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Formalism of traces in derived algebraic geometry

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1 Categorical preliminaries

We start with the following well-known

Definition 1.1. Let $(\mathcal{E}, \otimes, 1_{\mathcal{E}})$ be a symmetric monoidal $(\infty,1)$-category, and let $X \in \mathcal{E}$ be a dualizable object together with a map $X \xrightarrow{f} Y \otimes X$ where $Y \in \mathcal{E}$ is some object. We then define the **twisted trace** of f as the point in space $\mathrm{Hom}_{\mathcal{E}}(Y, Y)$ given by the composite

$$Y \xrightarrow{\mathsf{coev}_X} X \otimes X^\vee \xrightarrow{f \otimes \mathrm{Id}_{X^\vee}} Y \otimes X \otimes X^\vee \xrightarrow[\sim]{\mathrm{Id}_Y \otimes \mathsf{Twist}} Y \otimes X^\vee \otimes X \xrightarrow{\mathrm{Id}_Y \otimes \mathsf{ev}_X} Y.$$

In the special case $Y = 1_{\mathcal{E}}$, we obtain the well-known notion of trace $\mathsf{Tr}(f) \in \mathrm{Hom}_{\mathcal{E}}(1_{\mathcal{E}}, 1_{\mathcal{E}})$ of an endomorphism of the dualizable object X.

Now note that if $\mathcal{E}$ is a symmetric monoidal $(\infty,2)$-category, we obtain a whole $(\infty,1)$-category $\mathrm{Hom}_{\mathcal{E}}(1_{\mathcal{E}}, 1_{\mathcal{E}})$. One might ask how traces can be used to obtain morphisms in this $(\infty,1)$-category. An answer is provided by the following:

Proposition 1.2 (Morphism of traces)**.** Let $(\mathcal{E}, \otimes, 1_{\mathcal{E}})$ be a symmetric monoidal $(\infty,2)$-category and suppose we are given a (not necessary commutative) diagram

$$\begin{tikzcd}
A \arrow[r, "F_A"] \arrow[d, bend right, "\varphi"'] & A \arrow[d, bend right, "\varphi"'] \arrow[dl, Rightarrow, "T"'] \\
B \arrow[r, "F_B"'] \arrow[u, bend right, "\psi"'] & B \arrow[u, bend right, "\psi"']
\end{tikzcd}$$

in $\mathcal{E}$, where $A, B \in \mathcal{E}$ are dualizable object, the morphism φ is left adjoint to ψ and

$$\varphi \circ F_A \xrightarrow{T} F_B \circ \varphi$$

is a 2-morphism in $\mathcal{E}$. Then there exist a natural morphism

$$\mathsf{Tr}_{\mathcal{E}}(F_A) \xrightarrow{\mathsf{Tr}(\varphi, T)} \mathsf{Tr}_{\mathcal{E}}(F_B)$$

in the $(\infty,1)$-category $\mathrm{Hom}_{\mathcal{E}}(1_{\mathcal{E}}, 1_{\mathcal{E}})$ called a **morphism of traces induced by** T.
Moreover, given a diagram

$$\begin{tikzcd}
A \arrow[r, "F_A"] \arrow[d, bend right, "\varphi_1"'] & A \arrow[d, bend right, "\varphi_1"'] \arrow[dl, Rightarrow, "T_1"'] \\
B \arrow[r, "F_B"'] \arrow[u, bend right, "\psi_1"'] \arrow[d, bend right, "\varphi_2"'] & B \arrow[u, bend right, "\psi_1"'] \arrow[d, bend right, "\varphi_2"'] \arrow[dl, Rightarrow, "T_2"'] \\
C \arrow[r, "F_C"'] \arrow[u, bend right, "\psi_2"'] & C \arrow[u, bend right, "\psi_2"']
\end{tikzcd}$$

in $\mathcal{E}$, where φ_1 is left adjoint to ψ_1, φ_2 is left adjoint to ψ_2 and

$$\varphi_1 \circ F_A \xrightarrow{T_1} F_B \circ \varphi_1$$

$$\varphi_2 \circ F_B \xrightarrow{T_2} F_C \circ \varphi_2$$

are 2-morphisms, there is an equivalence

$$\mathsf{Tr}(\varphi_2 \circ \varphi_1, T_2 \circ_{\mathsf{vert}} T_1) \simeq \mathsf{Tr}(\varphi_2, T_2) \circ \mathsf{Tr}(\varphi_1, T_1)$$

where $\circ_{\mathsf{vert}}$ is the vertical composition of 2-morphisms.

Example 1.3 (Categorical Chern character)**.** Consider the case $\mathcal{E} = 2\,\mathrm{Cat}_k$ is the $(\infty, 2)$-category of k-linear stable presentable $(\infty, 1)$-categories and continuous functors, whose monoidal unit is given by the $(\infty, 1)$-category Vect_k of unbounded cochain complexes, and let $\mathcal{C} \in 2\,\mathrm{Cat}_k$ be some dualizable object together with an endofunctor $\mathcal{C} \xrightarrow{F} \mathcal{C}$. Note that there is a canonical equivalence

$$\mathrm{Fun}_{2\,\mathrm{Cat}_k}(\mathrm{Vect}_k, \mathcal{C})^{\mathsf{ladj}} \xrightarrow[\sim]{\mathsf{ev}_k} \mathcal{C}^{\mathsf{comp}}$$

where $\mathrm{Fun}_{2\,\mathrm{Cat}_k}(\mathrm{Vect}_k, \mathcal{C})^{\mathsf{ladj}} \subseteq \mathrm{Fun}_{2\,\mathrm{Cat}_k}(\mathrm{Vect}_k, \mathcal{C})$ is the full $(\infty, 1)$-subcategory spanned by those morphisms in $2\,\mathrm{Cat}_k$ which admit a right adjoint, and $\mathcal{C}^{\mathsf{comp}} \subseteq \mathcal{C}$ is the full $(\infty, 1)$-subcategory of compact objects.

