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Newton polytopes and numbers of roots of non-algebraic systems

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Contents

Introduction.

The relationship between the geometry of polyhedra and the numbers of roots of systems of polynomial equations was established by A.G. Kushnirenko and specified by D. N. Bernstein; see [Ko, B]. Let $f_1, \dots, f_n$ be Laurent polynomials in n variables and let Δ_i be a Newton polytope of f_i. It turns out that the number of common zeroes of a generic collection of polynomials $f_1, \dots, f_n$ is $n!\mathrm{vol}(\Delta_1, \dots, \Delta_n)$, where $\mathrm{vol}(\Delta_1, \dots, \Delta_n)$ is a mixed volume of polytopes $\Delta_1, \dots, \Delta_n$.

Kushnirenko–Bernstein formula (also called the BKK or Bernstein–Kushnirenko–Khovanskii formula) stimulated interest in the algebraic geometry of the complex torus $(\mathbb{C} \setminus 0)^n$. The practice of using Newton polytopes that arose after the BKK theorem led to the creation and application of concepts such as toric variety, tropical algebraic geometry, Newton-Okounkov body, ring of conditions of spherical space, etc.

It is known that the algebraic geometry of a torus, compared with affine algebraic geometry in $\mathbb{C}^n$, has some significant differences. Previously, similar differences were manifested, e.g., in Diophantine algebraic geometry; see [BMZ]. These differences are explained by the group structure of the complex torus. BKK-type theorems were also discovered for arbitrary complex linear reductive Lie groups, as well as for spherical homogeneous spaces; see [KK2, K04, VK1, VK2, B1, K1, O1].

If $\alpha \in \mathbb{C}^{n*}$ is a linear functional in $\mathbb{C}^n$, then the function $z \mapsto \mathrm{e}^{\alpha(z)}$ is a character of the additive group $\mathbb{C}^n_+$ of $\mathbb{C}^n$. A finite linear combination of characters of the form $\mathrm{e}^{\alpha(z)}$ is called the exponential sum (ES) in the space $\mathbb{C}^n$. The first part of the thesis is based on the following analogy between the torus $(\mathbb{C} \setminus 0)^n$ and the group $\mathbb{C}^n_+$. *Representation of $\mathbb{C}^n_+$ in the space of exponential sums, as well as the representation of the torus in the space of Laurent polynomials, decomposes into a direct sum of characters.* The set of common zeroes of the finite system of ESs is called exponential analytic set (EAS). The thesis describes our findings on intersections of EASs, similar to the previous findings on intersections of algebraic subvarieties of the torus. In particular, we define the intersection index of EASs and we construct an analogue of the ring of conditions of the torus $(\mathbb{C} \setminus 0)^n$ – the ring of the corresponding intersection theory of EASs, called the *ring of conditions for* $\mathbb{C}^n$. As for the torus $(\mathbb{C} \setminus 0)^n$, the ring of conditions of the space $\mathbb{C}^n$ turns out to be closely related to the geometry of convex polytopes. The calculation of this ring is based, mostly, on the use of technology of tropical geometry.

Our analogue of the BKK theorem is the computation of the intersection index of n exponential hypersurfaces $\{f_i = 0\}$. We define a homogeneous polynomial of degree n on the space of convex polytopes in the space $\mathbb{C}^n$, called *pseudovolume of polytope.* It turns out that this intersection index is equal to the mixed pseudovolume of Newton polytopes of exponential sums $f_1, \dots, f_n$ (in this case, a full description of the results is in preparation, and the construction of the ring of conditions in the text of the dissertation is not given). If the ESs f_i are quasialgebraic then the mixed volume of their Newton polytopes is equals to their mixed volume.

In the second part of the thesis, a connection is established between the numbers of roots of smooth systems of equations and mixed volumes of some families of convex bodies. More precisely, for any collection of n finite-dimensional function spaces $V_i \subset C^\infty(X)$ on the n-dimensional differentiable manifold X we consider the systems of equations

$$f_1 - a_1 = \ldots = f_i - a_i = \ldots = f_n - a_n = 0, \qquad (1)$$

where $f_i \in V_i$, $f_i \neq 0$, $a_i \in \mathbb{R}$. The equation $f_i - a_i = 0$ corresponds to an affine hyperplane in the space of linear functionals V_i^*. To any smooth non-negative translation invariant measure in the space of affine hyperplanes in V_i^* we associate a *Banach body* in X, which is a family of centrally symmetric convex bodies in the fibers of the cotangent bundle $T^*X \to X$. The union of these convex bodies is a domain in the manifold T^*X. The symplectic volume of this domain is called the Banach body volume. This volume is analogous to the volume of the Newton polytope in the BKK theorem.

