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*as a manuscript*

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# Bruhat decomposition in Morse theory

Summary of a PhD thesis
for the purpose of obtaining academic degree
Doctor of Philosophy in Mathematics

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## 1. Algebraic preliminaries

In this section we introduce a certain data, called B-data, which will be used as a container which stores information about a Morse function. The actual process of extraction is given in Subsection 2.2. The letter B stands simultaneously for Barannikov, Bruhat and barcode.

**Definition 1.1.** An $n \times m$ matrix is called a *rook* matrix if in every row and in every column there is at most one non-zero entry.

B-data consists of the following parts:

i) A non-negative integer $N$ along with a $\mathbb{Z}_{\geqslant 0}$-grading on a set $\{1, \ldots, N\}$, denoted by deg;

ii) Decomposition of $\{1, \ldots, N\}$ into the union of three disjoint sets $U, L, H$ (these letters stand for upper, lower and homological, for the reasons described below);

iii) Bijection $b \colon U \xrightarrow{1-1} L$ of degree $-1$ w.r.t. the grading. Map $b$ must satisfy $b(s) < s$;

iv) A function $\lambda \colon U \to \mathbb{F}^*$, where $\mathbb{F}$ is a field.

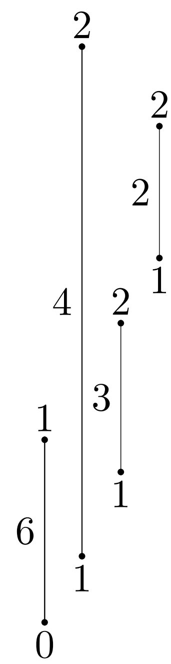

Figure 1

We call the image of $\lambda$ "Bruhat numbers". Two numbers $s$ and $b(s)$ are said to form a Barannikov pair (or simply a pair). It's convenient to think of each Bruhat number as being "written" on a Barannikov pair. Roughly speaking, B-data is a decomposition of some subset of $\{1, \ldots, N\}$ into Barannikov pairs (the rest of the elements are homological). Each pair consists of an upper element, a lower one and carries a Bruhat number. In other words, B-data is a grading on $\{1, \ldots, N\}$ together with a finite sequence of rook matrices $\{R_k\}$ over $\mathbb{F}$ (see Definition 1.1), where $R_k$ is of size $(\#\{s | \deg s = k-1\}) \times (\#\{s | \deg s = k\})$ and $R_{k-1}R_k = 0$.

Figure 1 gives an example of B-data over $\mathbb{Q}$ and describes pictorial format which we will use in future. Elements of the set $\{1, \ldots, N\}$ are drawn as dots, from bottom to top, pairs correspond to segments. Either to the left or to the right of a middle of a segment we write a Bruhat number. The degree of an element is written either above or below this element, whatever is more convenient. In the example $N = 8$, degree of 1 is 0, degree of $2, 3, 4$ and 6 is 1 and degree of $5, 7, 8$ is 2. Next, $U = \{4, 5, 7, 8\}$, $L = \{1, 2, 3, 6\}$, $H = \varnothing$. Bruhat numbers are $6, 3, 2, 4$ (i.e. values of $\lambda$ on 4,5,7,8 respectively). The map $b$ is defined by the segments. Finally, two rook matrices are

$$R_1 = \begin{pmatrix} 0 & 0 & 6 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 0 & 4 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

**Construction 1.2.** Let $R$ be a rook $n \times m$ matrix. We will now define a subset $\mathcal{T}(R)$ of a set $\mathrm{Mat}_{n,m}$ of all $n \times m$ matrices.

$\triangleright$ Let $M \in \mathrm{Mat}_{n,m}$ be a matrix. We say that its entry $M_{i,j}$ is covered if there exists a pair of indices $(i', j')$ s.t. the following two conditions hold:

1) $R_{i',j'} \neq 0$,

2) $(i < i' \text{ AND } j \geqslant j') \text{ OR } (i \leqslant i' \text{ AND } j > j')$.

