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Birational automorphisms of varieties

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Let X be a projective algebraic variety defined over a field k. The set of invertible algebraic self-maps $f\colon X \to X$ forms the group $\mathrm{Aut}(X)$ of regular automorphisms. Given a regular automorphism $f\colon X \to X$ one can consider its graph:

$$\Gamma_f = \{(x, f(x))|\ x \in X\} \subset X \times X.$$

It is a subvariety of $X \times X$ such that the projections $\mathrm{pr}_i\colon \Gamma_f \to X$ are isomorphisms for $i = 1$ and 2. More generally, we may consider the set of all birational self-maps $f\colon X \dashrightarrow X$. Such maps are determined by a subvariety Γ_f of $X \times X$ such that both projections $\mathrm{pr}_i\colon \Gamma_f \to X$ induce an isomorphism between a Zariski open subset of Γ_f onto a Zariski open subset of X. Then for any point $x \in X$ one can define its total image: it is the set $f(x) = \mathrm{pr}_2\left(\Gamma_f \cap {\mathrm{pr}_1}^{-1}(x)\right)$. A birational automorphism may have *indeterminacy locus*: it is an algebraic subvariety $\mathrm{Ind}(f) \subset X$ such that the total image of any point $p \in \mathrm{Ind}(f)$ has positive dimension. If X is normal then one has $\mathrm{codim}(\mathrm{Ind}(f)) \geqslant 2$. The union of all irreducible subvarieties $Z \subset X$ such that $\dim(f(Z)) < \dim(Z)$ is called the *exceptional locus* of f and we denote it by $\mathrm{Exc}(f) \subset X$. When X is smooth, then $\mathrm{Exc}(f)$ is of pure codimension 1. The composition of two birational automorphisms remains birational hence the set of all birational automorphisms $\mathrm{Bir}(X)$ also forms a group.

By construction $\mathrm{Aut}(X)$ is a subgroup in $\mathrm{Bir}(X)$. Moreover, by [Han87] both groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ have natural structures of k-schemes. Namely, any regular or birational automorphism f of X is determined by a subscheme Γ_f in $X \times X$ and $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ can be considered as subschemes in the Hilbert scheme $\mathrm{Hilb}(X \times X)$.

Note that the composition of maps in $\mathrm{Aut}(X)$ induces a natural structure of a group scheme. Denote by $\mathrm{Aut}(X)^0$ the connected component of the identity map $\mathrm{id}_X\colon X \to X$. It is a group scheme of finite type and $\mathrm{Aut}(X)$ fits into the following exact sequence of groups:

$$1 \to \mathrm{Aut}(X)^0 \to \mathrm{Aut}(X) \to \mathrm{Aut}(X)/\mathrm{Aut}(X)^0 \to 1.$$

The quotient group $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$ is a constant group scheme over k associated to at most countable abstract group with an action of the Galois group of k. Thus, the study of automorphisms of X can be reduced to the understanding of $\mathrm{Aut}(X)^0$ and of the quotient group $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$.

It is a fact that $\mathrm{Bir}(X)$ cannot be in general endowed with a structure of a group scheme, see [Han87, Remark 2.9]. Complexity of groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ highly depends on the geometry of the algebraic variety X defined over the field k. We will cover various problems on $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ in this thesis, and roughly explore the differences and similarities between these two groups.

If X is a smooth curve, then $\mathrm{Aut}(X) = \mathrm{Bir}(X)$ and one can understand the structure of this group very well. For simplicity let us here discuss the case where k is algebraically closed. First, if the genus of X is zero, then $X = \mathbb{P}^1$ and $\mathrm{Aut}(X) = \mathrm{PGL}_2(k)$. If the genus of X equals 1 then X is an elliptic curve and the group of points of $X(k)$ form a subgroup of finite index in the group $\mathrm{Aut}(X)$, see [HKT08, Theorem 11.94]. If the genus of X equals $g \geqslant 2$ then by Hurwitz theorem $\mathrm{Aut}(X)$ is a finite group, see [HKT08, Theorem 11.50], and its cardinality is at most $84(g-1)$ when the characteristic of k equals 0. In the case when the characteristic of k is positive there are also uniform bounds for the cardinality of $\mathrm{Aut}(X)$ depending only of the genus of X, see [HKT08, Theorem 11.127].

If $\dim(X) \geqslant 2$ then $\mathrm{Aut}(X)$ does not coincide with $\mathrm{Bir}(X)$ in general. For instance, if X is a projective space $\mathbb{P}^n_k$ over a field k, then $\mathrm{Aut}(X) = \mathrm{PGL}_{n+1}(k)$. The group $\mathrm{Cr}_n(k)$ of birational automorphisms of $\mathbb{P}^n_k$, so-called *Cremona group of rank n*, is much bigger than the group of regular automorphisms $\mathrm{Aut}(\mathbb{P}^n_k)$ if $n \geqslant 2$. In particular, if we consider $\mathrm{Cr}_n(k)$ as a k-scheme then it has infinitely many components and their dimensions are not bounded, see [BF13].

