest weight, however the action of $\mathfrak{h}(z)$ still can be diagonalized, the eigenvectors of $\mathfrak{h}(u),\psi(u)$ in $\mathcal{F}_{\boldsymbol{x}}\otimes\mathcal{F}^\star_{\boldsymbol{x}}$ are enumerated by the collection of $2N$ Young diagrams and denoted by $\left|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\right\rangle$. The eigenvalues are given by the formulas:

$$\mathfrak{h}(u)\left|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\right\rangle=\prod_{\square\in\vec{\boldsymbol{\lambda}}}\frac{(u-c_\square)}{(u-c_\square-\epsilon_3)}\prod_{\square\in\vec{\boldsymbol{\mu}}}\frac{(u-c_\square-\epsilon_3)}{(u-c_\square)}\left|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\right\rangle,$$

where

$$c_\square=x_k-(i-1)\epsilon_1-(j-1)\epsilon_2,\quad\text{for }\square=(i,j)\in\vec{\boldsymbol{\lambda}},$$
$$c_\square=-\epsilon_3-x_k+(i-1)\epsilon_1+(j-1)\epsilon_2,\quad\text{for }\square=(i,j)\in\vec{\boldsymbol{\mu}}.$$

One can also find the action of $\mathfrak{e},\mathfrak{f}$ currents:

$$\mathfrak{e}(u)\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\Big\rangle = \sum_{\square\in\text{addable}(\vec{\boldsymbol{\lambda}})} \frac{E\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}+\square\\ \vec{\boldsymbol{\mu}} & \to & \vec{\boldsymbol{\mu}}\end{pmatrix}}{u-c_\square}\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}+\square\\ \vec{\boldsymbol{\mu}}\end{matrix}\Big\rangle + \sum_{\square\in\text{removable}(\vec{\boldsymbol{\mu}})} \frac{E\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}} & \to & \vec{\boldsymbol{\mu}}-\square\end{pmatrix}}{u-c_\square}\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}-\square\end{matrix}\Big\rangle,$$

$$\mathfrak{f}(u)\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\Big\rangle = \sum_{\square\in\text{removable}(\vec{\boldsymbol{\lambda}})} \frac{F\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}-\square\\ \vec{\boldsymbol{\mu}} & \to & \vec{\boldsymbol{\mu}}\end{pmatrix}}{u-c_\square}\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}-\square\\ \vec{\boldsymbol{\mu}}\end{matrix}\Big\rangle + \sum_{\square\in\text{addable}(\vec{\boldsymbol{\mu}})} \frac{F\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}} & \to & \vec{\boldsymbol{\mu}}+\square\end{pmatrix}}{u-c_\square}\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}+\square\end{matrix}\Big\rangle.$$

The E,F coefficients are given in (2.5.3),(2.5.4). Note that now operators $\mathfrak{e},\mathfrak{f}$ not only add or remove boxes, but do both.

Reflection property of the $\langle\mathcal{K}|$ state. The final ingredient which allows to calculate the matrix elements of Bethe vector the so called of-shell Bethe functions is the formula which describe the action of the operator $\mathfrak{f}$ on the state $\langle\mathcal{K}|$. In section 2.5.2 we derive the following reflection properties (2.5.11):

$$\langle\mathcal{K}|\mathfrak{h}(u) = \langle\mathcal{K}|\mathfrak{h}(-u)$$
$$\langle\mathcal{K}|\mathfrak{f}(u) = r(u)\langle\mathcal{K}|\mathfrak{f}(-\epsilon_3-u),$$

with

$$r(u-\epsilon_3/2) = -\frac{u+\epsilon_3/2}{u-\epsilon_3/2} \quad \text{for the D case,}$$
$$r(u-\epsilon_3/2) = -\frac{u+\epsilon_i/2}{u-\epsilon_i/2} \quad \text{for the BC case,}$$

where in the last line $i=1$ corresponds to the B case and $i=2$ corresponds to the C case.

This formula allows to compute the coupling between $\langle\mathcal{K}|$ and $\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\Big\rangle$ state (2.5.2),(2.5.2).