The matrix $M$ is said to be in $\mathcal{T}(R)$ if the the following two conditions hold:

1) if the entry $M_{i,j}$ is not covered and $R_{i,j} = 0$ then it equals to zero,

2) if the entry $M_{i,j}$ is not covered and $R_{i,j} \neq 0$ then $M_{i,j} = R_{i,j}$. $\lhd$

Here is an example, for $\mathbb{F} = \mathbb{Q}$, of the matrix $R$ and the general form of a matrix $M$ from the set $\mathcal{T}(R)$:

$$R = \begin{pmatrix} 0 & 0 & 4 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, M = \begin{pmatrix} * & * & * \\ 3 & * & * \\ 0 & * & * \\ 0 & 2 & * \end{pmatrix}.$$

## 2. Morse theory

2.1. *Setup.* In this subsection we recall basic notions of Morse theory and fix appropriate notations, setting the stage for our results.

Let $M$ be a smooth closed manifold fixed once and for all. A smooth function is called strong if all its critical points have different critical values. Fix a strong Morse function $f$ on $M$ once and for all. For $a \in \mathbb{R}$ the subspace $M^a := \{x \in M \mid f(x) \leqslant a\}$ is called a sublevel set.

The set of the critical points of $f$ is denoted by $\mathrm{Cr}(f) \subset M$. Since $f$ is strong those are in bijection with critical values of $f$ (this set is finite because of the compactness of $M$). Keeping this bijection in mind, we will freely switch between points and values without mentioning this explicitly. We denote by $\mathrm{Cr}_k(f)$ the set of critical points of index $k$. By $\varepsilon$ we will mean a sufficiently small positive real number.

It follows from foundational results of Morse theory that for $c \in \mathrm{Cr}(f)$ one has $\mathsf{H}_{\deg c}(M^{f(c)+\varepsilon}, M^{f(c)-\varepsilon}; \mathbb{Z}) \simeq \mathbb{Z}$. We say that a critical point is oriented if the generator of this free abelian group of rank one is chosen. A strong Morse function is called oriented if all its critical points are oriented.

Fix a field $\mathbb{F}$ once and for all. All the homologies are assumed to be over $\mathbb{F}$ unless stated otherwise. If the group of coefficients is given explicitly, it goes after a semicolon, e.g. $\mathsf{H}_2(M; \mathbb{Z})$.

2.2. *B-data associated with a strong Morse function.* In this subsection we present a way to associate B-data with an oriented strong Morse function (and a field).

Let $x$ and $y$ be two critical points s.t. $f(x) > f(y)$ and $\operatorname{ind} x - 1 = \operatorname{ind} y = k$. Consider the fundamental class of the attaching sphere for $x$, it lives in $\mathsf{H}_k(M^{f(x)-\varepsilon})$. Let $X$ be its image under the natural map $\mathsf{H}_k(M^{f(x)-\varepsilon}) \to \mathsf{H}_k(M^{f(x)-\varepsilon}, M^{f(y)-\varepsilon})$. Consider now an attaching disk for $y$. It has a relative fundamental class, which lives in $\mathsf{H}_k(M^{f(y)+\varepsilon}, M^{f(y)-\varepsilon})$. Let $Y$ be its image under the natural map $\mathsf{H}_k(M^{f(y)+\varepsilon}, M^{f(y)-\varepsilon}) \to \mathsf{H}_k(M^{f(x)-\varepsilon}, M^{f(y)-\varepsilon})$ induced by inclusion. Critical points $x$ and $y$ form a Barannikov pair with Bruhat number $\lambda$ if and only if $X = \lambda Y \neq 0$. An illustration for $k = 1$ is given in Figure 2.

In the context of Morse theory, Barannikov pairs were introduced (in a different, but equivalent form) in [Bar94]. Now it is a popular tool in applied and symplectic topology called barcodes, see [EH08] for a recent survey. A close idea of construction of Bruhat numbers over $\mathbb{Q}$ appeared independently in [LNV20].