For any arbitrary set of translation invariant measures ν_i in manifolds of affine hyperplanes in V_i^*, we consider $\nu_1 \times \ldots \times \nu_n$ as a measure in the space of systems (1). It turns out that the number of roots of systems (1), averaged over this measure is equal to the mixed symplectic volume of Banach bodies corresponding to the measures ν_i. This last result is a far-reaching generalization of Crofton's classical formula, approving, that the length of a plane curve is equal to the expected number of its intersection points with a random affine line.

The space of systems (1) is not compact. It is known from integral geometry that the computations of compact averaging are usually more meaningful and more convenient to use. Consider the system

$$f_1 = \ldots = f_i = \ldots = f_n = 0, \qquad (2)$$

as a point of $\mathrm{P}(V_1) \times \ldots \times \mathrm{P}(V_n)$, where $\mathrm{P}(V_i)$ is the projectivization of the space V_i. Suppose that in the spaces V_i the scalar products are given. In this case, for each of the subspaces V_i one can define a *Banach ellipsoid* in X, which is a Banach body consisting of ellipsoids in the cotangent spaces T_x^*X. Let ν_i denote the rotation invariant measure in $\mathrm{P}(V_i)$ and consider the averaging of the number of solutions of the systems (2) with respect to measure $\nu_1 \times \ldots \times \nu_n$. It turns out that in this case, the average number of solutions is also equal to the mixed volume of the corresponding Banach ellipsoids. This result is applied to the calculation of the average number of roots of random systems of eigenfunctions of the Laplace operator, random systems of trigonometric polynomials, and, more generally, random systems of functions on arbitrary compact Lie group.

1. Thesis subject

The subject of the first part of the thesis is the theory of intersections of exponential analytic sets, which are subsets of $\mathbb{C}^n$, given as sets of common zeros of finite tuples of exponential sums. In the second part, we consider smooth versions of the BKK theorem. In this section, we give some basic definitions and a brief description of our findings.

1.1. **Exponential analytic sets.**

Recall that *the exponential sum* (ES) is an entire function in $\mathbb{C}^n$ of the form

$$f(z) = \sum_{\lambda\in\Lambda\subset\mathbb{C}^{n*},\, c_\lambda\in\mathbb{C}} c_\lambda\, \mathrm{e}^{\langle z,\lambda\rangle},$$

where $\mathbb{C}^{n*}$ is the space of linear functionals in $\mathbb{C}^n$, and Λ is a finite set in $\mathbb{C}^{n*}$ called *the support of* ES. The convex hull of the support is called *the Newton polytope* of ES. The Newton polytope of a generic ES is of dimension $2n$. An analytical set of common zeros of finite tuple of ESs is called *an exponential analytic set* (EAS). If the support ES belongs to subspace $\mathrm{Re}\,\mathbb{C}^{n*}$, then ES is called *quasi-algebraic*. The corresponding EASs are also called quasi-algebraic.

The ring of ESs looks like a Laurent polynomial ring. However, many attempts to find the algebraic-geometric properties of the ring of ESs, similar to the properties of the ring of polynomials encountered great difficulties. The first result in this direction was obtained by J. Ritt about a hundred years ago [R]. Ritt proved that, if the result of dividing two ESs in one variable is an entire function, then this function is also ES. (Ritt's multidimensional theorem was proved later; see [AG].) On the other hand, the existence of a common zero of two ESs does not imply the existence of a common divisor. For example, ESs $\mathrm{e}^z-1,\ \mathrm{e}^{\sqrt{2}z}-1$, having a common zero at the point $z=0$ have no common divisor in the ES ring. It is probable that J. Ritt himself proposed the conjecture about the finiteness of the set of common zeros of two coprime ESs in one variable. Currently, this conjecture is very far from being proven. If one of the ESs is e^z-1, then the conjecture is true. In this case, it follows from a theorem called the "Mordell-Lang conjecture" for a complex torus.

Exponential sums are linear combinations of characters of the additive group $\mathbb{C}^n_+$ of $\mathbb{C}^n$. Below, we consider ESs as an analogue of Laurent polynomials, which are linear combinations of characters of the torus $(\mathbb{C}\setminus 0)^n$. Respectively, EASs are considered as analogs of algebraic varieties in the torus. Guided by this analogy, we construct the ring of conditions of the corresponding intersection theory. This construction, in the case of quasi-algebraic ESs, is described in [8] and [9]. For arbitrary EASs the geometry of EASs becomes more involved. In particular, in the general case, apart from standard tropical geometry, a certain complex extension of tropical concepts (see subsection 2.2) is used. The texts describing the theory of intersections of arbitrary EASs are in preparation. Accordingly, below we consider only the quasi-algebraic ESs and EASs.

The first obstacles one encounters when trying to construct the intersection theory are 1) absence of a regular concept of dimension EAS and 2) infinity of the zero set of EAS of dimension 0 (for example, the zero set of $e^z - 1$ is $2\pi i\mathbb{Z}$). Respectively, we define the concepts of 1) *algebraic codimension* of EAS and 2) *of weak density* of EAS of algebraic codimension n. Note that the definition of algebraic codimension is algebraic. It means, that the computation of the algebraic codimension reduces to calculating the dimension of some algebraic variety. To define the intersection index we use the concept of *a domain of relatively full measure* in the real vector space E.