In particular, given a compact object $E \in \mathcal{C}^{\mathsf{comp}}$ together with a morphism $E \xrightarrow{t} F(E)$ in $\mathcal{C}$ we can apply the $(\infty, 2)$-categorical trace construction 1.2 to the diagram

$$\begin{tikzcd}
\mathrm{Vect}_k \arrow[r, "\mathrm{Id}_{\mathrm{Vect}_k}"] \arrow[d, bend right, "\varphi"'] & \mathrm{Vect}_k \arrow[dl, Rightarrow, "T"'] \arrow[d, bend right, "\varphi"'] \\
\mathcal{C} \arrow[u, bend right, "\psi"'] \arrow[r, "F"'] & \mathcal{C} \arrow[u, bend right, "\psi"']
\end{tikzcd}$$

where φ is the functor obtained from the compact object $E \in \mathcal{C}^{\mathsf{comp}}$ and T is the 2-morphism obtained from t. The corresponding element

$$k \simeq \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{Vect}_k}) \xrightarrow{\mathsf{Tr}(\varphi, T)} \mathsf{Tr}_{2\,\mathrm{Cat}_k}(F) \in \mathrm{Hom}_{2\,\mathrm{Cat}_k}(\mathrm{Vect}_k, \mathrm{Vect}_k) \simeq \mathrm{Vect}_k$$

is called the **categorical Chern character of** E and is denoted by $\mathrm{ch}(E, t) \in \mathsf{Tr}_{2\,\mathrm{Cat}_k}(F)$.

2 2-traces in derived algebraic geometry

Convention. For the rest of the document we assume k is an algebraically closed base field of characteristic 0.

The notion of trace is extremely useful in the setting of derived algebraic geometry. For a prestack X we denote by $\mathrm{QCoh}(X)$ the $(\infty, 1)$-category of unbounded cochain complexes of quasi-coherent sheaves. By [BZFN10, Theorem 1.2] for any perfect derived stacks X, Y (see [BZFN10, Definition 3.2]) there is a canonical equivalence $\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y) \simeq \mathrm{QCoh}(X \times Y)$ obtained from the bicontinuous functor

$$\mathrm{QCoh}(X) \times \mathrm{QCoh}(Y) \xrightarrow[\sim]{} \mathrm{QCoh}(X \times Y)$$

$$(\mathcal{F}, \mathcal{G}) \longmapsto (q_1^* \mathcal{F}) \otimes (q_2^* \mathcal{G})$$

where

$$X \xleftarrow{q_1} X \times Y \xrightarrow{q_2} Y$$

are the projection maps. In particular, the object $\mathrm{QCoh}(X) \in \mathrm{Cat}_k$ is self-dual, with the unit and counit maps given by

$$\mathrm{Vect}_k \xrightarrow{\Delta_* \mathcal{O}_X} \mathrm{QCoh}(X \times X) \simeq \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X)$$

$$\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X \times X) \xrightarrow{\Gamma(\Delta^* -)} \mathrm{Vect}_k$$

where $X \xrightarrow{\Delta} X \times X$ is the diagonal map and $\mathrm{QCoh}(X) \xrightarrow{\Gamma(-)} \mathrm{Vect}_k$ is the (derived) global sections functor.

A convenient way to calculate traces of various endomorphisms of the dualizable object $\mathrm{QCoh}(X) \in \mathrm{Cat}_k$ is provided by the formalism of kernels. Namely, by [BZFN10, Theorem 1.2] there is an equivalence

$$\mathrm{QCoh}(X \times X) \xrightarrow[\sim]{} \mathrm{Fun}_{\mathrm{Cat}_k}\big(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)\big)$$

$$\mathcal{K} \longmapsto q_{2*}(\mathcal{K} \otimes (q_1^* -))$$

of $(\infty, 1)$-categories. The sheaf $\mathcal{K}$ is frequently called the **kernel** of the corresponding functor. Unwinding the constructions, one obtains the following

Lemma 2.1 (Trace via kernel)**.** Let X be a perfect derived stack and F be an endomorphism of $\mathrm{QCoh}(X)$. Then there is an equivalence

$$\mathsf{Tr}_{\mathrm{Cat}_k}(F) \simeq \Gamma(X, \Delta^* \mathcal{K}) \in \mathrm{Vect}_k$$

where $\mathcal{K} \in \mathrm{QCoh}(X \times X)$ is the kernel of F.

It is now straightforward to see that notion of trace allows us to recover derived fixed points schemes in the setting of derived algebraic geometry:

Proposition 2.2 (Fixed points from traces)**.** Let $X \xleftarrow{g} Y \xrightarrow{f} X$ be a correspondence of perfect stacks. Then for a sheaf $\mathcal{G} \in \mathrm{QCoh}(Y)$ there is a canonical equivalence

$$\mathsf{Tr}_{\mathrm{Cat}_k}\big(f_*(\mathcal{G} \otimes g^* -)\big) \simeq \Gamma(Y^{g=f}, j^* \mathcal{G})$$

in Vect_k, where $Y^{g=f}$ is the **derived fixed points stack** of (g, f) defined as the pullback

$$\begin{tikzcd} Y^{g=f} \arrow[r, "j"] \arrow[d, "i"'] & Y \arrow[d, "{(g,f)}"] \\ X \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

of derived stacks.