2.3. *Morse complex.* Let $g$ be a generic Riemannian metric on $M$ and $f$ be an oriented strong Morse function. Then one can define a Morse complex $\mathcal{M}(f, g)$

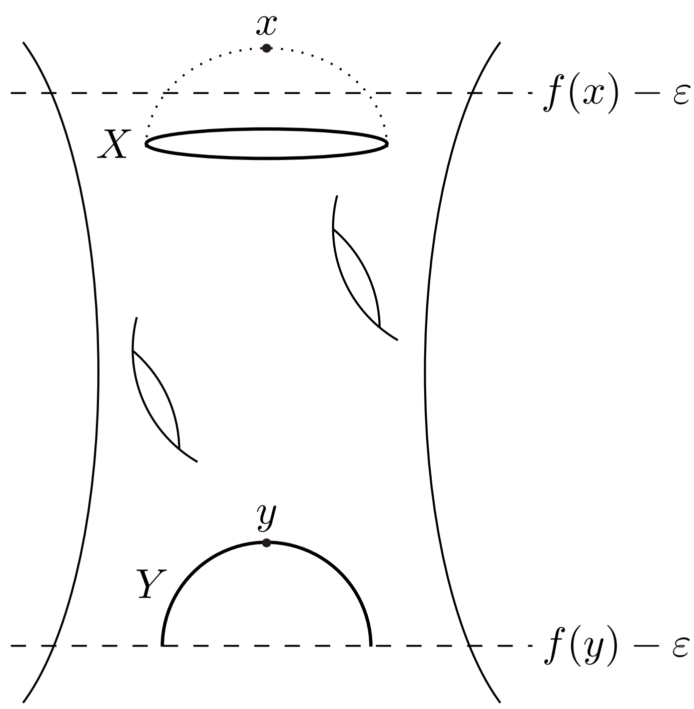


FIGURE 2. To Subsection 2.2. Classes $X$ and $Y$ are drawn in bold. Dotted dome depicts an attaching 2-disk for $x$.

whose integral homology is naturally isomorphic to that of $M$. Its matrix of differential, in general, depends on $g$.

For a B-data associated to $f$ let $R_k$ be the corresponding rook matrix of size $\mathrm{Cr}_{k-1}(f) \times \mathrm{Cr}_k(f)$. In other words, non-zero elements of $R_k$ equal to the Bruhat numbers on Barannikov pairs of points of degrees $k$ and $k-1$.

**Theorem 2.1.** Let $f$ be an oriented strong Morse function on a manifold $M$. Let also $R_k$ be the rook matrix associated to $f$ over $\mathbb{Q}$ (for $k \in \{1, \ldots, \dim M\}$). Then the matrix of Morse differential $\partial_k$ w.r.t. any Riemannian metric $g$ belongs to the set $\mathcal{T}(R_k)$.

For example suppose that $f$ has a B-data as depicted in Figure 1 and $k = 2$. Then the corresponding rook matrix and general form of a matrix of a second Morse differential $P$ are

$$R_2 = \begin{pmatrix} 0 & 0 & 4 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, P = \begin{pmatrix} * & * & * \\ 3 & * & * \\ 0 & * & * \\ 0 & 2 & * \end{pmatrix}.$$

2.4. *A few examples and properties.* In this subsection we quickly give several introductory examples and properties of Bruhat numbers.

Let $f$ be a function on $\mathbb{RP}^n$ which descends from the function $x_1^2 + 2x_2^2 + \ldots + (n+1)x_{n+1}^2$ defined on a unit sphere $S^n \subset \mathbb{R}^{n+1}$. It has $(n+1)$ critical points of all possible indices from 0 to $n$ (ordered by increasing of index). If $\mathrm{char}\,\mathbb{F} = 2$ then all of them are homological. Otherwise, $(2k)^{\mathrm{th}}$ and $(2k-1)^{\mathrm{th}}$ critical points form a Barannikov pair with Bruhat number $\pm 2$ (for any $k \in \{1, \ldots, [n/2]\}$, where brackets denote the integral part). See Figure 3 for an example for $n = 6$.

**Proposition 2.2.** Let $\mathbb{F}$ be either $\mathbb{Q}$ or $\mathbb{F}_p$ and $\lambda \in \mathbb{F}^*$ be any non-zero number. Let also $M$ be any closed manifold s.t. $\dim M \geqslant 4$. Then one can find an oriented strong Morse function $f$ on $M$ which has $\lambda$ as one of its Bruhat numbers.

In particular, Bruhat number over $\mathbb{F} = \mathbb{Q}$ may well be non-integer.

The next proposition is an incarnation of Poincare duality.

**Proposition 2.3.** Let $M$ be closed and orientable and $f$ be an oriented strong Morse function on it. Let also $\mathbb{F}$ be a field. Then B-data for $-f$ is B-data for $f$ turned upside down. Bruhat numbers on pairs remain the same.