Diagonalization of KZ and local IOMs. In section 2.5.3 we derive the Bethe ansatz equation for the diagonalization of KZ Integrals of Motion:

$$\begin{gathered}\text{BAE}(\boldsymbol{x}) \overset{\text{def}}{=} r^\alpha(x_i)r^\beta(x_i)A(x_i)A^{-1}(-x_i)\prod_{j\neq i} G(x_i-x_j)G^{-1}(-x_i-x_j) = 1,\\ G(x) = \frac{(x-\epsilon_1)(x-\epsilon_2)(x-\epsilon_3)}{(x+\epsilon_1)(x+\epsilon_2)(x+\epsilon_3)}, \qquad A(x) = \prod_{k=1}^n \frac{x-u_k+\frac{\epsilon_3}{2}}{x-u_k-\frac{\epsilon_3}{2}}, \quad r^\alpha(x) = -\frac{x+\epsilon_\alpha/2}{x-\epsilon_\alpha/2}.\end{gathered} \tag{0.3.14}$$

We also prove that the on-shell Bethe vector with shifted x parameters $|B(\boldsymbol{x}-\frac{\epsilon_3}{2})\rangle$ are the eigenvectors of KZ IOMs $\mathcal{I}_i^{\text{KZ}}$ (0.3.12):

$$\mathcal{I}_i^{\text{KZ}}|B(\boldsymbol{x}-\frac{\epsilon_3}{2})\rangle \overset{\text{BAE}(\boldsymbol{x})=1}{=} \prod_a \frac{(u_i+\frac{\epsilon_3}{2})^2-x_a^2}{(u_i-\frac{\epsilon_3}{2})^2-x_a^2}|B(\boldsymbol{x}-\frac{\epsilon_3}{2})\rangle. \tag{0.3.15}$$

Equations (0.3.14) and (0.3.15) together with the explicit form of off-shell Bethe vector (0.3.13) are the main results of the second chapter.

In contrast to the A case we will not provide a proof for the diagonalization of local Integrals of Motion, however we conjectured and checked numerically the formula for eigenvalues of $\mathbf{I}_3 = \frac{1}{2\pi}\int G_4(x)dx$, the local density G_4 is given by (0.3.9). Namely, on level N one has an eigenvalue:

$$\mathbf{I}_3^{\text{vac}} + \left(4N - 4\sum_{k=1}^n \frac{u_k^2}{\epsilon_1\epsilon_2} + \frac{\epsilon_1^2+\epsilon_2^2}{3\epsilon_1\epsilon_2}\left(2n-\frac{\epsilon_\alpha+\epsilon_\beta}{\epsilon_3}\right)\right)N + \frac{4}{\epsilon_1\epsilon_2}\left(2n-\frac{\epsilon_\alpha+\epsilon_\beta}{\epsilon_3}\right)\sum_{k=1}^N x_k^2,$$

where $\mathbf{I}_3^{\text{vac}} =_{\boldsymbol{u}}\langle\varnothing|\mathbf{I}_3|\varnothing\rangle_{\boldsymbol{u}}$ - is the vacuum expectation value.

More general integrable systems. One may note that affine Yangian commutation relations (1.3.9) are symmetric with respect to permutations of all ϵ_α. Nevertheless Bethe Ansatz equations (0.3.14) are not symmetric in all ϵ_α, because of the source term $A(x) = \prod_{k=1}^{n} \frac{x-u_k+\frac{\epsilon_3}{2}}{x-u_k-\frac{\epsilon_3}{2}}$. In fact this symmetry is broken by a choice of a particular Fock representation, in order to restore the symmetry back one should introduce three types of Fock modules $\mathcal{F}^\alpha$ (see [FJMM13, BFM18, LS16]). The whole machinery then may be applied to associate an integrable system to the chain of colored Fock spaces with two colored boundaries $\beta_L \Big| \mathcal{F}_1^{\alpha_1} \otimes \mathcal{F}_2^{\alpha_2} \cdots \otimes \mathcal{F}_n^{\alpha_n} \Big| \beta_R \quad , \quad \alpha_i, \beta_{L,R} = 1,2,3$. The corresponding systems of screenings are summarised in picture (B.1). We present the details in Appendix B.1, here we just mention a particular interesting model given by: $1 \Big| \mathcal{F}_1^1 \otimes \mathcal{F}_2^3 \cdots \otimes \mathcal{F}_{2n-1}^1 \otimes \mathcal{F}_{2n}^3 \Big| 3$. This model provides a UV limit for the (dual of) $O(2n+1)$ sigma model considered in [LS18]. Similarly the model $3 \Big| \mathcal{F}_1^3 \otimes \mathcal{F}_2^1 \cdots \otimes \mathcal{F}_{2n+1}^3 \Big| 3$ provides the UV limit of $O(2n)$ sigma model.

