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Equivariant singularity theory and mirror symmetry for Fermat polynomials with nonabelian groups of symmetry

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Introduction

This is a summary of the works [1],[2] and [9] of the author presented for the thesis defence.

The works are dedicated to the study of various homological ([1]) and differential geometric ([9]) invariants for a certain class of singularities with the action of nonabelian finite groups of symmetries as well as establishing examples of mirror symmetry for such singularities ([2]). More precisely, we study Fermat type singularities $f_{n,N} = x_1^n + \ldots + x_N^n$ with an action of groups, which are generated by rescalings and permutations of coordinates.

0.1. **Overview of the area.** Following late 80s works of physicists ([11], [21], [22]) the so-called Landau–Ginzburg orbifolds are widely studied both in mathematical physics and mathematics. These are essentially the pairs (f, G) of a polynomial f with an isolated singularity at the origin and G, its finite group of symmetries. After [4] Landau-Ginzburg orbifolds with f being a so-called invertible polynomial and G being a diagonal group of symmetries became an integral part of mirror symmetry. For this class of pairs (f, G) they construct a dual pair (f^T, G^T) and match some invariants for the dual pairs. However, most of the known results in this area considers only the case of an abelian group G.

The important homological invariant of such pair is the Hochschild cohomology $HH^*(MF(f, G))$ of the dg-category of G-equivariant matrix factorizations of f or in physical terms B-phase space. It has a natural structure of a Frobenius algebra with the multiplication given by the $\cup$-product. The main result of [20] explicitly describes the Frobenius algebra $HH^*(MF(f, G))$ for abelian G. This space also possesses the left-right charge bigrading introduced in [11], which plays the role of the Hodge decomposition.

In [12] the mirror map on the level of phase spaces was constructed for invertible polynomials and their diagonal groups of symmetries (the setting of [4]). Under the mirror map, left and right charges are interchanged.

In [6] and subsequent works of the same authors the generalization of the Berglund–Hübsch–Henningson mirror symmetry was proposed and studied. The pairs (f, G) considered there are of the following form: f is an invertible polynomial and G is a semidirect product of a subgroup of permutations of coordinates, satisfying a specific condition called parity condition and a diagonal group of symmetries. The duality here transorms f and the diagonal part of G in the same way as in [4] and preserves the permutation part of G. Some variations and examples of the mirror symmetry for nonabelian Landau–Ginzburg orbifolds were also studied in [13] and [17].

It is interesting to extract information about classical (nonequivariant) singularities from the study of equivariant ones. Mirror symmetry serve as one of examples of such procedure. For a different example see for example [14].

An important example of a structure appearing in studying of the classical singularities is the structure of Frobenius manifolds on the space of versal deformations of a singularity and related notion of the primitive form. They were introduced in [18]. The key existence theorem for primitive forms for general singularity was proved in [19]. However, there are very few examples of explicit constructions of primitive forms.

0.2. **Overview of the results.** In [1] we compute the Hochschild cohomology as a bigraded Frobenius algebra of Landau-Ginzburg orbifolds $(f_{n,N}, G)$ with G being a subgroup of a group generated by rescalings and permutations of coordinates (see Section 1.1 for the details). Building up on this, in [2] we construct the mirror map for some singularities of this type (see Section 1.2 for the details). This results explore the proposal of [6] on generalization of the original conception of Berglund–Hübsch–Henningson. The crucial novelty of these results as that their are among the very first results of these type with the group of symmetries being nonabelian.

In [9] we provide one of the few explicit constructions of Saito primitive forms in case of the so called Gepner singularities. The interest to this question comes from the works of physicists [7],[3] providing the relations with Kazama-Suzuki models (see Section 1.3 for the details).

0.3. **Methods.** We drew inspiration from [20],[6] and [3] in pausing the questions. We apply and noticebly generalize the ideas of [20], [12] and [5].

0.4. **Future prospects and applications.** Although we provided answers for some essential questions, there is still a lot to be done. It is important to further generalize the computations of [1] as well as to further explore the mirror map of [2] beyond the considered cases, which we plan on doing.

Building on the obtained results we plan to approach the question of intrinsic construction of Saito primitive forms for equivariant singularities.

It is essential to better understand the categorical aspects of the story. In particular, there should be interesting relations with the categorical McKay correspondence and the semiorthogonal decompositions of [16]. The study of the connections with the McKay was already studied by the author in [10].

Finally, the intrinsic definition and the structures of A-phase space for Landau-Ginzburg orbifolds with nonabelian G remains mysterious and needs further investigation. Mirror symmetry conjectures, the computations of [1] and the mirror map of [2] could provide a hint in this direction. In this context the relations with [15] and [8] are of particular interest.