The proposition above combined with Lemma 2.1 also gives a convenient way calculate various categorical Chern characters (Example 1.3) in the setting of derived algebraic geometry:

Example 2.3 (Categorical Chern character for lax equivariant sheaf)**.** Let $X \xleftarrow{g} Y \xrightarrow{f} X$ be a correspondence of perfect derived stacks and $E \in \mathrm{Perf}(X)$ be a perfect sheaf (by [BZFN10, 3.1] equivalently compact/dualizable object of $\mathrm{QCoh}(X)$) equipped with a map $t\colon E \to f_*(\mathcal{G} \otimes g^* E)$ for some $\mathcal{G} \in \mathrm{QCoh}(Y)$. Then the categorical Chern character $\mathrm{ch}(E, t)$ (1.3) of E obtained from the diagram

$$\begin{tikzcd} \mathrm{Vect}_k \arrow[rr, "\mathrm{Id}_{\mathrm{Vect}_k}"] \arrow[d, bend right, "\varphi"'] & & \mathrm{Vect}_k \arrow[d, bend right, "\varphi"'] \arrow[dll, Rightarrow, "T"'] \\ \mathrm{QCoh}(X) \arrow[u, bend right, "\psi"'] \arrow[rr, "f_*(\mathcal{G} \otimes g^* -)"'] & & \mathrm{QCoh}(X) \arrow[u, bend right, "\psi"'] \end{tikzcd}$$

is equivalent to the twisted trace (see Definition 1.1) of the induced map

$$i^* E \simeq j^* f^* E \xrightarrow{j^*(b)} j^*(\mathcal{G} \otimes g^* E) \simeq j^* \mathcal{G} \otimes j^* g^* E \simeq j^* \mathcal{G} \otimes i^* E$$

in $\mathrm{QCoh}(Y^{g=f})$, where $b\colon f^* E \to \mathcal{G} \otimes g^* E$ is the morphism which corresponds to $t \in \mathrm{Hom}_{\mathrm{QCoh}(X)}\big(E, f_*(\mathcal{G} \otimes g^* E)\big)$ via the adjunction $f^* \dashv f_*$.

3 Holomorphic Atiyah–Bott formula for correspondences

As a first application of the above techniques, we prove a version of Holomorphic Atiyah–Bott formula for correspondences. Let X, Y be a pair of smooth proper k-schemes, and let $(g, f)\colon Y \to X \times X$ be a correspondence such that:

- The underlying classical scheme $Y^{g=f,\mathsf{cl}}$ of the corresponding derived fixed point stack 2.2 is discrete.
- The morphism g is étale at the fixed points (that is, the morphism g is étale at each point $y \in Y^{g=f,\mathsf{cl}}$).
- The induced map on tangent spaces $1 - d_y f \circ (d_y g)^{-1}$ is invertible for all $y \in Y^{g=f,\mathsf{cl}}$.

For any lax (g, f)-equivariant perfect sheaf $(E \in \mathrm{Perf}(X), b\colon f^*E \to g^!E)$ we obtain the diagram

$$\begin{tikzcd}
\mathrm{Vect}_k \arrow[r, "\mathrm{Id}_{\mathrm{Vect}_k}"] \arrow[d, "E"'] & \mathrm{Vect}_k \arrow[d, "E"] \arrow[dl, Rightarrow, "T_1"'] \\
\mathrm{QCoh}(X) \arrow[r, "f_* g^!"'] \arrow[d, "\Gamma"'] & \mathrm{QCoh}(X) \arrow[d, "\Gamma"] \arrow[dl, Rightarrow, "T_2"] \\
\mathrm{Vect}_k \arrow[r, "\mathrm{Id}_{\mathrm{Vect}_k}"'] & \mathrm{Vect}_k
\end{tikzcd}$$

in $2\,\mathrm{Cat}_k$, and hence by applying the 2-trace formalism 1.2 a commutative triangle

$$\begin{tikzcd}
k \arrow[r, "\mathrm{ch}(E,t)"] \arrow[dr, "\mathsf{Tr}(\Gamma(X,E), T_2 \circ_{\mathsf{vert}} T_1)"'] & \mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_* g^!) \arrow[d, "\mathsf{Tr}(\Gamma, T_2)"] \\
& k
\end{tikzcd}$$

in Vect_k, that is, an equality

$$\mathsf{Tr}(\Gamma, T_2) \circ \mathrm{ch}(E, t) \simeq \mathsf{Tr}(\Gamma(X, E), T_2 \circ_{\mathsf{vert}} T_1) \tag{1}$$

of two numbers.

Since under our assumptions by Proposition 2.2 we have

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_* g^!) \simeq \Gamma(Y^{g=f}, j^*\omega_g) \simeq \Gamma(Y^{g=f}, \mathcal{O}_{Y^{g=f}}) \simeq \bigoplus_{f(y)=g(y)} k e_y$$

where $e_y := \Gamma(\{y\}, \mathcal{O}_y)$, and by Example 2.3 we have that

$$\mathrm{ch}(E, t) = \textstyle\sum_{f(y)=g(y)} \mathrm{ch}(E, t)_y e_y, \qquad \mathrm{ch}(E, t)_y \simeq \mathsf{Tr}_{\mathrm{Vect}_k}(E_{f(y)} \xrightarrow{b_y} E_{g(y)}),$$

to get a complete description of the equality 1 it suffices to understand the value of the map

$$\int_{Y^{g=f}} : \bigoplus_{f(y)=g(y)} k e_y \simeq \mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_* g^!) \xrightarrow{\mathsf{Tr}(\Gamma, T_2)} k$$

on e_y. By using the functoriality of the construction, it suffices to consider the special case $E := x_* k$ is a skyscraper sheaf at a fixed point $x = f(y) = g(y)$. By plugging $x_* k$ in 1 and unwinding the construction, this gives

Theorem 3.1 (Holomorphic Atiyah–Bott formula for correspondences)**.** The equality 1 is concretely given by

$$\mathsf{L}(E, b) = \sum_{f(y)=g(y)} \frac{\mathsf{Tr}_{\mathrm{Vect}_k}(E_{f(y)} \xrightarrow{b_y} E_{g(y)})}{\det(1 - d_y f \circ (d_y g)^{-1})}. \tag{2}$$

where here $\mathsf{L}(E, b) \in k$ is the **Lefschetz number** of (E, b), defined as the trace in Vect_k of the corresponding endomorphism

$$\Gamma(X, E) \longrightarrow \Gamma(X, f_* f^* E) \simeq \Gamma(Y, f^* E) \xrightarrow{\Gamma(Y, b)} \Gamma(Y, g^! E) \simeq \Gamma(X, g_* g^! E) \longrightarrow \Gamma(X, E)$$

on global sections of E.