q-deformation of local and KZ IOMs. In the last chapter we provide the q-deformation of objects considered in first two chapter. In section 3.2 we review the definition of the q-deformed W algebra as a commutant of the screenings. In section 3.3 we provide a construction of a commutant of affine set of screenings, it turns out that in a q-deformed case the commutant can be found explicitly. We provide explicit formulas for q-deformed Integrals of Motion of arbitrary high spin (3.4.23) for W algebras of BCD cases, and considered in details an example of affine Lie algebra of type D in section 3.4. We found that all W algebras of BCD case fits into the same scheme, which allows to introduce a new algebra $\mathcal{K}$ which unifies the W algebras of type BCD. The detailed study of algebra $\mathcal{K}$ is reported in paper [FJMV21], while in this thesis we restrict ourselves to a more elementary approach. Finally in section 3.5 we provide a construction for a q-deformed versions of R and K reflection operators, as well as q-deformed KZ IOMs.

The results of the thesis are based on three publications.

1. Alexey Litvinov; Ilya Vilkoviskiy. *Liouville reflection operator, affine Yangian and Bethe ansatz.* JHEP, 12:100, 2020.

2. Alexey Litvinov; Ilya Vilkoviskiy. *Integrable structure of* BCD *conformal field theory and boundary Bethe ansatz for affine Yangian.* JHEP, 141 (2021).

3. B. Feigin, M. Jimbo, E. Mukhin and I. Vilkoviskiy. *Deformation of W algebras via quantum toroidal algebra* Selecta Mathematica, 27(4):1-62, 2021

Bibliography

[AKOS96] Hidetoshi Awata, Harunobu Kubo, Satoru Odake, and Junichi Shiraishi. Quantum W_N-algebras and Macdonald polynomials. *Communications in mathematical physics*, 179(2):401–416, 1996.

[AO17] Mina Aganagic and Andrei Okounkov. Quasimap counts and Bethe eigenfunctions. *Moscow Math. J.*, 17(4):565–600, 2017.

[BFM18] M. Bershtein, B. Feigin, and G. Merzon. Plane partitions with a "pit": generating functions and representation theory. *Sel. Math. New Ser.*, 24:21, 2018.

[BLZ96] Vladimir V. Bazhanov, Sergei L. Lukyanov, and Alexander B. Zamolodchikov. Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz. *Comm. Math. Phys.*, 177:381–398, 1996.

[BLZ97] Vladimir V. Bazhanov, Sergei L. Lukyanov, and Alexander B. Zamolodchikov. Integrable structure of conformal field theory II. Q-operator and DDV equation. *Comm. Math. Phys.*, 190:247–278, 1997.

[BLZ99] Vladimir V. Bazhanov, Sergei L. Lukyanov, and Alexander B. Zamolodchikov. Integrable structure of conformal field theory III: The Yang-Baxter relation. *Comm. Math. Phys.*, 200:297–324, 1999.

[BLZ01] Vladimir V. Bazhanov, Sergei L. Lukyanov, and Alexander B. Zamolodchikov. Spectral determinants for Schrodinger equation and Q operators of conformal field theory. *J. Stat. Phys.*, 102:567–576, 2001.

[BLZ04] Vladimir V. Bazhanov, Sergei L. Lukyanov, and Alexander B. Zamolodchikov. Higher level eigenvalues of Q operators and Schroedinger equation. *Adv. Theor. Math. Phys.*, 7:711–725, 2004.

[DF93] Jin Tai Ding and Igor B. Frenkel. Isomorphism of two realizations of quantum affine algebra $U_q(\mathfrak{gl}(n))$. *Comm. Math. Phys.*, 156:277–300, 1993.

[DT99a] Patrick Dorey and Roberto Tateo. Anharmonic oscillators, the thermodynamic Bethe ansatz, and nonlinear integral equations. *J. Phys.*, A32:L419–L425, 1999.

[DT99b] Patrick Dorey and Roberto Tateo. On the relation between Stokes multipliers and the $T-Q$ systems of conformal field theory. *Nucl. Phys. B*, 563:573–602, 1999. [Erratum: Nucl.Phys.B 603, 581–581 (2001)].

[FF95] Boris Feigin and Edward Frenkel. Kac-Moody groups and integrability of soliton equations. *Inventiones mathematicae*, 120(1):379–408, 1995.