On the other hand there are many varieties X for which the groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ coincide. Varieties satisfying this condition are said to be *birationally super-rigid.* If we assume that X is a *minimal model*, i.e. that the canonical class K_X of X is nef, and if there are no other minimal models in the birational class of X then one has $\mathrm{Aut}(X) = \mathrm{Bir}(X)$. Important classes of such varieties include abelian varieties and minimal surfaces of non-negative Kodaira dimensions, see [BHPVdV04, Theorem VI.1.1]. If $\dim(X) \geqslant 3$ we know fewer examples of birationally super-rigid varieties. If X is a variety of general type then by [HMX13] the cardinality of $\mathrm{Bir}(X)$ can be bounded solely in terms of the dimension and volume of the canonical class, thereby generalizing Hurwitz theorem. Therefore, there exists a birational model $\widetilde{X}$ of X such that $\mathrm{Aut}(\widetilde{X}) = \mathrm{Bir}(\widetilde{X})$. In [BCHM10] there is a construction of such variety $\widetilde{X}$, it is the canonical model of X. If the Kodaira dimension of a variety X is negative, then X does not admit a minimal model. However, among these varieties, some of them are still birationally super-rigid. It is a deep fact due to [IM71] that any smooth quartic hypersurface in $\mathbb{P}^4$ is a birationally super-rigid Fano threefold. More examples of such phenomenon have been subsequently found, see, e.g., [Puk98], [dF13], [CP17].

Given a variety X, the construction of its minimal model is the subject of the *minimal model program* (*MMP*). The idea of this method is to single out curves intersecting negatively K_X, and to contract them (then maybe compose this with a small birational transformation). The result of MMP is a model X_0 of one of the following types:

- X_0 is a minimal model of X i.e. the canonical class K_{X_0} is nef;
- there exists a dominant morphism $\pi\colon X_0 \to B$ where $\dim(B) < \dim(X)$, the rank of the relative Picard group $\mathrm{Pic}(X_0)/\pi^* \mathrm{Pic}(B)$ equals 1 and the relative anticanonical class $-K_{X_0/B}$ is ample.

Recall that the variety X_0 as in the second case and with a restriction on its singularities is called a *Mori fiber space.* The case when we get a Mori fiber space corresponds to the situation when X admits no minimal model; nevertheless, MMP produces a model of X with nice properties which can be used in the study of birational and regular automorphisms of X.

Here is a brief list of the main topics of this thesis.

In the first chapter, we focus on the description of finite subgroups of $\mathrm{Bir}(X)$ when X is a rationally connected complex threefold. We shall also describe $\mathrm{Aut}(X)$ when X is a quasi-projective surface defined over a field k such that $\mathrm{char}(k) > 0$. One of the main ingredients here is MMP, which allows us to reduce questions about finite subgroups of $\mathrm{Bir}(X)$ to classifying groups of automorphisms of very special algebraic varieties that arise as the final result of MMP.

In the second chapter we consider birational automorphisms of infinite order, and try to understand when it is possible to construct a birational model where the induced automorphism is regular. We are mainly interested in the example of a birational automorphism of a rational threefold introduced by J. Blanc in [Bla13]. The main result here is that it is not conjugate to a regular automorphism. Approach which we take in this part is dynamical in nature, and the action of birational maps on the cohomology groups plays an important role.

Finally the third chapter is concerned with the description of the automorphism groups of non-Kähler manifolds introduced by D. Guan [Gua94] and further studied by F. Bogomolov [Bog96]. These manifolds are non-Kähler analogues of hyperkähler manifolds; thus, we expect that their properties are similar. By Bogomolov's construction these manifolds fiber over the projective space with abelian varieties as generic fibers; thus, algebraic tools can be used to study their geometry.

1. Finite groups of automorphisms

In this section, we summarize the results that will be presented in the first chapter of this thesis. Recall that the Cremona group $\mathrm{Cr}_n(k)$ of rank n is the group of birational automorphisms of the projective space $\mathbb{P}^n_k$. Two striking results about the Cremona group of rank 2 were nearly simultaneously published in 2009. On the one hand, I. Dolgachev and V. Iskovskikh in [DI09] gave a complete classification of all finite subgroups of $\mathrm{Cr}_2(\mathbb{C})$. On the other hand, J.-P. Serre in [Ser09] proved that the group $\mathrm{Cr}_2(k)$ satisfies the Jordan property for any field k of characteristic 0.

Definition 1.1. A group Γ is said to satisfy the *Jordan property* if there exists $J > 0$ such that any finite subgroup $G \subset \Gamma$ contains a normal abelian subgroup $A \subset G$ with $[G : A] \leqslant J$.

Serre subsequently conjectured that the Cremona group of any rank satisfies the Jordan property over a field of characteristic 0. Serre's conjecture was proved by Yu. Prokhorov and C. Shramov in [PS16]: they established the Jordan property for Cremona groups of all ranks over a field of characteristic 0 assuming the Borisov–Alexeev–Borisov conjecture which was later proved by C. Birkar in [Bir21]. The Serre's conjecture inspired study of Jordan property for automorphisms groups of different varieties. An interesting statement of this type was proved by V. Popov in [Pop11, Theorem 2.32]: he showed that in characteristic 0 the group of birational automorphisms of a surface satisfies the Jordan property for all but concretely described birational classes of surfaces. Then S. Meng and D.-Q. Zhang in [MZ18] showed that the Jordan property holds for groups of regular automorphisms of all projective varieties over a field of characteristic 0. The Jordan property was also established for groups of regular automorphisms of Kähler manifolds and manifolds in Fujiki class $\mathscr{C}$, see [Kim18] and [MPZ20]. Groups of birational automorphisms of complex surfaces and threefolds were also proved to be Jordan, see [PS21a], [PS20], [PS21b]. Also T. Bandman and Yu. Zarkhin in [BZ15] showed that the group of regular automorphisms of a quasi-projective complex surface always satisfies the Jordan property.

Some of these results are true over a field k of positive characteristic p. For instance, Prokhorov and Shramov managed to show that the Cremona group $\mathrm{Cr}_2(k)$ is Jordan if k is a finite field. However, if k is an algebraically closed field of positive characteristic the situation is much harder. Actually, many Lie groups over such field do not satisfy the Jordan property. In view of this F. Hu suggested the following analogue of the Jordan property:

Definition 1.2 ([Hu20, Definition 1.6])**.** We say that a group Γ is *p-Jordan*, if there exist constants $J(\Gamma)$ and $e(\Gamma)$ depending only on Γ such that any finite subgroup $G \subset \Gamma$ contains a normal abelian subgroup A with

$$[G : A] \leqslant J(\Gamma) \cdot |G_p|^{e(\Gamma)},$$

where G_p is a Sylow p-subgroup of G.