4 Ind-coherent sheaves

In order to proceed further we briefly review some basic facts concerning ind-coherent sheaves from [GR17a, Part II] and [Gai13]. For $X \in \mathrm{Sch}_{\mathsf{aft}}$ (see [GR17a, Chapter 4, 1.1.1]) we define the $(\infty,1)$-category of **ind-coherent sheaves on** X denoted by $\mathrm{ICoh}(X)$ simply as

$$\mathrm{ICoh}(X) := \mathrm{Ind}(\mathrm{Coh}(X)),$$

where we denote by $\mathrm{Coh}(X)$ the $(\infty,1)$-category of coherent sheaves on X.

We will be interested by the following properties of this construction:

Proposition 4.1.
1) ([GR17a, Chapter 4, Proposition 2.1.2, Proposition 2.2.3]) The assignment of ind-coherent sheaves can be lifted to a functor

$$\mathrm{Sch}_{\mathsf{aft}} \xrightarrow{\mathrm{ICoh}_*} \mathrm{Cat}_k$$

such that, moreover, for every morphism $X \xrightarrow{f} Y$ in $\mathrm{Sch}_{\mathsf{aft}}$ the diagram

$$\begin{tikzcd} \mathrm{ICoh}(X) \arrow[r, "\Psi_X"] \arrow[d, "f_*"'] & \mathrm{QCoh}(X) \arrow[d, "f_*"] \\ \mathrm{ICoh}(Y) \arrow[r, "\Psi_Y"'] & \mathrm{QCoh}(Y) \end{tikzcd}$$

commutes, where $\mathrm{ICoh}(X) \xrightarrow{\Psi_X} \mathrm{QCoh}(X)$ is obtained by ind-extending the natural inclusion $\mathrm{Coh}(X) \subseteq \mathrm{QCoh}(X)$ (and similar for Y).

2) ([GR17a, Chapter 4, Corollary 5.1.12]) The assignment of ind-coherent sheaves can be lifted to a functor

$$\mathrm{Sch}^{\mathsf{op}}_{\mathsf{aft,proper}} \xrightarrow{\mathrm{ICoh}^!} \mathrm{Cat}_k,$$

such that, moreover, given a proper morphism $X \xrightarrow{f} Y$ in $\mathrm{Sch}_{\mathsf{aft}}$ the induced pullback functor $f^! := \mathrm{ICoh}^!(f)$ is right adjoint to f_*.

3) ([GR17a, Chapter 4, Proposition 6.3.7; Chapter 5, Theorem 4.2.5]) For every $X \in \mathrm{Sch}_{\mathsf{aft}}$ the $(\infty,1)$-category $\mathrm{ICoh}(X)$ is symmetric monoidal, and for every proper $X \xrightarrow{f} Y$ the induced functor $f^!$ is symmetric monoidal. The monoidal unit is given by $\omega_X^{\mathrm{ICoh}} \simeq p^! k$, where $X \xrightarrow{p} *$ is the projection and $k \in \mathrm{ICoh}(*) \simeq \mathrm{Vect}_k$. Moreover, $\mathrm{ICoh}(X)$ is self-dual as an object of Cat_k.

Example 4.2. Let X be a smooth classical scheme. By [GR17a, Lemma 1.1.3] in this case the canonical functor $\mathrm{ICoh}(X) \xrightarrow{\Psi_X} \mathrm{QCoh}(X)$ is an equivalence of $(\infty,1)$-categories. In particular we can identify $\mathrm{ICoh}(X)$ with $\mathrm{QCoh}(X)$ with the twisted monoidal structure

$$\mathcal{F} \overset{!}{\otimes} G \simeq \mathcal{F} \otimes \mathcal{G} \otimes \omega_X^{-1},$$

where here $\omega_X \in \mathrm{QCoh}(X)$ is the QCoh-dualizing sheaf.

We now discuss an example of morphism of traces between the categories of ind-coherent sheaves. By [GR17a, Chapter 5, Theorem 4.1.2] parts $2,3$ of the above proposition can be strengthened: the ind-coherent sheaves functor can be lifted to a symmetric monoidal functor

$$\mathsf{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}} \longrightarrow 2\,\mathrm{Cat}_k,$$

where $\mathsf{Corr}(\mathrm{Sch}_{\mathsf{aft}})$ is a symmetric monoidal $(\infty,2)$-category of correspondences. We refer to [GR17a, Chapter 7, Chapter 5] for a throughout discussion of the category of correspondences.