**Proposition 2.4.** The number of homological critical points of $f$ of index $k$ equals to $\dim \mathsf{H}_k(M;\mathbb{F})$.

## 3. Bruhat numbers and the theory of torsions

As we saw in Proposition 2.2 any number may appear as a Bruhat number of some function; in a sense, there is no control over the individual Bruhat number. However, sometimes the alternating product of all these numbers turns out to be independent of $f$. Thus this product depends only on the manifold $M$. In the present section we make this statement precise (in Subsection 3.1) and provide a framework where the mentioned product of Bruhat numbers equals to the Reidemeister torsion of $M$ (in Subsection 3.2).

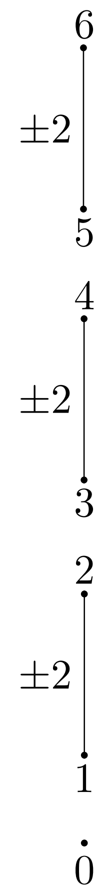


Figure 3

3.1. *Torsion of a Morse function.*

**Definition 3.1.** Let $f$ be an oriented strong Morse function on $M$ and $\mathbb{F}$ be a field. The number

$$\tau(f,\mathbb{F}) = \prod_{s\in U} \lambda(s)^{(-1)^{\deg s}} \in \mathbb{F}^*/\pm 1$$

is called the *torsion* of $f$ over $\mathbb{F}$.

We refer to the r.h.s. as "alternating product" of all Bruhat numbers, in analogy with alternating sum, which is used to define Euler characteristic.

**Theorem 3.2.** Let $f$ be a strong Morse function on $M$ and $\mathbb{F}$ be a field. Suppose that $\mathsf{H}_k(M) = 0$ for all $0 < k < \dim M$. Then the alternating product of all Bruhat numbers (as an element from $\mathbb{F}^*/\pm 1$) is independent of $f$.

For example, taking $M$ to be $\mathbb{RP}^n$ one sees that $\tau(f,\mathbb{Q}) = \pm 2^{[n/2]}$, where brackets denote integral part. Indeed, one has to calculate such a $\tau$ for some particular Morse function on $\mathbb{RP}^n$. They do so for a standard one from Subsection 2.4.

3.2. *Reidemeister torsion and Bruhat numbers.* Suppose now one is given not only an oriented function $f$ but also a one-dimensional representation $\rho : \pi \to GL_1(\mathbb{F}) = \mathbb{F}^*$, where $\pi = \pi_1(M)$. In other words, one is now given a one-dimensional local system on $M$. Then arguing similarly as in Subsection 2.2 one can construct Barannikov pairs and Bruhat numbers, which are elements of $\mathbb{F}^*/\rho(\pi)$ (without choosing a particular orientation of $f$ these numbers live in $\mathbb{F}^*/\pm\rho(\pi)$). To emphasize the presence of $\rho$ we say "twisted Barannikov pairs" and "twisted Bruhat numbers". One then defines torsion $\tau(f,\rho)$ of $f$ exactly as in Definition 3.1. Generally $\tau(f,\rho)$ may well depend on $f$.

Suppose that twisted homology $\mathsf{H}_\bullet(M;\rho)$ vanishes. Then one can define the Reidemeister torsion of $M$, which is an element of the quotient group $\mathbb{F}^*/\pm\rho(\pi)$. It is a topological invariant, which is, however, not stable under homotopy equivalences. It can be used to classify lens spaces.

**Theorem 3.3.** Let $f$ be a strong Morse function on a manifold $M$, $\mathbb{F}$ be a field and $\rho\colon \pi \to \mathbb{F}^*$ be a one-dimensional representation. Suppose that twisted homology vanishes. Then the alternating product of twisted Bruhat numbers of $f$ equals to the Reidemeister torsion of $M$. In particular, it is independent of $f$.

## 4. One-parameter Morse theory

4.1. *Generalities on one-parameter Morse theory.* In this subsection we recall foundations of one-parameter Morse theory, initiated by Cerf [Cer70].

Fix a generic path $\{f_t\}$ in the space of functions on $M$ once and for all (here $t \in [-1, 1]$). Its endpoints $f_{-1}$ and $f_1$ are strong Morse functions on $M$. Moreover, the same holds for all but finitely many points of $\{f_t\}$. This subsection is devoted to describing what changes may occur to a function at these points.