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1. Formulation of the results

For fixed positive integers n and N consider a polynomial $f_{n,N} = x_1^n + \ldots + x_N^n$ and its full group of symmetries, the wreath product $G_{n,N} = S_N \wr \mu_n := S_N \ltimes (\mu_n)^N$, where the N copies of group of the n-th roots of unity μ_n acts by rescaling of the respective coordinates and S_N acts by permuting the coordinates. It is an invertible polynomial, which is self-dual in the sense that $f^T_{n,N} = f_{n,N}$.

1.1. **Hochschild cohomology of Fermat type polynomials with nonabelian symmetries.** In paper [1], joint with A. Basalaev, we expand the methods of [20] and compute the bigraded Frobenius algebra $HH^*(MF(f_{n,N}, G))$ for all $G \subset G_{n,N}$. This is one of the first times such results were obtained in case of nonabelian G.

Following [20] we observe a chain of ring isomorphisms

$$HH^*(MF(f,G)) \cong HH^*(\mathbb{C}[\mathbf{x}] \rtimes G, f) \cong HH^*(\mathbb{C}[\mathbf{x}], f; \mathbb{C}[\mathbf{x}] \rtimes G)^G,$$

where the second ting is the Hochschild cohomology of a curved algebra and the third ring is the subring of G-invariants in the Hochschild cohomology of a curved algebra with the coefficients in the bimodule. It is therefore, sufficient to understand $HH^*(\mathbb{C}[\mathbf{x}], f_{n,N}; \mathbb{C}[\mathbf{x}] \rtimes G_{n,N})$ together with action $G_{n,N}$ to compute $HH^*(MF(f_{n,N}, G))$ for any $G \subset G_{n,N}$, as $\mathbb{C}[\mathbf{x}] \rtimes G$ is a direct summand subbimodule of $\mathbb{C}[\mathbf{x}] \rtimes G_{n,N}$.

The decomposition with respect to the second factor $\mathbb{C}[\mathbf{x}] \rtimes G = \bigoplus_{g\in G} \mathbb{C}[\mathbf{x}]_g$ yield the G-grading

$$HH^*(\mathbb{C}[\mathbf{x}], f; \mathbb{C}[\mathbf{x}] \rtimes G) = \bigoplus_{g\in G} HH^*(\mathbb{C}[\mathbf{x}], f; \mathbb{C}[\mathbf{x}]_g),$$

compatible with the multiplications. The summands in the second sum are called *sectors*. For each sector as $\mathbb{C}[\mathbf{x}]$-module there is an isomorphism

$$HH^*(\mathbb{C}[\mathbf{x}], f; \mathbb{C}[\mathbf{x}]_g) \cong Jac(f^g)\xi_g,$$

where $Jac(f^g)$ is the Jacobi ring of f^g, the restriction of f to the fixed locus of g, and ξ_g is the generator representing the top exterior power of the conormal bundle to the fixed locus of g. It follows that it is sufficient to compute the products and G-action on the generator elements ξ_g.

For the sake of clarity we first present the answer for $G = S_N$.

Theorem 1.1. *1) (Corollary 21) Over $\mathbb{C}[\mathbf{x}]$ the algebra $HH^*(\mathbb{C}[\mathbf{x}], f_{n,N}; \mathbb{C}[\mathbf{x}] \rtimes S_N)$ is generated by the transposition elements $\xi_{(ij)}$.*

2) (Propositions 12, 17) For a permutation $\sigma \in S_N$ we have

$$\xi_\sigma \cup \xi_{(ij)} = \pm \xi_{\sigma(ij)}$$

if i and j lie in different cycles of σ and

$$\xi_\sigma \cup \xi_{(ij)} = \pm n \sum_{k+l=n-2} x_i^k x_j^l \xi_{\sigma(ij)}$$

if i and j lie in the same cycle of σ.

3) (Proposition 38) We have $(ij)^*(\xi_{(kl)}) = \pm \xi_{(ij)(kl)(ij)}$.

Let us also describe the full answer for $G = G_{n,N}$, which has similar structure, but is more technical. Let t_i be the group element multiplying the i-th coordinate by a fixed n-th primitive root of unity ζ_n. Now for the generators we consider the product of the transposition (ij) and $t_i^d t_j^{-d}$ as well as elements t_i^d (Proposition 22). Relations with the generators of the first type are similar to the relations with the transposition (Proposition 23). An additional relation for the generators of the second type are

$$\xi_{t_i^d} \cup \xi_{t_i^{-d}} = \frac{n}{\zeta_n^d - 1} x_i^{n-2} \xi_e$$

and

$$\xi_{t_i^{d_1}} \cup \xi_{t_i^{-d_2}} = 0$$

for $d_1 \neq d_2$ and both nonzero (Proposition 28). The group action is described by Propositions 33, 37, 38.