[FF96] Boris Feigin and Edward Frenkel. Integrals of motion and quantum groups. In *Integrable systems and quantum groups*, pages 349–418. Springer, 1996.

[FJM17] B Feigin, M Jimbo, and Eugene Mukhin. Integrals of motion from quantum toroidal algebras. *Journal of Physics A: Mathematical and Theoretical*, 50(46):464001, 2017.

[FJMM13] Boris Feigin, Michio Jimbo, Tetsuji Miwa, and Evgeny Mukhin. Representations of quantum toroidal gl_n. *Journal of Algebra*, 380:78–108, 2013.

[FJMV21] B Feigin, M Jimbo, E Mukhin, and I Vilkoviskiy. Deformations of W algebras via quantum toroidal algebras. *Selecta Mathematica*, 27(4):1–62, 2021.

[FL88] V. A. Fateev and Sergei L. Lukyanov. The models of two-dimensional conformal quantum field theory with $Z(n)$ symmetry. *Int. J. Mod. Phys.*, A3:507, 1988.

[FR97] Edward Frenkel and Nicolai Reshetikhin. Deformations of W-algebras associated to simple lie algebras. *arXiv preprint q-alg/9708006*, 1997.

[KL20] Gleb A. Kotousov and Sergei L. Lukyanov. Spectrum of the reflection operators in different integrable structures. *JHEP*, 02:029, 2020.

[KOJ06] Takeo KOJIMA. The integrals of motion for the deformed $W-$algebra $W_{q,t}(\hat{sl}_N)$. In *Proceedings of the Symposium on Representation Theory 2006* , pages 102–114. Publication Society of the Proceedings of the Symposium on Representation Theory, 2006.

[KS82] P.P. Kulish and E.K. Sklyanin. Quantum spectral transform method. Recent development. *Lect. Notes Phys.*, 151:61–119, 1982.

[LF91] S. L. Lukyanov and V. A. Fateev. Additional symmetries and exactly solvable models in two-dimensional conformal field theory. *Soviet Scientific Reviews, sec A, volume 15, part 3*, 1991.

[Lit19] A. V. Litvinov. Integrable $\mathfrak{gl}(n|n)$ Toda field theory and its sigma-model dual. *Pisma Zh. Eksp. Teor. Fiz.*, 110(11):723–726, 2019.

[LS16] Alexey Litvinov and Lev Spodyneiko. On W algebras commuting with a set of screenings. *JHEP*, 11:138, 2016.

[LS18] A. V. Litvinov and L. A. Spodyneiko. On dual description of the deformed $O(N)$ sigma model. *JHEP*, 11:139, 2018.

[Luk88] Sergei L. Lukyanov. Quantization of the Gel'fand–Dikii brackets. *Funct. Anal. Its Appl.*, 22:255–262, 1988.

[LV20] Alexey Litvinov and Ilya Vilkoviskiy. Liouville reflection operator, affine Yangian and Bethe ansatz. *JHEP*, 12:100, 2020.

[LV21] Alexey Litvinov and Ilya Vilkoviskiy. Integrable structure of BCD conformal field theory and boundary Bethe ansatz for affine Yangian. *JHEP*, 141, 2021.

[MO19] Davesh Maulik and Andrei Okounkov. Quantum Groups and Quantum Cohomology. *Astérisque*, 408, 2019.

[Nak01] Hiraku Nakajima. Quiver varieties and finite dimensional representations of quantum affine algebras. *J. Amer. Math. Soc.*, 14:145–238, 2001.

[Skl88] E.K. Sklyanin. Boundary conditions for integrable quantum systems. *J. Phys. A*, 21:2375–2389, 1988.

[Tsy14] A. Tsymbaliuk. *The affine Yangian of* $\mathfrak{gl}_1$ *and the infinitesimal Cherednik algebras.* Phd thesis, Department of Mathematics MIT, 2014.

[Tsy17] Alexander Tsymbaliuk. The affine Yangian of $\mathfrak{gl}_1$ revisited. *Adv. Math.*, 304:583–645, 2017.

[Var00] Michela Varagnolo. Quiver varieties and Yangians. *Lett. Math. Phys.*, 53(4):273–283, 2000.

[Zam89] A.B. Zamolodchikov. Integrable field theory from conformal field theory. *Adv. Stud. Pure Math.*, 19:641–674, 1989.