We will depict paths of functions in the following manner. The Cerf diagram of a path $\{f_t\}$ is a subset of $[-1, 1] \times \mathbb{R}$ consisting of points $(t, x)$ s.t. $x$ is a critical value of $f_t$. Topologically it is a set of (possibly self-intersecting and non-closed) curves in the plane.

As proven in [Cer70] in a generic one-parameter path there are two possible changes of isotopy class of a strong Morse function, which we call events. (Since there are only finitely many of them anyway, we assume for convenience that $f_t$ is strong Morse for all $t$ except for a single value $t = 0$.)

1) At the moment $t = 0$ the birth/death of two points of neighboring indices happens. On a Cerf diagram this corresponds to a (left or right) cusp. This event is called birth/death event.

2) The function $f_0$ is Morse, but not strong. This happens when two critical values collide. On the Cerf diagram this corresponds to a simple transversal self-intersection; in a sense a pair of critical values is swapped. We call this event a Maxwell event.

Now we may describe a Cerf diagram a bit more precisely: it is a set of plane arcs (smooth in the interior) whose endpoints are either at cusps or have $t$ coordinate equal to $\pm 1$. These arcs don't have vertical tangencies and may self-intersect.

4.2. *B-data in path of functions.* In this subsection we start describing how B-data behaves along the generic path of functions. In Subsection 4.3 we finish this description.

First of all, we will orient all the functions in the path in the following way. Pick a generic point on some arc of the Cerf diagram. It corresponds to a critical point of some $f_t$; orient it. Extend this orientation by continuity to all the critical points lying the same arc (excluding the cusps). Apply this procedure to all the arcs. This recipe allows us to orient all the functions in the path $\{f_t\}$ by making only finite number of binary choices, namely $2^l$ where $l$ is the number of arcs. We use the term "orientation of an arc" for short.

Recall that we have to fix a field $\mathbb{F}$ in order to define B-data. Next, if the path $\{f_t\}$ consists of only strong Morse functions, then this data stays the same for all the time.

We use the term "bifurcations" for the description of the way B-data changes after two events from Subsection 4.1. Disregarding the Bruhat numbers, this description was presented already in [Bar94] (see [Lau15] for a different proof). See also the paper [CEM06] and pictures in the survey [EH08]. Thus our job is to determine how Bruhat numbers change along the way. In the case of birth/death event we restrict ourselves to birth one for brevity (death one is obtained from birth one by reversing the time).

**Theorem 4.1.** After the birth event a Barannikov pair of two newborn critical points appears; its Bruhat number is $\pm 1$. All the other pairs and Bruhat numbers remain unaltered. See Figure 4.

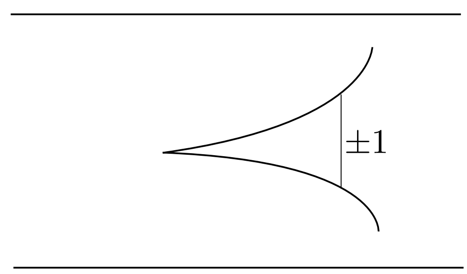


FIGURE 4. Birth of two critical points.

4.3. *Maxwell event.* In this subsection we consider the second type of event, namely self-intersection of a Cerf diagram (in other words, Maxwell event). This finishes the description of bifurcations of B-data in a path of functions started in Subsection 4.2.

Let us fix the notations first. Let $c_{s+1}$ and $c_s$ be two critical points of $f_{-1}$ participating in the bifurcation. Recall from Subsection 4.1 that $\mathrm{Cr}(f_1)$ coincides, as an ordered subset of $M$, with $\mathrm{Cr}(f_{-1})$ with the order of $c_{s+1}$ and $c_s$ reversed. As we will see in Theorem 4.2 some bifurcations can only happen provided that certain restrictions on the linear order of involved critical points are satisfied. These restrictions depend on types of critical points (upper, lower or homological); see also Remark 4.3.