This definition was motivated by the work of M. J. Larsen and R. Pink [LP11] in which they established the p-Jordan property for the group $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$ for any prime number p and any $n > 0$. Then Hu proved in [Hu20] that the group of regular automorphisms of any projective variety over a field of characteristic $p > 0$ satisfies the p-Jordan property. Moreover, Y. Chen and C. Shramov in [CS21] generalized Popov's result to positive characteristic; namely, they proved that the group of birational automorphisms of an algebraic surface over an algebraically closed field of characteristic $p > 0$ satisfies the p-Jordan property for all but concretely described birational classes of surfaces. The birational type of the surface S when $\mathrm{Bir}(S)$ is not Jordan is if S is the product $\mathbb{P}^1 \times E$ where E is an elliptic curve.

We study finite subgroups in groups of automorphisms of quasi-projective surfaces, thereby extending Bandman and Zarkhin's theorem to positive characteristic. Here is the first result of the thesis:

Theorem 1.3. *If S is a quasi-projective surface defined over a field of characteristic $p > 0$, then the group $\mathrm{Aut}(S)$ is p-Jordan.*

The idea of the proof is the following. Since the subgroup of a p-Jordan group is p-Jordan then in view of Chen and Shramov's result the theorem can be reduced to the case when S is birationally equivalent to the product $\mathbb{P}^1 \times E$ where E is an elliptic curve. We construct a compactification $\overline{S}$ of S and consider the Albanese map $\pi\colon \overline{S} \to E$. If there is an irreducible component of $\overline{S} \setminus S$ whose image under π is a point on E then we prove that $\mathrm{Aut}(S)$ is Jordan. If $\overline{S} \setminus S$ consists of multisections of π then we show that the action of any element of the group $\mathrm{Aut}(S)$ induces a regular automorphism of $\overline{S}$. The proof of Theorem 1.3 highly relies on the fact that any unirational curve is rational. Note that this fact is not true in higher dimensions in positive characteristic, there exist many examples of unirational non-rational surfaces; see, for instance, [Shi74], [Kat81], [Miy76].

We now turn to a more precise discussion of finite groups of birational automorphisms of projective varieties. For $n \geqslant 3$ a complete description of all finite subgroups of $\mathrm{Cr}_n(\mathbb{C})$ is out of reach. We shall thus focus on bounding the cardinality of the generating sets of p-subgroups in $\mathrm{Cr}_3(\mathbb{C})$. Recall that if p is a prime number then a *p-group* is a finite group of order p^m for some $m \geqslant 0$.

The idea of considering such groups comes from the work [BB00] by L. Bayle and A. Beauville where they classified all birational involutions of $\mathbb{P}^2$. Then T. de Fernex in [dF04] studied birational automorphisms of $\mathbb{P}^2$ of prime order, and Blanc in [Bla09] described all conjugacy classes of finite abelian subgroups in $\mathrm{Cr}_2(\mathbb{C})$. Beauville in [Bea07] proved sharp bounds on ranks of abelian p-subgroups of $\mathrm{Cr}_2(\mathbb{C})$ for all prime numbers p. Prokhorov in [Pro11] and [Pro14] extended this result to dimension 3 and to a wider class of varieties; he proved bounds on the rank of abelian p-subgroups in the group $\mathrm{Bir}(X)$ of birational automorphisms of any rationally connected threefold X.

Prokhorov and Shramov in [PS18] proved that if $p \geqslant 17$ is a prime number and X is a rationally connected threefold, then a p-subgroup in $\mathrm{Bir}(X)$ is necessarily abelian, its rank is at most 3 and this bound is sharp. In the work by J. Xu [Xu20] this result was generalized to all prime numbers $p \geqslant 5$. Moreover, Prokhorov in [Pro14] gave a sharp bound on the number of generators of any 2-subgroup in $\mathrm{Bir}(X)$ for a rationally connected threefold X. Thus, we have a sharp bound on the number of generators of a p-subgroup in $\mathrm{Bir}(X)$ for a rationally connected threefold X and all prime numbers except $p = 3$.

In our work we study the last remaining case of 3-subgroups in the group $\mathrm{Bir}(X)$ for a rationally connected threefold X. Prokhorov in [Pro11] proved that any abelian 3-subgroup can be generated by at most 5 elements. We extend this result to not necessarily abelian groups and prove the following theorem:

Theorem 1.4. *Let X be a projective rationally connected complex threefold and let G be a 3-subgroup in $\mathrm{Bir}(X)$. Then the following is true:*

1. *The group G can be generated by at most 5 elements.*

2. *If G cannot be generated by 4 elements, then $G \subset \mathrm{Aut}(X_0)$ where X_0 satisfies one of the following properties:*

   (*a*) *X_0 is a Fano threefold with terminal singularities, the number of its non-Gorenstein singular points is 9 and all these points are cyclic quotient singularities of type $\frac{1}{2}(1,1,1)$.*

   (*b*) *X_0 is a Fano threefold with terminal Gorenstein singularities with $\mathrm{Pic}(X_0) = \mathbb{Z} K_{X_0}$ of genus 7 or 10 and the number of singular points of X_0 is 9 or 18.*

The second assertion of Theorem 1.4 relies heavily on the G-equivariant version of MMP. Recall that the G-equivariant MMP which starts with a variety X with a faithful regular action of a finite group G and its result is another variety X_0 with a regular action of G which is G-birational to X. Moreover, X_0 is either a minimal model or it is a *G-Mori fiber space* i.e. an equivariant analogue of a Mori fiber space. Note that a rationally connected threefold cannot have a minimal model by [KMM92].