Definition 1.1. Let $\mathfrak{I} = \{I\}$ be a finite set of proper subspaces of a real vector space E. Put $B_{\mathfrak{I}} = E \setminus \bigcup_{I\in\mathfrak{I}} I$. When $0 < R \in \mathbb{R}$, let $B^R_{\mathfrak{I}}$ denote the subset of E consisting of the points lying at the distance $\geq R$ from $\bigcup_{I\in\mathfrak{I}} I$. We call a domain $U \subset E$ that contains a subdomain of the form $B^R_{\mathfrak{I}}$ a domain of relatively full measure.

Note that 1) the property of being a domain of relatively full measure does not depend on the choice of the metric in the space E and 2) the union or intersection of a finite number of domains of relatively full measure is also a domain of relatively full measure.

Theorem 1.1. *Let the sum of algebraic codimensions* EASs X, Y *be equal to* n. *Then there is a domain of relatively full measure* $U(X,Y)$ *in the space* $\operatorname{Re}\mathbb{C}^n$, *such that for all* $z \in U(X,Y) + \operatorname{Im}\mathbb{C}^n$

(1) *the algebraic codimensions of* EASs $(z+X)\cap Y$ *are equal to* n

(2) *the weak densities of* EASs $(z+X)\cap Y$ *are the same.*

Definition 1.2. Intersection index $I(X,Y)$ is defined as the weak density of EAS $X \cap (z+Y)$ for $z \in U(X,Y) + \operatorname{Im}\mathbb{C}^n$.

Definition 1.3. We say that two EASs with algebraic codimension $k \leq n$ are *numerically equivalent* if $I(X,Z) = I(Y,Z)$ for each EAS Z of algebraic codimension $n-k$. All EASs of algebraic codimension $> n$ are also said to be numerically equivalent.

Furthermore, following [dCP, dC], we construct a quasi-algebraic ring of conditions for the space $\mathbb{C}^n$. This ring is a commutative graded $\mathbb{Z}$-algebra, whose elements are formal differences of numerical equivalence classes of EASs. In order to establish that equivalence classes form a ring, we must prove the following statement: *Let* μ, ν *be the classes of numerical equivalence. Then for almost all* $X \in \mu$, $Y \in \nu$ *the equivalence classes of* EASs $X\cup Y$ *and* $X\cap Y$ *do not depend on the choice of* X, Y. In the quasi-algebraic case, the proof of the statement is based entirely on applying the methods of tropical algebraic geometry.

The degree of EAS as an element of the ring of conditions is equal to its algebraic codimension. A posteriori, the ring of conditions turns out to be an algebra over the field $\mathbb{R}$. If the numerical equivalence class ν contains EAS X, defined by equations $\{f_i(z) = 0\}$, then for $t > 0$ the class $t\nu$ contains EAS, defined by the equations $\{f_i(t^{\frac{1}{k}} z) = 0\}$, where k is the algebraic codimension of X.

The structure of the ring of conditions of a complex torus can be described in different ways in the language of lattice polytope geometry (geometry of Newton polytopes). These descriptions of the ring of conditions remain valid when we move from polynomials to ESs but only if the condition that polytopes are integral is abandoned. Here we give one statement of the BKK type formula for exponential sums. It is a refinement of the previously known results; see [Kh1, K3].

For any quasi-algebraic ESs $f_1, \ldots, f_n$ with Newton polytopes $\Delta_1, \ldots, \Delta_n$ there is an exponential hypersurface D in the space $\mathbb{C}^{n\times n}$, such that, if $(w_1, \ldots, w_n) \notin D$, then the weak density of EAS, defined by equations

$$f_1(z + w_1) = \ldots = f_n(z + w_n) = 0,$$

is equal to the mixed volume of the polytopes Δ_i, multiplied by $n!$.

1.2. **Smooth versions of the BKK theorem.** The relationship between the numbers of common zeroes and mixed volumes also exists for any systems of smooth real functions.

Consider a smooth n-dimensional manifold X as an analogue of the torus $(\mathbb{C} \setminus 0)^n$. The formulation of Theorem BKK involves n spaces of Laurent polynomials V_i, consisting of linear combinations of n fixed finite sets Λ_i in the lattice of characters of the torus $(\mathbb{C} \setminus 0)^n$. Let us replace these n spaces by n arbitrary finite-dimensional spaces of smooth real functions $V_1, \ldots, V_n$ in n-dimensional differentiable manifold X.