We also verify the compatibility of the product and the G-action with the bigrading (Proposition 30) and the residue symmetric bilinear form (Theorem 32, Proposition 45).

1.2. **Mirror map for Fermat polynomial with nonabelian group of symmetries.** In paper [2], joint with A. Basalaev, we apply the results of the previous work to generalize the mirror map of [12] to the case of $f_{n,N}$ with $n = N$ (necessary for mirror symmetry Calabi-Yau condition) being a prime number (technical condition, satisfied, for example, in the essential case of $n = N = 5$ of [13]) and groups of symmetries suggested by [6]. We provide a homological interpretation of the parity condition of [6], by noticing that it is equivalent to saying that the canonical line bundle is locally trivial as an S-equivariant sheaf for the group of permutation symmetries $S \subset S_N$. This leads us to the definition of a subspace of stable sectors $HH^*(MF(f_{n,N}, G))^{\mathrm{st}} \subseteq HH^*(MF(f_{n,N}, G))$ as the subspace of sectors, whose generator ξ_g is stabilized by the centralizer of $g \in G$.

Fix $S \subset S_N$, let T_0 be a group of diagonal symmetries of $f_{n,N}$ with the determinant 1 (denoted SL_f in loc.cit.) and $\langle J \rangle \simeq \mu_n$ be a group of scalar symmetries of $f_{n,N}$, so that the pairs $(f_{n,N}, \langle J\rangle)^T = (f_{n,N}, T_0)$ are dual. We construct a mirror map

$$\tau\colon HH^*(MF(f_{n,N}, S \ltimes T_0))^{\mathrm{st}} \to HH^*(MF(f_{n,N}, S \times \langle J\rangle))^{\mathrm{st}}$$

and prove

Theorem 1.2. *1) (Theorem 1) The mirror map τ is an isomorphism and interchanges the gradings.*

2) (Theorem 1) If S satisfies the parity condition we have $HH^*(MF(f_{n,N}, G))^{\mathrm{st}} = HH^*(MF(f_{n,N}, G))$.

3) (Theorem 2) If $n = N = 5$ the mirror map τ extends to an isomorphism

$$\widehat{\tau}\colon HH^*(MF(f_{5,5}, S \ltimes T_0)) \xrightarrow{\sim} HH^*(MF(f_{5,5}, S \times \langle J\rangle)).$$

The latter mirror map covers several of the cases listed in [6] (see also [13]) and provides an additional insight on their proposal.

1.3. **Primitive forms for Gepner singularities.** The following summarizes the results of [9].

In this paper we study the pair $(f_{n,N}, S_N)$ to obtain the information about the quotient singularity of $\{f_{n,N} = 0\}/S_N$ called the Gepner singularity $g_{n,N}$ (the pair $f_{n,N}$ is $F_{k,n}$ in loc.cit., renamed here to be consistent with the notations of other paper; $g_{n,N}$ is $G_{n,N}$ in loc.cit. renamed here for not to be confused with the group $G_{n,N}$ above). Namely, we provide a construction of Saito primitive forms (Theorem 7.1) and Frobenius manifolds (Theorem 6.5) for $g_{n,N}$ by studying the relation between Saito primitive forms for the Gepner singularities and primitive forms for $f_{n,N}$ invariant under S_N-action. More precisely, we use the construction of [5] of a Frobenius manifold from a data of a Frobenius manifold with a finite group action to construct primitive forms for $g_{n,N}$ starting from the primitive form for $f_{n,N}$ invariant with respect to S_N-action. It is a natural singularity theory analogue of one of the cases of the abelian/nonabelian correspondence in [5] relating the quantum cohomology of the Grassmannian $\mathrm{Gr}(N, n)$ and the product of projective spaces $(\mathbb{P}^{n-1})^{\times N}$.

The interest in the Gepner singularities appeared after the isomorphism of the chiral ring of a $SU(N+1)_{n-N-1}/(SU(N)_{n-N} \times U(1))$ Kazama–Suzuki model, the Milnor ring of the Gepner singularity $g_{n,N}$ and the cohomology ring of the Grassmannian $\mathrm{Gr}(N, n)$ were established in [7]. The interest in primitive for Gepner singularities arose after [3]. Remarkably, the relation we study should also impose a relation between the $SU(N+1)_{n-N-1}/(SU(N)_{n-N} \times U(1))$ Kazama–Suzuki model and the tensor product of N copies of the minimal $SU(2)_n/U(1)$ model, which is to be investigated.

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