If X is a rationally connected threefold and G is a finite subgroup in $\mathrm{Bir}(X)$, then one can construct a birational model $\widetilde{X}$ of X such that $G \subset \mathrm{Aut}(\widetilde{X})$. Then we apply G-equivariant MMP to $\widetilde{X}$ with the action of G and obtain a G-Mori fiber space X_0. Thus, Theorem 1.4 is a consequence of the following proposition.

Proposition 1.5. *Let G be a 3-group and let X_0 be a G-Mori fiber space of dimension 3. Then G can be generated by at most 4 elements unless X_0 is a Fano threefold which satisfies properties* (a) *or* (b) *in Theorem* 1.4.

The equivariant MMP works in the class of complex threefolds with terminal singularities endowed with an action of a finite group. Mori fiber spaces with terminal singularities are very well studied. Moreover, by [Isk79], [MM82] and other works there exists a complete classification of smooth Fano threefolds. In the proof of Proposition 1.5 we use also many results on the geometry of distinct types of Fano threefolds and on the properties of terminal singularities.

Recently Loginov in [Log21] has studied in more details Fano varieties which satisfy properties (a) and (b) in Theorem 1.4. He was able to prove that in both cases the group G can be generated by at most 4 elements. This leads us to the following corollary.

Corollary 1.6. *Let G be a 3-subgroup in a group $\mathrm{Bir}(X)$ where X is a complex rationally connected threefold. Then G can be generated by at most 4 elements and this bound is sharp.*

2. Regularization of birational automorphisms

In the second chapter of this thesis we shall focus our attention on birational automorphisms of infinite order. The set-up will be as follows. Let X be a normal projective variety defined over an algebraically closed field k of characteristic 0. We say that a birational automorphism $f\colon X \dashrightarrow X$ is *regularizable on Y* if there exists a birational map $\alpha\colon X \dashrightarrow Y$ to a projective variety Y and $g \in \mathrm{Aut}(Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ Y & \xrightarrow{g} & Y \end{array}$$

The question whether one can regularize an given birational automorphism $f\colon X \dashrightarrow X$ becomes increasingly difficult when dimension of X grows. In the curve case the question is trivial: any birational automorphism is obviously regularizable. In order to recall the known results in higher dimensions we need the following definitions.

Recall that $N^i(X)$ is the $\mathbb{R}$-vector space generated by classes of irreducible subvarieties of codimension i in X modulo numerical equivalence. Let $H \in N^1(X)$ be an ample divisor class on X and let $\dim(X) = d$. One can define the class $f^*(H^i) \in N^i(X)$ by taking the class of the proper preimage under f^{-1} for a general subvariety in the class of $H^i \in N^i(X)$. Then the *i-th degree of f* for $0 \leqslant i \leqslant d$ is defined as the following number:

$$\deg_i(f) = (f)^*(H^i) \cdot H^{d-i}.$$

By [DS05] the growth rate of the sequence $(\deg_i(f^n))_{n>0}$ is a birational invariant of the pair (X, f). In particular, it does not depend on the choice of the ample divisor H, see also [Tru20]. Moreover, the sequence $(\deg_i(f^n))_{n>0}$ is submultiplicative in n; thus, we can define the *i-th dynamical degree of f* as:

$$\lambda_i(f) = \lim_{n\to\infty} \left(\deg_i(f^n)\right)^{\frac{1}{n}}.$$

By [DN11] and [Tru20] the numbers $\lambda_i(f)$ are real, satisfy $\lambda_i(f) \geqslant 1$ and they are birational invariants of the pair (X, f) for $0 \leqslant i \leqslant d$. In particular, they do not depend on the choice of the ample divisor H. Moreover, $\lambda_1(f) = \lambda_d(f) = 1$ and dynamical degrees are log-concave i.e. one has the following inequality for all $1 \leqslant i \leqslant d-1$:

$$\lambda_{i-1}(f) \cdot \lambda_{i+1}(f) \leqslant \lambda_i(f)^2.$$

Using the terminology coined in [BV99] we say that a birational automorphism f is of *positive entropy* if for some $0 \leqslant i \leqslant d$ one has $\lambda_i(f) > 1$. This terminology is justified by the fact that the topological entropy of f equals $\log\left(\max_{0\leqslant i\leqslant d}(\lambda_i(f))\right)$ when f is a regular automorphism of a compact projective complex variety by theorems of Gromov and Yomdin [Gro03] and [Yom87], see also [DS05]. Note that by the log-concavity of dynamical degrees one has f is a positive entropy automorphism if and only if $\lambda_1(f) > 1$.

By [Wei55] (see also [Dés21, Section 3.5]) if the sequence $(\deg_i(f^n))_{n>0}$ is bounded, then there exists a birational model X_0 of X such that f is regularizable on X_0. Otherwise, one has an additional structure associated with the automorphism f which allows us to understand better its properties.

If $\dim(X) = 2$ and the sequence $(\deg_1(f^n))_{n>0}$ is not bounded then by [BC16] the dynamical degree $\lambda_1(f)$ is an algebraic number with special properties, i.e. it is either a Salem or a Pisot number. Moreover, by [DF01] if the sequence $(\deg_1(f^n))_{n>0}$ is not bounded and $\lambda_1(f) = 1$, then $(\deg_1(f^n))_{n>0}$ grows as n or as n^2. The birational types of complex surfaces which admit a positive entropy birational automorphism are described in [Can99]; moreover, there exists many examples of surface positive entropy automorphisms, see for instance [McM07], [Bla08] and [BK09]. The first dynamical degree is proved to be lower semi-continuous in families of birational automorphisms of surfaces by [Xie15].