Suppose that for every $x \in X$ we are given a centrally symmetric convex body $\mathcal{B}(x) \subset T^*_x X$ depending continuously on $x \in X$. We call the collection $\mathcal{B} = \{\mathcal{B}(x) \mid x \in X\}$ a Banach set in X. The volume of a Banach set $\mathcal{B}$ is defined as the volume of $\cup_{x\in X}\mathcal{B}(x) \subset T^*X$ with respect to the standard symplectic structure on the cotangent bundle. More precisely, if ω is a simplectic form, then the volume form is $\omega^n/n!$. Using Minkowski sum and homotheties, we consider linear combinations of convex sets with non-negative coefficients. The linear combination of Banach sets is defined by

$$(\sum_i \lambda_i \mathcal{B}_i)(x) = \sum_i \lambda_i \mathcal{B}_i(x).$$

The symplectic volume of the Banach set $\lambda_1\mathcal{B}_1 + \ldots + \lambda_n\mathcal{B}_n$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_n$. Its coeficient at $\lambda_1 \cdot \ldots \cdot \lambda_n$ divided by $n!$ is called the *mixed volume of Banach sets* $\mathcal{B}_1, \ldots, \mathcal{B}_n$ and is denoted by $\mathrm{vol}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$.

We denote by $\mathrm{AGr}^k(E)$ the affine Grassmanian, whose points are affine subspaces of codimension k in vector space E. Let ν_i be a translation invariant smooth non-negative measure in $\mathrm{AGr}^1(V_i^*)$. For $f_i \in V_i$, $a_i \in \mathbb{R}$ we identify the equation $f_i - a_i = 0$ with an affine hyperplane

$$H_i = \{v^* \in V_i^* \colon v^*(f_i) = a_i\} \in \mathrm{AGr}^1(V_i^*).$$

Respectively, we identify the set of equations $\{f_i - a_i = 0\}$ with the set of affine hyperplanes $\{H_i \in \mathrm{AGr}^1(V_i^*)\}$. Consider the measure $\nu = \nu_1 \times \ldots \nu_n$ on the manifold $\mathrm{AGr}^1(V_1^*) \times \ldots \times \mathrm{AGr}^1(V_n^*)$. The integral of the number of roots of systems

$$f_1 - a_1 = \ldots = f_n - a_n = 0. \tag{1.1}$$

over the measure ν is called *the average number of roots of systems* (1.1). To each of the measures ν_i there corresponds (see section 3.2) a Banach body $\mathcal{B}_i$ in X. We consider the following statement as a smooth version of the BKK theorem.

Theorem 1.2. *The average number of roots* (1.1) *is equal to* $n!\mathrm{vol}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$.

In applications of the theorem, it often turns out that there is a scalar product defined on each of the spaces V_i. In such cases, instead of systems of equations (1.1), one can consider systems of the form

$$f_1 = \ldots = f_n = 0. \tag{1.2}$$

In these cases, where scalar products are defined, we define Banach ellipsoids $\mathcal{B}_i$ which are Banach bodies, consisting of ellipsoids in the cotangent spaces T_x^*X. Let ν_i be a normalized rotation invariant measure in the projective space $\mathrm{P}(V_i)$. Consider the averaging of the number of roots of the systems (1.2) over the measure $\nu_1 \times \ldots \times \nu_n$. It turns out that the average number of solutions is also equal to $n!\mathrm{vol}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$.

If a transitive action of a compact group G on a manifold X preserves the spaces V_i and their scalar products, then for any fixed $x \in X$ the mixed volume of Banach ellipsoids $\mathcal{B}_1, \ldots, \mathcal{B}_n$, coincides, up to a constant multiplication factor, with the mixed volume of the ellipsoids $\mathcal{B}_i(x)$. An unexpected consequence of this last statement are the following inequalities for the averaged numbers of roots of systems of equations on homogeneous spaces of compact groups.

Theorem 1.3. *Let the transitive action of a compact group on the manifold X preserve the Euclidean function spaces $V_1, \ldots, V_n$. Let $\mathfrak{M}(V_1, \ldots, V_n)$ be the average number of solutions of* (1.2). *Then*

$$\mathfrak{M}^2(V_1, \ldots, V_n) \geq \mathfrak{M}(V_1, \ldots, V_{n-1}, V_{n-1}) \cdot \mathfrak{M}(V_1, \ldots, V_n, V_n),$$

$$\mathfrak{M}^n(V_1, \ldots, V_n) \geq \mathfrak{M}(V_1) \cdots \mathfrak{M}(V_n),$$

where $\mathfrak{M}(V_k) = \mathfrak{M}(V_k, \ldots, V_k)$.

These inequalities are similar to the Hodge inequalities for the intersection indices of divisors in projective algebraic varieties. Just like the Hodge inequalities, they follow from the Aleksandrov-Fenchel inequalities for mixed volumes of convex bodies; see, for example, [KK]. If the action of the group on X is isotropically irreducible, then these inequalities become equalities.