Now by calculating the morphism of traces 1.2 in the $(\infty,2)$-category of correspondences (where it is a pure diagram chase), we obtain the following

Proposition 4.3. Let $(X, g_X) \xrightarrow{f} (Y, g_Y)$ be an equivariant proper morphism in $\mathrm{Sch}_{\mathsf{aft}}$. Then the induced morphism of traces

$$\Gamma(X^{g_X}, \omega_{X^{g_X}}) \simeq \mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_{X*}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_*)} \mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_{Y*}) \simeq \Gamma(Y^{g_Y}, \omega_{Y^{g_Y}})$$

can be obtained by applying the global sections functor $\Gamma(Y^{g_Y}, -)$ to the morphism

$$(f^g)_* \omega_{X^{g_X}} \simeq (f^g)_*(f^g)^! \omega_{Y^{g_Y}} \longrightarrow \omega_{Y^{g_Y}}$$

in $\mathrm{ICoh}(X)$ induced by the counit of the adjunction $(f^g)_* \dashv (f^g)^!$, where $X^{g_X} \xrightarrow{f^g} Y^{g_Y}$ is the induced by f morphism on derived fixed points.

5 The categorical Chern character as the classical one

In this section we discuss how the categorical Chern character Example 1.3 is related to the classical one.

Let X be a quasi-compact scheme and (E, t) be a pair consisting of a perfect sheaf $E \in \mathrm{QCoh}(X)$ and an endomorphism $t : E \longrightarrow E$. By applying the formalism of 2-traces to the induced diagram

$$\begin{tikzcd}
\mathrm{Vect}_k \arrow[r, "\mathrm{Id}_{\mathrm{Vect}_k}"] \arrow[d, bend right, "\varphi"'] & \mathrm{Vect}_k \arrow[dl, Rightarrow, "T"'] \arrow[d, bend right, "\varphi"'] \\
\mathrm{QCoh}(X) \arrow[r, "\mathrm{Id}_{\mathrm{QCoh}(X)}"'] \arrow[u, bend right, "\psi"'] & \mathrm{QCoh}(X) \arrow[u, bend right, "\psi"']
\end{tikzcd}$$

we obtain the categorical Chern character

$$\mathrm{ch}(E, t) \in \Gamma(X^{\mathrm{Id}_X = \mathrm{Id}_X}, \mathcal{O}_{X^{\mathrm{Id}_X = \mathrm{Id}_X}}),$$

where here

$$X^{\mathrm{Id}_X = \mathrm{Id}_X} := X \times_{X \times X} X$$

is the derived self-intersection of the diagonal. We will further denote this pullback by $\mathcal{L}X$ and call it the **derived loop space of** X motivated by the equivalence $\mathcal{L}X \simeq \mathrm{Map}(S^1, X)$ of derived stacks.

By using Example 2.3 we in instantly obtain the following

Proposition 5.1. There is an equivalence

$$\mathrm{ch}(E, t) \simeq \mathsf{Tr}_{\mathrm{QCoh}(\mathcal{L}X)}\left(i^*E \overset{\beta}{\simeq} i^*E \xrightarrow{i^*(t)} i^*(E) \right),$$

where here the equivalence β is obtained from the canonical equivalence $\mathrm{Id}_X \circ i \simeq i$ on derived fixed points of the identity morphism.

Our goal is now to give a concrete description of the equivalence β. The main idea here is that derived loop space

$$\mathcal{L}X \xrightarrow{i} X$$

of X is a formal group over X (where the group structure is given by composition of loops). This is relevant due to the following

Theorem 5.2 ([GR17b, Chapter 7, Theorem 3.6.2, Proposition 5.1.2, and Corollary 3.2.2])**.**

1. There is an equivalence of $(\infty, 1)$-categories

$$\mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X}) \xrightarrow[\sim]{\mathrm{Lie}_X} \mathrm{LAlg}(\mathrm{ICoh}(X)),$$

   between the $(\infty, 1)$-category of formal groups over X and Lie algebras in $\mathrm{ICoh}(X)$. Moreover, for a formal group $\widehat{G} \in \mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X})$ the underlying ind-coherent sheaf of $\mathrm{Lie}_X(\widehat{G}) \in \mathrm{LAlg}(\mathrm{ICoh}(X))$ is equivalent to $\mathbb{T}_{\widehat{G}/X,e} := e^!\mathbb{T}_{\widehat{G}/X}$, where $X \xrightarrow{e} \widehat{G}$ is the unit section and $\mathbb{T}$ denotes tangent sheaf.

2. For $\widehat{G} \in \mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X})$ there is an equivalence of $(\infty, 1)$-categories

$$\mathrm{Rep}_{\widehat{G}}(\mathrm{ICoh}(X)) \xrightarrow[\sim]{} \mathrm{Mod}_{\mathrm{Lie}_X(\widehat{G})}(\mathrm{ICoh}(X)).$$

3. Let $\widehat{G} \in \mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X})$ be a formal group over X. Then there is a functorial equivalence

$$\mathbb{V}(\mathrm{Lie}_X(\widehat{G})) \xrightarrow[\sim]{\exp_{\widehat{G}}} \widehat{G}$$

   of formal moduli problems over X, where $\mathbb{V}(\mathrm{Lie}_X(\widehat{G}))$ is the vector prestack of $\mathrm{Lie}_X(\widehat{G})$.

Corollary 5.3. [Hochschild-Kostant-Rosenberg] By combining 4.2 and the third part of the above theorem we see that for a smooth scheme X we have an equivalence

$$\mathrm{Spec}_{/X}\,\mathrm{Sym}(\mathbb{L}_X[1]) \xrightarrow[\sim]{} \mathcal{L}X$$

of formal moduli problems over X. In particular, we obtain an equivalence

$$\pi_0\Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) \simeq \bigoplus_{p=0}^{\dim X} H^p(X, \Omega^p_X).$$

Now note that any $E \in \mathrm{QCoh}(X)$ admits canonical $\mathcal{L}X$-equivariant structure given by

$$\mathrm{QCoh}(X) \xrightarrow{q_2^*} \mathrm{QCoh}(X \times X) \xrightarrow{c^*} \mathrm{QCoh}(B_{/X}\mathcal{L}X) = \mathrm{Rep}_{\mathcal{L}X}(\mathrm{QCoh}(X))$$

where here $B_{/X}\mathcal{L}X \simeq (\widehat{X \times X})_\Delta \in \mathrm{PreStack}_{/X}$ is the delooping of $\mathcal{L}X$ over X (as a formal moduli problem), and $B_{/X}\mathcal{L}X \xrightarrow{c} X \times X$ is the canonical morphism. Moreover, by diagram chase we obtain the following

Proposition 5.4. The endomorphism $i^*E \overset{\beta}{\simeq} i^*E$ is equivalent to the pullback $i^*\alpha_E$, where α_E is the action morphism of $\mathcal{L}X$ on E.