If $\dim(X) = 2$ the growth rate of the sequence $(\deg_i(f^n))_{n>0}$ in many situations determines whether f is regularizable or not. By [DF01] if $\lambda_1(f) = 1$ then f is regularizable if and only if the sequence $(\deg_i(f^n))_{n>0}$ is bounded or grows as n^2. By [BC16] if $\lambda_1(f)$ is a Salem number then f is regularizable. Moreover, there is a more complicated criterion by [DF01]. It claims that if $\lambda_1(f) > 1$ and there exists a divisor class θ on X such that $f^*\theta = \lambda_1(f)\theta$ and $\theta^2 = 0$, then f is a regularizable automorphism.

Positive entropy automorphisms in higher dimensions are much more complicated. Unlike the case of surfaces a positive entropy automorphism $f\colon X \dashrightarrow X$ of a smooth projective variety X such that $\dim(X) \geqslant 3$ can *preserve a fibration* i.e. there can exist a dominant map $\pi\colon X \dashrightarrow B$ to some variety B and $g \in \mathrm{Bir}(B)$ such that $1 \leqslant \dim(B) \leqslant \dim(X) - 1$ and $\pi \circ f = g \circ \pi$. If f preserves a fibration we say that it is *imprimitive*. By J. Lesieutre [Les18] we get that if X is a smooth complex threefold and $f\colon X \dashrightarrow X$ is a positive entropy birational non-regular automorphism which can be regularized on a variety constructed by an iterated blow-up of X in smooth subvarieties, then f is imprimitive.

Note that for any regular automorphism $f\colon X \to X$ the canonical class K_X is f^*-invariant. If K_X is an ample or an anti-ample divisor, then one can use it in order to compute the dynamical degree of f. Therefore, if X is a Fano threefold then there is no positive entropy automorphism of X. Moreover, by Lesieutre's result any iterated blow-up of a smooth complex Fano threefold in smooth subvarieties admits no regular primitive positive entropy automorphism. Thus, it is very complicated to construct an example of a primitive positive entropy regular automorphism on a rational threefold. At the moment there is known only one example of such automorphism described in [OT15].

Attempts to generalize regular positive entropy automorphisms of rational surfaces resulted in constructions of birational automorphisms [BK14], [PZ14], [Bla13] which turn out to be *pseudo-automorphisms*. Recall that a birational map $f\colon X \dashrightarrow X$ of a smooth variety X is called a pseudo-automorphism if neither f nor f^{-1} contract any divisor in X. Note that in the case of surfaces any pseudo-automorphism is a regular automorphism. Thus, pseudo-automorphisms form a class of birational automorphisms which are very close to being regular. One might expect that any pseudo-automorphism can be regularized and under appropriate assumptions this indeed is true. however, it may be false. Here we list some known constructions of positive entropy pseudo-automorphisms of rationally connected threefolds:

Example 2.1 ([BK14])**.** This example is obtained as a generalization of a surface automorphism construction from [BK09]. Fix $a \in \mathbb{C}\setminus\{0\}$ and a primitive third root of unity ζ and consider the birational automorphism of $\mathbb{P}^3$:

$$f_{a,\zeta}\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3; \qquad f_{a,\zeta}(x_0 : x_1 : x_2 : x_3) = (x_0x_1 : x_1x_2 : x_1x_3 : ax_0^2 + \zeta x_0x_2 + x_0x_3).$$

Then $\lambda_1(f_{a,\zeta}) = \lambda_2(f_{a,\zeta}) > 1$ and $f_{a,\zeta}$ induces a pseudo-automorphism on a blow-up of $\mathbb{P}^3$ in several points and curves. Moreover, $f_{a,\zeta}$ is imprimitive and if $a \neq 1$ then $f_{a,\zeta}$ is non-regularizable.

Example 2.2 ([BCK14])**.** Let a, c be complex numbers and let $f_{a,c}\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a birational automorphism defined as $f_{a,c} = L_{a,c} \circ J$ where

$$L_{a,c}(x_0 : x_1 : x_2 : x_3) = (x_3 : x_0 + ax_3 : x_1 : x_2 + cx_3);$$
$$J(x_0 : x_1 : x_2 : x_3) = (x_0^{-1} : x_1^{-1} : x_2^{-1} : x_3^{-1}).$$

Thus, $f_{a,c}$ is the composition of a regular automorphism $L_{a,c}$ of $\mathbb{P}^3$ and the Cremona involution J. If a and c satisfy a certain quadratic equation then $\lambda_1(f_{a,c}) = \lambda_2(f_{a,c}) > 1$ and $f_{a,c}$ induces a pseudo-automorphism on a blow-up of $\mathbb{P}^3$ in several points. Moreover, $f_{a,c}$ is primitive and non-regularizable.

Example 2.3 ([PZ14], [BDK], [DO88])**.** This example is obtained as a generalization of a surface automorphism construction from [McM07]. There exists a blow-up $\delta\colon X \to \mathbb{P}^3$ of several points $p_1, \ldots, p_k$ in $\mathbb{P}^3$ and a bilinear form $(,)$ on the lattice $H^2(X, \mathbb{Z})$ which induces the structure of the root lattice. Thus, there is a Weyl group W with a natural representation in $H^2(X, \mathbb{Z})$. Moreover, for any element $w \in W$ there is a pseudo-automorphism

$$f_w\colon X \dashrightarrow X,$$

such that f_w^* acts on $H^2(X, \mathbb{Z})$ as w. We denote by w_0 the Coxeter element in W. If W is an infinite group we get that $\lambda_1(f_{w_0}) = \lambda_2(f_{w_0}) > 1$ and f_{w_0} is imprimitive.