**Theorem 4.2.** After the Maxwell event two types of bifurcations possible.

1) Trivial bifurcation. After it points $c_{s+1}$ and $c_s$ keep their initial pairs (if any) and Bruhat numbers on them. No restrictions on the linear order of points are placed. The values $\deg c_{s+1}$ and $\deg c_s$ may be any.
2) Non-trivial bifurcation. The necessary condition is $\deg c_{s+1} = \deg c_s$. The list of five possible variants is given in Figure 5. Restrictions on the linear order can be deduced from the pictures, see Remark 4.3.

All the points not participating in the bifurcation keep their initial pairs (if any) and Bruhat numbers on them.

**Remark 4.3.** As seen on Figure 5 pairing and Bruhat numbers may well change after the non-trivial bifurcation. As for the restrictions on the linear order, suppose, for example, that both $c_s$ and $c_{s+1}$ are of upper type (picture 3). Then the restriction says that $b(s+1) < b(s)$ (where $b$ is a bijection from the definition of B-data). Note that the same restrictions are involved in the definition of the ruling of a Legendrian knot [Fuc03; CP05]. See [CP05] for discussion.

**Remark 4.4.** Suppose that (twisted) homology of $M$ vanishes in degree $k$. Then there are no homological critical points of index $k$ (see Proposition 2.4). Therefore, non-trivial bifurcation of such points can only be one of the first three types on Figure 5. In turn, this implies that the alternating product of (twisted) Bruhat numbers stays the same after the bifurcation. This provides an alternative proof of the fact that alternating product of twisted Bruhat numbers doesn't depend on the function (assuming homology vanishes).

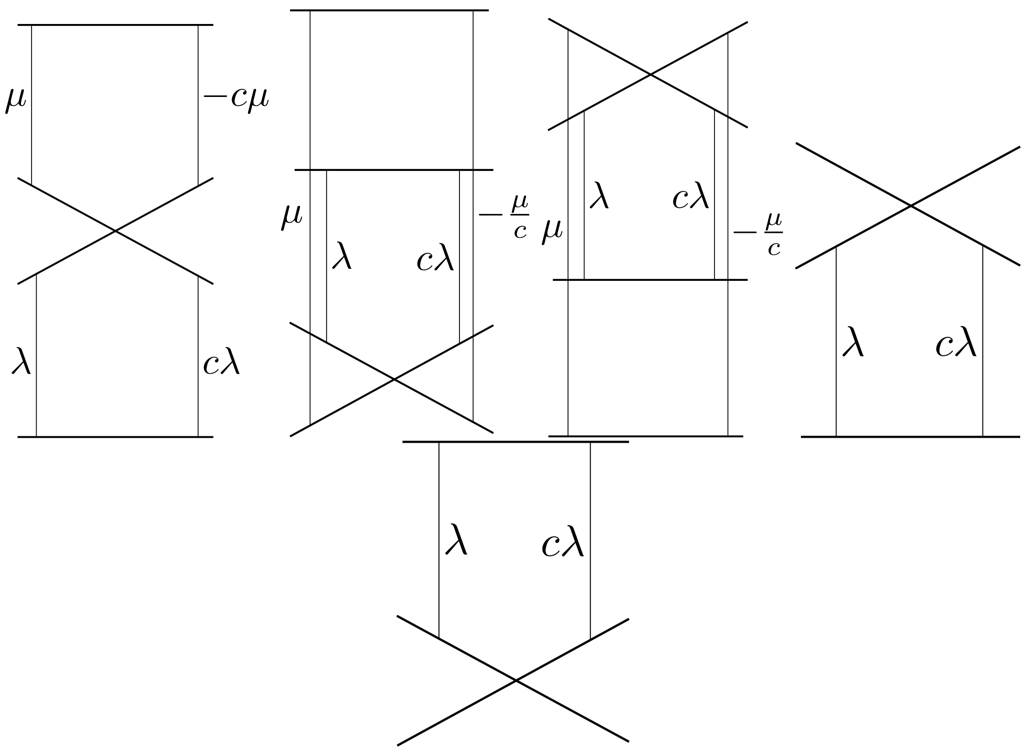


FIGURE 5. Non-trivial bifurcations at the self-intersection of a Cerf diagram

4.4. *A theorem of Akhmetev-Cencelj-Repovs.* In this subsection we state a theorem of Akhmetev-Cencelj-Repovs [ACR05] in greater generality. Roughly it says that two numerical invariants of a generic path of functions must satisfy a certain equation mod 2. It is proven by using Bruhat numbers and analyzing their behaviour in paths of functions (but the formulation doesn't involve these notions).