For example, in this case

$$\mathfrak{M}^n(V_1,\dots,V_n)=\mathfrak{M}(V_1)\cdots\mathfrak{M}(V_n). \tag{1.3}$$

Example 1.1. Let V be the space of trigonometric polynomials

$$a_0+\sum_{1\le k\le m} a_k\cos(k\theta)+b_k\sin(k\theta)$$

of degree $\le m$ on the circle $X=S^1=\{\,\mathrm{e}^{i\theta}\colon\ 0\le\theta\le 2\pi\}$. Consider a metric in the space V with an orthonormal basis

$$\frac{1}{\sqrt{2\pi}},\frac{\cos(\theta)}{\sqrt{\pi}},\frac{\sin(\theta)}{\sqrt{\pi}},\dots,\frac{\cos(m\theta)}{\sqrt{\pi}},\frac{\sin(m\theta)}{\sqrt{\pi}}.$$

In this case, the action of the circle S^1 preserves the Banach ellipsoids. The Banach ellipsoid at the point $\theta=0$ is a segment $\left[-\sqrt{\frac{m(m+1)}{3}},\sqrt{\frac{m(m+1)}{3}}\right]$. This implies that the average number of zeros of trigonometric polynomials of degree $\le m$ equals $2\sqrt{\frac{m(m+1)}{3}}$. On the other hand, trigonometric polynomial $a_0+\sum_{k\le m}a_k\cos(k\theta)+b_k\sin(k\theta)$ is a restriction of a Laurent polynomial on the circle. This Laurent polynomial is defined as

$$P(z)=a_0+\sum_{k\le m}\alpha_k z^k+\overline{\alpha_k}z^{-k}, \tag{1.4}$$

where $\alpha_k=\frac{a_k+ib_k}{2}$. This implies, that the average fraction of real zeros (i.e. of the zeroes in S^1) of polynomials (1.4) is $\sqrt{\frac{m+1}{3m}}$, and for $m\to\infty$ converges to $\sqrt{\frac{1}{3}}$.

2. Results and publications

2.1. **Tropical geometry [2,3,5].**

2.1.1. *Tropical bases of ideals* [3]. To apply the tropical mathematics in geometry of zero varieties of Laurent polynomials or of exponential sums, it is convenient to use some special bases in ideals of the Laurent polynomial ring. These bases are called *the tropical bases of ideal.* Recall the definition of the tropical basis of the ideal.

The Laurent polynomial is the function $P=\sum_{m\in\Lambda\subset\mathbb{Z}^n,\,c_m\ne 0}c_m z^m$ on the complex torus $(\mathbb{C}\setminus 0)^n$. The finite subset Λ of the character lattice $\mathbb{Z}^n$ of the torus $(\mathbb{C}\setminus 0)^n$ is called the *support of the polynomial* P. We put $H=\mathbb{Z}^n\otimes_{\mathbb{R}}\mathbb{R}$ and denote by H^* the space of linear functionals in the H.

To each linear functional $\xi\in H^*$ we associate *truncation* $P^{(\xi)}$ *of order* ξ of Laurent polynomial P. By definition, $P^{(\xi)}=\sum_{m\in B}c_m z^m$, where B is a subset of the support Λ of P on which the function ξ attains its maximum. For every ideal I in the ring of Laurent polynomials and every order ξ we have an ideal $I^{(\xi)}$ generated by the truncations of order ξ of all Laurent polynomials in I.

Definition 2.1. The finite set $\{Q_j\} \subset I$ is called the tropical basis of the ideal I, if for every order $\xi \in H^*$ the ideal $I^{(\xi)}$ is generated by the Laurent polynomials $\{Q_j^{(\xi)}\}$.

The Laurent polynomial ring has the following tropical Noetherian property: *there is a tropical basis in every ideal of the Laurent polynomial ring.* Some versions of this statement were previously known; see [K6]. Since the introduction of tropical algebraic geometry, related statements were found by several authors, for example, [EKL]. The strongest version and its complete proof are found in [3]. We list some additional properties of tropical ideal bases.

(1) For any order $\xi \in H^*$ the truncations $Q_i^{(\xi)}$ of elements of the tropical basis $\{Q_i\}$ of the ideal I form a *tropical basis* of the truncation ideal $I^{(\xi)}$

(2) Recall that the convex hull $\Delta(P)$ of the support of the polynomial P is said to be its *Newton polytope.* Let K_P be a fan of dual cones of faces of nonzero dimension of the polytope $\Delta(P)$. Then the support of the fan-intersection $K_{\{Q_j\}} = \bigcap_j K_{Q_j}$ coincides with the Bergman cone of the zero variety of the ideal I.

(3) A toric variety is called good with respect to an m-dimensional algebraic variety $X \subset (\mathbb{C} \setminus 0)^n$, if the closure of X does not intersect toric orbits of codimension greater than m. If, moreover, the closure of X is compact, then the toric variety is called a good compactification of X. The toric variety, corresponding to the fan of cones $K_{\{Q_j\}}$, is a good compactification of the zero variety of the ideal I. A completely geometric proof of the good compactification existence theorem recently found in [Kh2].