Example 2.4 ([BL15])**.** Let $Y \subset \mathbb{P}^4$ be a smooth cubic threefold and let C_1 be a smooth curve of genus 2 and degree 6 on Y. The base locus of a general pencil of hyperquadric sections containing C_1 is the union $C_1 \cup C_2$, where C_2 is a smooth curve of genus 2 and degree 6 on Y. Then there is a birational automorphism:

$$f_{Y,C_1}\colon Y \dashrightarrow Y,$$

such that $\lambda_1(f_{Y,C_1}) = \lambda_2(f_{Y,C_1}) > 1$ and f induces a pseudo-automorphism on the subsequent blow-up of curves C_1 and C_2. Moreover, f preserves the pencil of quadrics passing through $C_1 \cup C_2$; thus, f is imprimitive.

Example 2.5 ([Bla13])**.** This example is obtained as a generalization of a surface automorphism construction from [Bla08]. Let $S \subset \mathbb{P}^3$ be a smooth cubic surface. With each point $p \in S$ one can associate a birational involution $\sigma_p\colon \mathbb{P}^3 \to \mathbb{P}^3$. For any collection $p_1, \ldots, p_k$ of k points on S consider the following birational automorphism:

$$f_{p_1,\ldots,p_k} = \sigma_{p_1} \circ \cdots \circ \sigma_{p_k}\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3.$$

If the points $p_1, \ldots, p_k$ are general and $k \geqslant 3$ then $\lambda_1(f_{p_1,\ldots,p_k}) = \lambda_2(f_{p_1,\ldots,p_k}) > 1$ and $f_{p_1,\ldots,p_k}$ induces a pseudo-automorphism on a blow-up of $\mathbb{P}^3$ in several points and curves.

Examples 2.1 and 2.2 are proved to be non-regularizable. Examples 2.3 and 2.4 are non-primitive. Thus, we concentrate on the last example and prove that it is non-regularizable under appropriate assumptions. Here is the main result of the second chapter of this thesis.

Theorem 2.6. *Let $S \subset \mathbb{P}^3$ be a very general smooth complex cubic surface and let p_1, p_2, p_3 be general points on S. Then the birational automorphism $f_{p_1,p_2,p_3}\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ described in Example* 2.5 *is non-regularizable and does not preserve a fibration over a surface.*

There are several criteria which allow us to prove that a birational automorphism of a threefold is non-regularizable. By [CDX21] if $\deg_1(f^n)$ grows as n^k where k is odd, then f is non-regularizable; also if $\lambda_1(f) > 1$ is an integer then then f is non-regularizable. By [LB19] if X is a threefold and if the sequence $(\deg(f^n))_{n>0}$ grows as n^k where $k > 4$ then f is non-regularizable. Also if the number $\lambda_1(f)$ does not satisfy several conditions given in [LB19, Proposition 4.6.7, 4.7.2, 5.0.1] then f is non-regularizable. Another criterion which does not use dynamical properties of the birational automorphism f was used to prove that Examples 2.1 and 2.2 are non-regularizable. It is based on [BK14, Corollary 1.6] which says that if $f\colon X \dashrightarrow X$ is a birational automorphism of a smooth threefold X and if Y is an f-invariant surface in X such that the birational automorphism $f|_Y\colon Y \dashrightarrow Y$ is non-regularizable then f is non-regularizable. In both Examples 2.1 and 2.2

one can find an f-invariant surface Y such that $\lambda_1(f|_Y)$ is a Pisot non-quadratic number; thus, by [BC16] we get that $f|_Y$ is non-regularizable.

All these arguments do not work in Example 2.5. Thus, in order to prove Theorem 2.6 we establish a new criterion. In order to formulate it recall that if $f\colon X \to X$ is a pseudo-automorphism such that one has an inequality $\lambda_1(f)^2 > \lambda_2(f)$, then by [Tru14] there exists a unique up to proportionality pseudo-effective divisor class $\theta_1(f)$ such that:

$$f^*\theta_1(f) = \lambda_1(f)\theta_1(f). \tag{2.7}$$

Such class was successfully used in [DF01] for the necessary condition on the existence of a regularization of a surface birational automorphism. Now we can formulate our criterion:

Theorem 2.8. *Let $f\colon X \dashrightarrow X$ be a pseudo-automorphism of a smooth projective threefold X such that*

(1) $\lambda_1(f)^2 > \lambda_2(f)$; *thus, there exists a class $\theta_1(f)$ as in* (2.7);

(2) *there exists a curve C such that $\theta_1(f) \cdot [C] < 0$;*

(3) *there exist infinitely many integers $m > 0$ such that $C \not\subset \operatorname{Ind}(f^{-m})$.*

Then f is non-regularizable and it does not preserve a fibration over a surface.

Some comments are in order. By log-concavity of dynamical degrees one has $\lambda_1(f)^2 \geqslant \lambda_2(f)$. Thus, the condition (1) is always true either for f or for f^{-1} since $\lambda_2(f) = \lambda_1(f^{-1})$. Condition (2) implies that the class $\theta_1(f)$ is not nef and the last condition is required to avoid some situations which we describe in the second chapter of the thesis.

To prove Theorem 2.6 we show that the pseudo-automorphism model $\widetilde{f}_{p_1,p_2,p_3}\colon X \dashrightarrow X$ of f_{p_1,p_2,p_3} constructed in [Bla13] satisfies the condition of Theorem 2.8. The first condition is obviously true. To show the second condition we consider a curve of indeterminacy of $\widetilde{f}_{p_1,p_2,p_3}$ and prove that it intersects $\theta_1(\widetilde{f}_{p_1,p_2,p_3})$ negatively. This curve is the proper transform of a line L from $\mathbb{P}^3$ to X. The verification of the third condition is quite difficult; we prove that the line L does not lie in the indeterminacy locus of $f^{-m}_{p_1,p_2,p_3}$ using explicit formulas for the involutions σ_{p_i}. Most of our computations were done in Sage. Dealing with three involutions already makes our proof tricky. We expect that our theorem is valid for any composition of at least three involutions associated to general points on S.