Now by combining the second part of Theorem 5.2 and Example 4.2 we see that for smooth X the action morphism α_E can be described in terms of the action of corresponding Lie algebra $\mathrm{Lie}_X(\mathcal{L}X) \simeq \mathbb{T}_X[-1]$ in $\mathrm{QCoh}(X)$. This action can be described concretely: it is straightforward to see that in the special case $X = B\mathbb{G}_m$ it is given by the first Chern class, and hence by using the splitting principle we obtain

Proposition 5.5 (Categorical Chern character explicitly)**.** Let X be a smooth proper scheme and $E \in \mathrm{QCoh}(X)$ is a perfect sheaf with an endomorphism $E \xrightarrow{t} E$. Then under the Hochschild-Kostant-Rosenberg identification 5.3 we have an equality

$$\mathrm{ch}(E, t) = \mathsf{Tr}_{\mathrm{QCoh}(\mathcal{L}X)}\left(i^*E \xrightarrow[\sim]{\exp(\mathrm{At}(E))} i^*E \xrightarrow{i^*(t)} i^*E \right)$$

of elements of $\bigoplus_p H^p(X, \Omega^p_X)$, where here $\mathrm{At}(E)$ is the classical Atiyah class of E.

Corollary 5.6. Let E be a dualizable object of $\mathrm{QCoh}(X)$. Then under the Hochschild-Kostant-Rosenberg isomorphism the categorical Chern character $\mathrm{ch}(E, \mathrm{Id}_E)$ coincides with the classical one $\mathrm{ch}(E)$.

6 From morphisms of traces to Todd class

Let X be a smooth, proper scheme. Our goal in this section is to understand the morphism of traces

$$\Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) \xrightarrow[\sim]{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes\mathcal{O}_X)} \Gamma(\mathcal{L}X, \omega_{\mathcal{L}X})$$

induced by the diagram

$$\begin{tikzcd} \mathrm{QCoh}(X) \arrow[r, "\mathrm{Id}_{\mathrm{QCoh}(X)}"] \arrow[d, "-\otimes\mathcal{O}_X"'] & \mathrm{QCoh}(X) \arrow[d, "-\otimes\mathcal{O}_X"] \\ \mathrm{ICoh}(X) \arrow[r, "\mathrm{Id}_{\mathrm{ICoh}(X)}"'] & \mathrm{ICoh}(X). \end{tikzcd}$$

We start with the following

Definition 6.1. For an almost finite type scheme Z an **orientation** on Z is a choice of an equivalence $\mathcal{O}_Z \simeq \omega_Z$ in $\mathrm{QCoh}(Z)$.

Remark 6.2. Note that any orientation $u : \mathcal{O}_Z \simeq \omega_Z$ produces an equivalence

$$\Gamma(Z, \mathcal{O}_Z) \xrightarrow[\sim]{u} \Gamma(Z, \omega_Z) \simeq \Gamma(Z, \mathcal{O}_Z)^\vee.$$

Example 6.3 (Serre orientation)**.** Let $\mathcal{L}X \xrightarrow{i} X$ be the canonical map. Projection to the top summand

$$i_*\mathcal{O}_{\mathcal{L}X} \simeq \mathrm{Sym}_{\mathrm{QCoh}(X)}(\Omega_X[1]) \longrightarrow \omega_X$$

produces an equivalence

$$u_S : \mathcal{O}_{\mathcal{L}X} \xrightarrow[\sim]{} i^!\omega_X \simeq \omega_{\mathcal{L}X}$$

called **Serre orientation**.

The induced equivalence

$$\bigoplus_p \Gamma(X, \Omega_X^p[p]) \simeq \Gamma(X, \mathcal{O}_{\mathcal{L}X}) \simeq \Gamma(X, \mathcal{O}_{\mathcal{L}X})^\vee \simeq \left(\bigoplus_p \Gamma(X, \Omega_X^p[p])\right)^\vee$$

is given by the Poincaré duality.

Example 6.4 (Canonical orientation)**.** For any endomorphism $X \xrightarrow{g} X$ the derived fixed-point scheme X^g from Proposition 2.2 admits an orientation given by the series of equivalences

$$u_C : \mathcal{O}_{X^g} \simeq i^*\omega_X \otimes i^*\omega_X^{-1} \simeq i^*\omega_X \otimes i^*\omega_{X/X\times X} \simeq i^*\omega_X \otimes \omega_{X^g/X} \simeq i^!\omega_X \simeq \omega_{X^g}.$$

We call this orientation **canonical**.

Corollary 6.5. We see that $\mathcal{L}X = X^{\mathrm{Id}_X}$ also admits canonical orientation.

By using a version of the $(\infty, 2)$-category of correspondences, we prove the following

Theorem 6.6. The morphism of traces

$$\bigoplus_p \Gamma(X, \Omega_X^p[p]) \simeq \mathsf{Tr}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \xrightarrow{\mathsf{Tr}(\mathrm{Id}_{(-\otimes\mathcal{O}_X)})} \mathsf{Tr}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \simeq \left(\bigoplus_p \Gamma(X, \Omega_X^p[p])\right)^\vee$$

induced by $\mathrm{QCoh}(X) \xrightarrow{-\otimes\mathcal{O}_X} \mathrm{ICoh}(X)$ is obtained from the canonical orientation u_C on $\mathcal{L}X$.