First of all we need to pass to a bit more general setting. Cobordism is a manifold $M$ with boundary $\partial_0 M \sqcup \partial_1 M$. By a function $f$ on a cobordism $(M, \partial_0 M, \partial_1 M)$ we will mean a function $f\colon M \to [0,1]$ s.t. $f^{-1}(0) = \partial_0 M$ and $f^{-1}(1) = \partial_1 M$. The function $f$ on cobordism is called Morse if all its critical points are non-degenerate and lie in the interior of $M$. Strongness property is defined in the same manner as in the closed case. Trivial cobordism is a cylinder $(N \times [0,1], N \times \{0\}, N \times \{1\})$, where $N$ is a closed manifold.

We will now introduce two invariants of a generic path $\{f_t\}$. The first one is the number of self-intersections of the Cerf diagram (or, in our terminology, the number of Maxwell events), call it $\mathsf{X}$. To get to the second one recall that in Subsection 4.2 we described the procedure of orienting the arcs of a Cerf diagram, which outputs an orientation of each strong Morse function in a path. After orienting the arcs somehow one can assign a sign to each cusp of a Cerf diagram as follows. Let $t_0$ be a point of birth (resp. death) event. Pick any value $t_1 > t_0$ (resp. $t_1 < t_0$) s.t. all functions between $t_1$ and $t_0$ ($t_0$ excluded) are strong Morse. Denote by $c_{s+1}$ and $c_s$ two newborn (resp. about to die) critical points of $f_{t_1}$. It follows from classical results that differential of $c_{s+1}$ contains $c_s$ with coefficient either 1 or $-1$ regardless of choices made (e.g. the choice of a Riemannian metric). The sign of a cusp is now defined as the sign of this number. Let $\mathsf{C}$ be the number of negative cusps. Changing orientation of an arc changes the sign of each cusp which serves as this arc's endpoint (obviously, there are at most two such cusps). Therefore, if both $f_{-1}$ and $f_1$ have no critical points, then the parity of $\mathsf{C}$ is a well-defined invariant of a path $\{f_t\}$. The following corollary asserts a certain relation between two introduced invariants of a path (the number $\mathsf{X}$ and the parity of $\mathsf{C}$).

**Corollary 4.5.** Let $\{f_t\}$ be a generic path of functions on a cylinder $N \times [0,1]$ s.t. both $f_{-1}$ and $f_1$ have no critical points. Let $\mathsf{X}$ be the number of self-intersections of its Cerf diagram and $\mathsf{C}$ be the number of negative cusps. Then one has

$$\mathsf{X} + \mathsf{C} = 0 \pmod 2.$$

**Remark 4.6.** In [ACR05] Corollary 4.5 was proved using two different methods, both requiring additional assumptions on $N$.

See the full text, where we prove a general theorem, which implies the above corollary. This theorem deals a with general cobordism $(M, \partial_0 M, \partial_1 M)$ such that $\mathsf{H}_*(M, \partial_0 M; \mathbb{F}) = 0$ (instead of a cylinder $N \times [0,1]$).

## 5. Approbation of results

Results of this thesis were presented on the following seminars:

- Seminar on geometric topology, Steklov Institute, December 2020
- Topology seminar, University of Georgia, April 2021
- Seminar of Laboratory of algebraic geometry and its applications, HSE, April 2021
- Topology seminar, Dartmouth College, September 2021
- Symplectic geometry seminar, Stanford University, March 2022
- Topology seminar, University of Notre Dame, April 2022

Results were also presented on the following conferences:

- LUTSINOfest, Lutsino, Moscow region, July 2021
- South Central Topology Conference, College Station, TX (rodeo talk), September 2021
- Bridging applied and quantitative topology, online (poster), May 2022
- Richmond geometry festival, online (poster), May 2022

## 6. Publications

Results of this thesis are contained in two papers accepted for publication:

- Petr Pushkar and Mikhail Tyomkin. *On the matrix of differential in the Morse complex*, accepted to Uspekhi Matematicheskikh Nauk (brief communications)
- Petya Pushkar and Misha Tyomkin. *Enhanced Bruhat decomposition and Morse theory*, accepted International Mathematics Research Notices

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