3. Automorphisms of Bogomolov–Guan manifolds

In the third chapter of this thesis we shall explore the properties of a special class of non-Kähler complex compact manifolds. These manifolds are particularly interesting in view of their similarity with hyperkähler manifolds. Recall that a *hyperkähler manifold* is a Riemannian manifold (M, g) equipped with three Kähler complex structures $I, J, K : TM \to TM$, satisfying the quaternionic relation:

$$I^2 = J^2 = K^2 = IJK = -\operatorname{id}.$$

Any hyperkähler manifold is holomorphically symplectic i.e. admits a non-degenerate $(2, 0)$-form. Conversely, a compact holomorphically symplectic manifold is hyperkähler, provided that it is

Kähler. This follows from the Calabi-Yau theorem [Yau78], see also [Bea83]. A hyperkähler manifold M is called *irreducible holomorphic symplectic* (*IHS*) if it is compact, complex, simply connected and the group $H^{2,0}(M)$ is 1-dimensional.

An example of non-Kähler manifolds which are very close to IHS manifolds was constructed in several papers by D. Guan [Gua94], [Gua95a] and [Gua95b]. Later F. Bogomolov in [Bog96] gave a more geometric construction for these manifolds. We recall here the main steps of the Bogomolov's construction.

Let S be a *primary Kodaira surface*, i.e. a smooth complex compact holomorphic symplectic surface which admits a structure of an isotrivial elliptic fiber space:

$$\pi\colon S \to E,$$

over an elliptic curve E such that any algebraic subvariety in S is either a point or a fiber of π. All fibers of π are isomorphic to an elliptic curve F. The map π induces a structure of F-torsor on S. Denote by $\pi^{[n]}\colon S^{[n]} \to E^{[n]}$ the induced map beetween Douady spaces of length n of S and E respectively. Denote by $\mathrm{Alb}\colon E^{[n]} \to E$ the Albanese morphism of the algebraic variety $E^{[n]}$ which is isomorphic to the symmetric power $\mathrm{Sym}^n(E)$. The following variety

$$W = (\pi^{[n]})^{-1}(\mathrm{Alb}^{-1}(0))$$

is a complex manifold with an action of F induced from the diagonal action of F on $S^{[n]}$. Then by [Bog96, Corollary 4.10] under appropriate conditions on the Kodaira surface S there exists a smooth, compact, complex, simply connected manifold Q such that the group $H^{2,0}(Q)$ is generated by a non-degenerate holomorphic symplectic form and which is a finite cover of W/F:

$$p\colon Q \to W/F.$$

If $n = 2$ then Q is a K3-surface; if $n \geqslant 3$ then Q is a non-Kähler $(2n-2)$-dimensional manifold.

The manifold Q constructed as described above for $n \geqslant 3$ is the main object of study in the third chapter of this thesis, we call it the *BG-manifold* (for Bogomolov–Guan). Since the fiber $\mathrm{Alb}^{-1}(0)$ is isomorphic to the $(n-1)$-dimensional projective space then the map $\pi^{[n]}$ induces the map $\Pi\colon W/F \to \mathbb{P}^{n-1}$. Thus, the BG-manifold Q admits a surjective map to a projective space:

$$\Phi = \Pi \circ p\colon Q \to \mathbb{P}^{n-1}. \tag{3.1}$$

Since many properties of BG-manifols are similar to those of IHS manifolds we expect that many results about hyperkähler manifolds can be extended to the case of BG-manifolds. Recall here several significant results about hyperkähler manifolds. First, if M is a hyperkähler manifold then by [Fuj87] there exists an important non-degenerate symmetric quadratic form on the cohomology group $H^2(M,\mathbb{Z})$. This form is called the *Beauville–Bogomolov–Fujiki form* or *BBF-form* and it is very useful in the study of the geometry of hyperkähler manifolds and their moduli spaces. Another important result which later led to the proof of the Torelli theorem for hyperkähler manifolds is the Bogomolov–Tian–Todorov theorem [Bog78], which says that the deformation theory of a Kähler manifold with trivial canonical class is unobstructed. Groups of biholomorphic $\mathrm{Aut}(M)$ and bimeromorphic $\mathrm{Bim}(M)$ automorphisms of a hyperkähler manifold M were studied by N. Kurnosov and E. Yasinsky in [KY19] and by A. Cattaneo and L. Fu in [CF19]. They proved in particular that the order of finite subgroups in the groups $\mathrm{Aut}(M)$ and $\mathrm{Bim}(M)$ are bounded. Moreover, there are only finitely many conjugacy classes of finite subgroups in $\mathrm{Aut}(M)$ and $\mathrm{Bim}(M)$.

There are some partial extensions of these results to BG-manifolds. In [KV19] N. Kurnosov and M. Verbitsky proved the existence of a symmetric quadratic form on the cohomology group $H^2(Q,\mathbb{Z})$ on a BG-manifold Q analogues to the BBF-form. They conjectured that this form is non-degenerate and that it satisfies all properties of a BBF-form. Moreover, the study of holomorphic symplectic defiormations and this symmetric form led them to the generalization of the Bogomolov–Tian–Todorov theorem to the class of BG-manifolds.