In conclusion we present one conjecture from [K6]. A fan of cones $\mathcal{K}$ is said to be *minimal for* X, if the fan of cones corresponding to any toric compactification of X contains some partition of the fan $\mathcal{K}$. For example, for any algebraic curve X there exists a minimal fan. If X is a shift of a subtorus of dimension > 1, then there is no minimal fan.

In Diophantine geometry, the variety is considered generic if it does not contain shifted subthors of nonzero dimension (for example, it is known that the intersection of a generic variety with any finitely generated subgroup of the torus is finite). In [K6], it was suggested that for such varieties a minimum fan exists. This assumption is not proven, except for the case $\dim X = 2$, see [K6]. It is connected with the tropical bases of ideals in the following way. We call the points ξ, ψ in the space H^* equivalent, if there is a curve $K = \{K(t) \subset H^* \colon 0 \leq t \leq 1\}$, such that 1) $K(0) = \xi$, 2) $K(1) = \psi$ and 3) $\forall x \in K \colon I^{(x)} = I^{(\xi)}$. The equivalence classes are called the truncation chambers of the ideal I. The above assumption is equivalent to the following statement about truncation chambers. *If the zero variety of the ideal I does not contain shifted subtori of nonzero dimension, then the set of truncation chambers of the ideal I is a fan, consisting of strictly convex cones.*

2.1.2. *Multiplication of cocycles in polyhedral complex* ([5]). The main finding of the article [5] is the algorithm for multiplying cochains of the polyhedral complex (hereinafter P-complex) X, located in the space V. This algorithm depends on the choice of the functional $v \in V^*$. For any choice of v, the cocycles form a subring, and coboundaries form an ideal in the ring of cocycles. The quotient ring of cocycles along coboundaries is independent of the choice of v and coincides with the cohomology ring of the P-complex. The algorithm is obtained by transferring the multiplication algorithm for tropical varieties to the context of arbitrary polyhedral complexes. Recall the definition of the P-complex.

Definition 2.2. Let X be a finite set of closed convex polyhedra of dimension $\leq k$ in a real vector space V. We will call them polyhedra cells. We call X a k-dimensional P-complex if
(1) any face of any cell is a cell;
(2) any non-empty intersection of two cells is their common face.

The multiplication of cohomology classes was discovered independently by Kolmogorov and Alexander, and published at the conference on topology in Moscow in 1935 [Ko1, Ko2, A36]. However, in their reports, multiplication formulas for cocycles of a simplicial complex were incorrect. The first correct formula of multiplication cochains of a simplicial complex, inducing the multiplication in the cohomology ring, was suggested by Cech [Ch] (apparently during the same conference). Cech's rule for multiplication of cochains is as follows. Let r_p and r_q be respectively p-cochain and q-cochains of simplicial complex X. Then

$$(r_p \smile r_q)([u_0, \cdots, u_{p+q}]) = r_p([u_0, \cdots, u_p])\, r_q([u_p, u_{p+1}, \cdots, u_{p+q}]),$$

where $[u_0, \cdots, u_m]$ is an m-dimensional simplex with ordered vertices $u_0, \ldots, u_m$ (some order of vertices X is assumed to be defined). If the P-complex X is simplicial, then the multiplication algorithm proposed here practically coincides with Cech's algorithm. Equivalence of the definitions of Cech multiplication and Kolmogorov-Alexander multiplication is proved by Whitney in 1938 [Wh]. See the details of this story in [M].

Topologists sometimes consider the manifolds, related to the geometry of convex polyhedra; see [B]. Such manifolds can arise together with P-complexes due to their origin. The proposed algorithm allows multiplying the cohomology of such manifolds without simplicial triangulation, i.e. without destroying the original geometry of the problem.

The algorithm for multiplying the cochains of a polyhedral complex has geometric origin. For simplicial complexes, most of the geometry is trivialized. The algorithm is also used in convex geometry, for example, to calculate the mixed volumes of polyhedra or to construct a stable intersection of tropical varieties. In these applications, cocycles with values in the Grassmann algebra of the space are considered. Therefore, we assume that the value ring S is supercommutative. Recall that a supercommutative ring is a $\mathbb{Z}_2$-graded ring, such that for homogeneous elements it is true that $xy = (-1)^{|x||y|}yx$, where $|x|$ is the parity (i.e. $\mathbb{Z}_2$-degree) of x. (A commutative ring is a supercommutative ring without odd elements.) Below, wherever the notation $|r|$ is used, it is assumed that the parities of all values of the cochain r are the same and equal to $|r|$.

In the abovementioned geometrical applications, we sometimes consider homotopically trivial P-complexes. In this case, it turns out to be essential that the product of some special cocycles does not depend on the choice of parameter v. Therefore, we operate with the multiplication of cocycles more accurately than it is necessary to construct the multiplication of cohomology. The topology also sometimes require such accuracy when multiplying cocycles; see for example [St].