Corollary 6.7. The morphism of traces

$$\bigoplus_p \Gamma(X, \Omega_X^p[p]) \xrightarrow{\mathsf{Tr}(\mathrm{Id}_{(-\otimes\mathcal{O}_X)})} \left(\bigoplus_p \Gamma(X, \Omega_X^p[p])\right)^\vee \overset{\text{Poincaré}}{\simeq} \bigoplus_p \Gamma(X, \Omega_X^p[p]).$$

is obtained from the composite $u_S^{-1} \circ u_C$. To put it differently, it measures the difference between the canonical and the Serre orientation.

Finally, we give an explicit description of the composite $u_S^{-1} \circ u_C$.

Recall that $\mathcal{L}X \xrightarrow{i} X$ is a formal group over X, with the corresponding Lie algebra given by $\mathbb{T}_X[-1]$. We have:

1. Given a trivialization $\mathbb{T}_{\mathcal{L}X/X} \simeq i^*\mathbb{T}_X[-1]$ of $\mathbb{T}_{\mathcal{L}X/X}$ one can build an orientation of $\mathcal{L}X$.
2. In terms of this construction, the canonical and Serre orientations of $\mathcal{L}X$ can be obtained by using the canonical and the abelian Lie algebra structure on $\mathbb{T}_X[-1]$.

Now unwinding the constructions, one checks that the composite $u_S^{-1} \circ u_C$ in fact measures the difference between the canonical and the abelian Lie algebra structures on $\mathbb{T}_X[-1]$:

Corollary 6.8. The equivalence $u_S^{-1} \circ u_C$ can be obtained by applying the determinant map to the equivalence

$$d\exp_{\mathcal{L}X} : i^*\mathbb{T}_X[-1] \overset{can}{\simeq} \mathbb{T}_{\mathcal{L}X/X} \overset{ab}{\simeq} i^*\mathbb{T}_X[-1].$$

Analogous to a similar statement in the world of real Lie groups and corresponding Lie algebras, we prove that in the world of formal groups we have

Theorem 6.9. Let $\widehat{G}$ be a formal group over X such that $\mathfrak{g} := \mathrm{Lie}_X(\widehat{G}) \in \mathrm{Coh}^{<0}$. Then

$$d\exp_{\widehat{G}} = \frac{1 - e^{-\operatorname{ad}_{\mathfrak{g}}}}{\operatorname{ad}_{\mathfrak{g}}}.$$

By combining Corollary 6.7, Corollary 6.8 and Theorem 6.9, this gives

Theorem 6.10. Let X be a smooth, proper scheme. Then under the Serre orientation and HKR identifications the morphism of traces

$$\bigoplus_p \Gamma(X, \Omega_X^p[p]) \simeq \pi_* \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes\mathcal{O}_X)} \pi_* \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)}) \simeq \bigoplus_p \Gamma(X, \Omega_X^p[p])$$

is given by multiplication with the Todd class td_X.

7 Grothendieck-Riemann-Roch theorem

We now show how one obtains the classical and the equivariant versions of the Grothendieck-Riemann-Roch theorem by using the formalism of 2-traces.

Let $X \xrightarrow{f} Y$ be a morphism of smooth proper k-schemes and let E be a perfect sheaf on X. By applying functoriality of traces to the diagram

$$\begin{tikzcd}
\mathrm{Vect}_k \arrow[r, "E\otimes-"] & \mathrm{QCoh}(X) \arrow[r, "f_*"] \arrow[d, "\sim"', "\otimes\mathcal{O}_X"] & \mathrm{QCoh}(Y) \arrow[d, "\sim"', "\otimes\mathcal{O}_Y"] \\
& \mathrm{ICoh}(X) \arrow[r, "f_*"'] & \mathrm{ICoh}(Y)
\end{tikzcd}$$

in $2\,\mathrm{Cat}_k$, we obtain a commutative diagram of traces

$$\begin{tikzcd}
k \arrow[r, "\mathsf{Tr}_{2\,\mathrm{Cat}_k}(E\otimes-)"'] \arrow[rr, bend left, "\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_*(E)\otimes-)"] & \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \arrow[r] \arrow[d, "\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes\mathcal{O}_X)"] & \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(Y)}) \arrow[d, "\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes\mathcal{O}_Y)"] \\
& \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)}) \arrow[r, "\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_*)"'] & \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)})
\end{tikzcd} \tag{3}$$

in Vect_k. Now:

- By Corollary 5.6 under the identifications

$$\pi_0 \operatorname{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \simeq \bigoplus_p H^p(X, \Omega^p_X) \qquad \pi_0 \operatorname{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(Y)}) \simeq \bigoplus_p H^p(Y, \Omega^p_Y).$$

the traces $\operatorname{Tr}_{2\,\mathrm{Cat}_k}(E \otimes -)$ and $\operatorname{Tr}_{2\,\mathrm{Cat}_k}(f_*(E) \otimes -)$ are given by the classical Chern characters of E and f_*E respectively.

- By Proposition 4.3 under the identifications

$$\pi_0 \operatorname{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)}) \simeq \bigoplus_p H^p(X, \Omega^p_X)^\vee \qquad \pi_0 \operatorname{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(Y)}) \simeq \bigoplus_p H^p(Y, \Omega^p_Y)^\vee$$

the morphism of traces induced by the pushforward functor $\mathrm{ICoh}(X) \xrightarrow{f_*} \mathrm{ICoh}(Y)$ coincides with the usual pushforward in homology (defined as the Poincaré dual of the pullback).