In this thesis we are going to explore the groups of biholomorphic and bimeromorphic automorphisms of BG-manifolds. In order to do this we find several structures on a BG-manifold Q which should be preserved under automorphisms. Since a BG-manifold Q is non-algebraic one can consider algebraic submanifolds in Q. The image of an algebraic submanifold under an automorphism is necessarily an algebraic submanifold. One can consider an *algebraic reduction* of Q i.e. a meromorphic map $f\colon Q \dashrightarrow X$ to an algebraic variety X such that any meromorphic map from Q to an algebraic variety factors through f. A map f with this property is unique up to birational conjugations. Our first result is the following description of the algebraic reduction of Q:

Theorem 3.2. *Let $n \geqslant 3$ be an integer and let Q be a BG-manifold of dimension $2n-2$. Then the map $\Phi\colon Q \to \mathbb{P}^{n-1}$ described in* (3.1) *is an algebraic reduction of Q.*

Then we study subvarieties of BG-manifolds. Recall that a manifold X is called *Moishezon* if its algebraic reduction is a generically finite map. In particular, any algebraic variety is Moishezon. We prove the following result:

Theorem 3.3. *Let Q be a BG-manifold of dimension $2n-2$ and let $\Phi\colon Q \to \mathbb{P}^{n-1}$ be its algebraic reduction as in* (3.1)*. There exists a divisor $D \subset \mathbb{P}^{n-1}$ of degree $2n$ such that for any point $x \in \mathbb{P}^{n-1}$ one has:*

(1) *If $x \in \mathbb{P}^{n-1} \setminus D$, then the fiber $\Phi^{-1}(x)$ is an abelian variety.*

(2) *If $x \in D$, then the fiber $\Phi^{-1}(x)$ is a uniruled Moishezon manifold.*

Moreover, if $X \subset Q$ is a submanifold such that $\dim(\Phi(X)) \geqslant 2$ then X is not Moishezon.

By this theorem if $X \subset Q$ is a submanifold of a BG-manifold and $\Phi(X)$ is a point then X is Moishezon; in the case where $\dim(\Phi(X)) \geqslant 2$ one has X is not Moishezon. We also consider the case where $\dim(\Phi(X)) = 1$ and obtain that X may or may not be a Moishezon manifold depending on the curve $\Phi(X)$.

By definition of an algebraic reduction any bimeromorphic or biholomorphic automorphism of a complex manifold is compatible with algebraic reduction. Therefore, we conclude that the group $\mathrm{Aut}(Q)$ fits into the following exact sequence:

$$1 \to G'' \to \mathrm{Aut}(Q) \to G' \to 1, \tag{3.4}$$

where G' is a subgroup of $\mathrm{Aut}(\mathbb{P}^{n-1})$ and G'' is a subgroup of $\mathrm{Aut}\,(A)$ where A is an abelian variety $\Phi^{-1}(x)$ and x is a point in $\mathbb{P}^{n-1} \setminus D$. To get a more clear understanding of the group $\mathrm{Aut}(Q)$ we use a description of the fibers of the map Φ.

Theorem 3.3 implies that a biholomorphic automorphism of Q induces a regular automorphism of the projective space $\mathbb{P}^{n-1}$ which preserves the divisor D. Thus, we study the geometry of D in details and prove that the group $\mathrm{Aut}(Q)$ satisfies the Jordan property, see Definition 1.1. Here is the main result of the third chapter of this thesis:

Theorem 3.5. *Let Q be a BG-manifold of dimension $2n-2$ and let $\Phi\colon Q \to \mathbb{P}^{n-1}$ be its algebraic reduction as in* (3.1)*. Then the group* $\mathrm{Aut}(Q)$ *fits into the exact sequence* (3.4) *where G' is a finite group, $G'' \subset \mathrm{Aut}(A)$ where A is an $(n-1)$-dimensional abelian variety. In particular,* $\mathrm{Aut}(Q)$ *is a Jordan group.*

This result follows from a description of the divisor D and its singular locus. We prove that D contains a finite set $\mathcal{Z}$ of n^2 points of multiplicity $n-1$ and that $\mathcal{Z}$ does not lie in a hyperplane in $\mathbb{P}^{n-1}$. Thus, the group of automorphisms of $\mathbb{P}^{n-1}$ fixing D should fix also $\mathcal{Z}$; therefore, it is finite.

It would be extremely interesting to prove a similar result for the group of bimeromorphic automorphisms of a BG-manifold Q. By the same reasons as in the case of biholomorphic automorphisms the group $\mathrm{Bim}(Q)$ fits into the following exact sequence:

$$1 \to H'' \to \mathrm{Bim}(Q) \to H' \to 1,$$

where $H'' \subset \mathrm{Aut}(A)$ and A is an $(n-1)$-dimensional abelian variety isomorphic to a general fiber of Φ and H' is a subgroup in the group $\mathrm{Cr}_{n-1}(\mathbb{C})$ of birational automorphisms g of $\mathbb{P}^{n-1}$ such that either D lies in $\mathrm{Exc}(g)$ or D is g-invariant.

In the simplest case when $n=3$ and $\dim(Q)=4$ we managed to establish the Jordan property for the group $\mathrm{Bim}(Q)$. However, in higher dimensions it is still unclear whether the group $\mathrm{Bim}(Q)$ is Jordan or not.

4. Publications reflecting the main scientific results of the thesis

(1) *Finite* 3*-subgroups in Cremona group of rank* 3, Math. Notes, 108:5 (2020), 697–715.

(2) *Automorphisms of quasi-projective surfaces over fields of finite characteristic*, J. Algebra 595 (2022), 271–278.

(3) *Geometry and automorphisms of non-Kähler holomorphic symplectic manifolds*, joint work with F. Bogomolov, N. Kurnosov and E. Yasinsky, published online in International Mathematics Research Notices, https://doi.org/10.1093/imrn/rnab043.

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