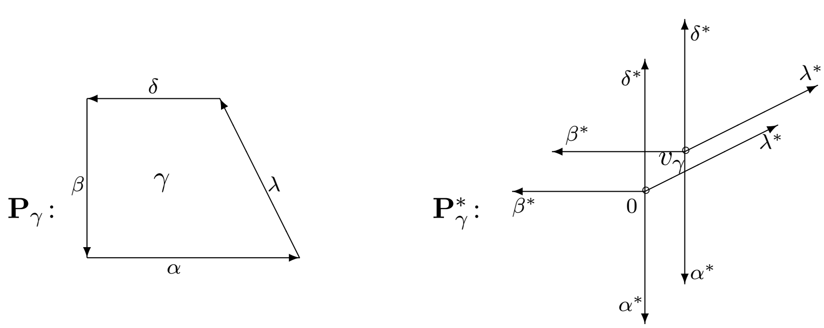
In the space V^* there are two subsets $D' \supset D$, each of them being the union of a finite set of subspaces of codimension 1. When the parameter v changes in the connected component of $V^* \setminus D$, the products of cocycles do not change. For parameters belonging to different components, the product of cocycles can differ by the cobound. When the parameter v changes in the connected component of $V^* \setminus D'$ the products of cochains also do not change.

By definition, the product of cochains $r_p \smile_v r_q$ on the $(p+q)$-dimensional cell γ of P-complex X has a value

$$(r_p \smile_v r_q)(\gamma) = \sum_{(\delta,\lambda)\in\mathcal{P}(p,q,\gamma,v)} r_p(\delta) r_q(\lambda), \tag{2.1}$$

where $\mathcal{P}(p,q,\gamma,v)$ is some subset of the set of pairs of oriented faces (δ,λ) of dimensions p,q of polytope γ. The subset $\mathcal{P}(p,q,\gamma,v)$ depends on the choice of parameter $v \in V^*$. A cochain multiplication algorithm is an algorithm for choosing a subset $\mathcal{P}(p,q,\gamma,v)$. Let us explain the choice of this set for $p = q = 1$.

Example 2.1. Let r,s be the 1-cochains of the P-complex X, $v \in V^*$, $\gamma \in X$, $\dim\gamma = 2$.



Draw the polygon γ on its tangent plane $\mathbf{P}_\gamma$ (in the picture on the left). On its cotangent plane $\mathbf{P}^*_\gamma$ (in the picture on the right) there is a point v_γ, equal to the restriction of $v \in V^*$ onto the plane $\mathbf{P}_\gamma$. On the plane $\mathbf{P}^*_\gamma$ we draw the exterior normals of the sides of the polygon twice: starting at points 0 and

v_γ. Intersections of the normals from the first set with the normals from the second set define the pairs of oriented sides from the set $\mathcal{P}(p,q,\gamma,v)$ (the set $\mathcal{P}(p,q,\gamma,v)$ depends on the choice of the parameter $v \in V^*$). In the picture, these are the pairs (δ,β) and (λ,α). In this case, the multiplication formula is $(r \smile_v s)(\gamma) = r(\delta)s(\beta) + r(\lambda)s(\alpha)$.

Example 2.2. Let $\bigwedge^* V$ be a Grassmann algebra of V. Let the cocycle $r_1 \in C^1(X, \bigwedge^* V)$ be defined as follows: $r_1(\alpha) = \alpha$, where $\alpha \in X$ is any oriented edge of the P-complex. Then the 2-cocycle $r_1 \smile_v r_1$ does not depend on the choice of v. Hence it follows, that the area of the polygon γ is equal to the sum of the areas (depending on the choice of v) triangles, formed by pairs of vectors from the set $\mathcal{P}(p,q,\gamma,v)$. In the picture, these are pairs of vectors (δ,β) and (λ,α). A multidimensional version of this statement is one of the formulations of the tropical BKK theorem; see [8].

The source of the multiplication rule (2.1) is the multiplication of some special cocycles, arising in the geometry of polyhedra. To construct these cocycles, the following definition of volume is used.

Let U be a bounded domain in oriented q-dimensional affine subspace of the space V and $\beta_U \in \bigwedge^q V$ is such a multivector, that $\int_U \omega = \omega(\beta_U)$ for any $\omega \in \bigwedge^q V^*$. The multivector β_U changes sign when changing the orientation of U. We consider β_U as the q-dimensional volume of U.