- By Theorem 6.10 under the Poincaré self-duality

$$\bigoplus_p H^p(X, \Omega^p_X) \simeq \bigoplus_p H^p(X, \Omega^p_X)^\vee$$

the morphism $\operatorname{Tr}_{2\,\mathrm{Cat}_k}(- \otimes \mathcal{O}_X)$ is given by the multiplication with the Todd class td_X and analogously for Y.

Summarizing, we obtain

Theorem 7.1 (Grothendieck-Riemann-Roch)**.** Let $X \xrightarrow{f} Y$ be a morphism of smooth proper k-schemes and let E be a perfect sheaf on X. Then the diagram 3 gives an equality

$$f_*(\mathrm{ch}(E)\,\mathrm{td}_X) = \mathrm{ch}(f_*(E))\,\mathrm{td}_Y \quad \in \quad \bigoplus_p H^p(Y, \Omega^p_Y).$$

In fact, the techniques above also prove an equivariant version of Grothendieck-Riemann-Roch theorem. Specifically, let

$$g_X \circlearrowright X \xrightarrow{\quad f \quad} Y \circlearrowleft g_Y$$

be an equivariant morphism between smooth proper schemes.

Then for a lax g_X-equivariant perfect sheaf E on X we can form the diagram

$$\begin{tikzcd}
\mathrm{Vect}_k \arrow[r, "E"] & \mathrm{QCoh}(X) \arrow[loop above, "g_{X*}"] \arrow[r, "f_*"] \arrow[d, "\otimes\mathcal{O}_X", "\sim"'] & \mathrm{QCoh}(Y) \arrow[loop above, "g_{Y*}"] \arrow[d, "\otimes\mathcal{O}_Y", "\sim"'] \\
 & \mathrm{ICoh}(X) \arrow[loop below, "g_{X*}"] \arrow[r, "f_*"'] & \mathrm{ICoh}(Y) \arrow[loop below, "g_{Y*}"]
\end{tikzcd} \tag{4}$$

in Cat_k and then apply the formalism of 2-traces.

In order to get an explicit description of the result, we need a concrete description of derived fixed point stack (2.2). Let (W, g) be a smooth scheme with an endomorphism $W \xrightarrow{g} W$ such that the reduced classical scheme $\overline{W^g} := \mathcal{H}^0(W^g)^{\mathsf{red}}$ is smooth (but not necessarily connected), and let us denote by $\overline{W^g} \xrightarrow{j} W$ the canonical embedding and by $\mathcal{N}_g^\vee$ its conormal bundle. Note that the action of g on Ω^1_W in particular restricts to an endomorphism $\mathcal{N}_g^\vee \xrightarrow{g^*_{|\mathcal{N}_g}} \mathcal{N}_g^\vee$.

By combining [GR17b, Chapter 1, Proposition 8.3.2] with some direct calculations we have the following

Theorem 7.2 (Localization theorem)**.** The following conditions are equivalent:

1. The canonical morphism $j^g \colon \mathcal{L}\overline{W^g} \longrightarrow W^g$ is an equivalence.

2. The determinant $\det(1-g^*_{|\mathcal{N}^\vee_g}) \in \Gamma(\overline{W^g}, \mathcal{O}_{\overline{W^g}})$ is an invertible function.

Corollary 7.3. In the assumptions of the above theorem, we obtain a canonical equivalence

$$\Gamma(W^g, \mathcal{O}_{W^g}) \xrightarrow[\sim]{(j^g)^*} \Gamma(\mathcal{L}\overline{W^g}, \mathcal{O}_{\mathcal{L}\overline{W^g}}) \simeq \bigoplus_p \Gamma(\overline{W^g}, \Omega^p_{\overline{W^g}}[p])$$

By arguing similarly as in non-equivariant case and using Theorem 7.2, we obtain the following

Theorem 7.4 (Equivariant Grothendieck-Riemann-Roch)**.** Let $(X, g_X) \xrightarrow{f} (Y, g_Y)$ be an equivariant morphism between smooth proper schemes such that

- Reduced fixed loci $\overline{X^{g_X}}$ and $\overline{Y^{g_Y}}$ are smooth,
- The induced morphisms on conormal bundles $1-(g^*_X)_{|\mathcal{N}^\vee_{g_X}}$ and $1-(g^*_Y)_{|\mathcal{N}^\vee_{g_Y}}$ are invertible.

Then for a perfect lax g_X-equivariant sheaf (E, t) on X the morphism of traces applied to the diagram 4 produces an equality

$$(\overline{f^g})_*\left(\operatorname{ch}(E,t)\frac{\operatorname{td}_{\overline{X^{g_X}}}}{e_{g_X}}\right) = \operatorname{ch}\left(f_*(E,t)\right)\frac{\operatorname{td}_{\overline{Y^{g_Y}}}}{e_{g_Y}}$$

in $\bigoplus_p H^p(\overline{Y^{g_Y}}, \Omega^p_{\overline{Y^{g_Y}}})$, where here e_{g_X} is the **equivariant Euler class** defined as

$$\operatorname{ch}\left(\operatorname{Sym}(\mathcal{N}^\vee_{g_X}[1]), \operatorname{Sym}({g_X}^*_{|\mathcal{N}^\vee_{g_X}[1]})\right) \in \pi_0\Gamma\big(\mathcal{L}\overline{X^{g_X}}, \mathcal{O}_{\mathcal{L}\overline{X^{g_X}}}\big),$$

and similarly for Y.

8 Approbation

The results of the papers were presented at the talk "Formalism of traces in derived algebraic geometry" on the seminar "Laboratory of algebraic geometry and its applications", May 6, 2022.

9 Publications

The main results are published in [KP18], [KP21] and [KP22].

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