Let X be a P-complex in the space V and $S = \bigwedge^* V$ be a Grassmann algebra of the space V. We denote by V_δ the subspace, generated by the point differences of the cell $\delta \in X$. Consider the q-dimensional cochain X

$$\mathrm{vol}_q^X(\delta) = \beta_\delta \tag{2.2}$$

with values in $\bigwedge^q V \supset \bigwedge^q V_\delta$. From Pascal's equations for $(q+1)$-dimensional cells of X it follows that the cochain vol_q^X is a cocycle with values in S. (Pascal's equation for the convex polytope Δ is the equality $\sum_\lambda \eta_\lambda = 0$, where λ is a facet of Δ, and η_λ is the vector of its exterior normal with length, equal to area of λ). It is true that for any choice of the functional v

$$\mathrm{vol}_p^X \smile_v \mathrm{vol}_q^X = \frac{(p+q)!}{p!q!}\mathrm{vol}_{p+q}^X. \tag{2.3}$$

Hence we get that

$$(\mathrm{vol}_1^X)^p = p!\mathrm{vol}_p^X. \tag{2.4}$$

The equation (2.1) reduces the statement (2.3) to the case of a P-complex X, which consists of the faces of the convex polytope. In this latter case, it is formulated in the language of tropical varieties and becomes the so-called "tropical formula BKK"

Let the P-complex $\bar{\Delta}$ consist of the faces of the convex polytope Δ. Further, from the properties of the cocycles vol_p^X, only the following simple property of the 1-cocycles $\mathrm{vol}_1^{\bar{\Delta}}$ is used. Let $\Delta = \Lambda + \Gamma$ be a Minkowski sum of convex polytopes. Any edge δ of polytope Δ is uniquely represented as

$\delta = \lambda + \gamma$, where λ, γ are the faces of the summand polytopes Λ, Γ. Then

$$\mathrm{vol}_1^{\bar{\Delta}}(\delta) = \mathrm{vol}_1^{\bar{\Lambda}}(\lambda) + \mathrm{vol}_1^{\bar{\Gamma}}(\gamma) \tag{2.5}$$

(if $\dim \lambda = 0$ then by definition $\mathrm{vol}_1^X(\lambda) = 0$).

In conclusion, recall the definition of a tropical variety and explain its connection with cocycles vol_p^X. Let $\mathcal{K}$ be a k-dimensional fan of cones in N-dimensional vector space E. In other words, $\mathcal{K}$ is a P-complex, whose cells are convex polyhedral cones. For $K \in \mathcal{K}$, we denote by E_K the subspace in E generated by the cone K. Let $W \colon K \mapsto W(K) \in \bigwedge^q E^*$ be a p-chain of P-complex $\mathcal{K}$. Let's say that W is a p-chain of degree q.

Definition 2.3. **(1)** Closed k-chain W of degree $N-k$ is called the weighted chain of the k-dimensional fan of the cones $\mathcal{K}$, if

$$\forall \{v_1 \in E_K, v_2, \ldots, v_{N-k} \in E\} : \; W(K)(v_1 \wedge \ldots \wedge v_{N-k}) = 0.$$

The value of $W(K)$ is called the weight of the cone K.

(2) Fan of cones with a closed weight chain is called *a tropical fan.*

Remark 2.1. In publications on tropical geometry, the closedness of the chain W sometimes called the *balance relations* or *additive relations*.

Example 2.3. Let $\bar{\Delta}$ be a P-комплекс of faces of the convex polytope $\Delta \subset E^*$. We denote by $\mathcal{K}_{d,\Delta}$ the d-dimensional fan of cones, consisting of the cones dual to the faces Δ of dimension $\geq N-d$. For the d-dimensional cone K dual to the $(N-d)$-dimensional face $\Lambda \subset \Delta$ we put

$$W(K) = \mathrm{vol}_d^X(\Lambda).$$

Then *the fan $\mathcal{K}_{d,\Delta}$ with the weight chain W is a d-dimensional tropical fan.*

Definition 2.4. Any partition of tropical fan $\mathcal{K}$ with weights inherited from $\mathcal{K}$ also is a tropical fan. Two tropical fans are said to be equivalent, if they have a common tropical partition. The equivalence class of tropical fans is called *tropical variety.*

Remark 2.2. Sometimes (see [A1]), in the tradition of classical intersection theory (see, for example, [8], subsection 4.1.5), instead of the term "tropical variety" the term "tropical cycle" is used.

It is known that tropical varieties in the space E form a commutative graded ring $\mathbf{T}(E)$ with the following properties (see [K5], [9]):

(1) Equidimensional tropical fans of dimension $N-k$ form homogeneous component $\mathbf{T}_k(E)$ of degree k in the ring $\mathbf{T}(E)$ (a fan $\mathcal{K}$ of dimension k is called equidimensional, if any of its cones is a face of some k-dimensional cone $K \in \mathcal{K}$).

(2) The spaces $\mathbf{T}_0(E)$, $\mathbf{T}_N(E)$ are one-dimensional

(3) If Δ is a convex polytope in E^* then (see example 2.3)

$$\mathcal{K}_{p,\Delta} \cdot \mathcal{K}_{q,\Delta} = \mathcal{K}_{p+q,\Delta}.$$

(4) Tropical varieties $\mathcal{K}_{p,\Delta}$, corresponding to convex polytopes $\Delta \subset E^*$, generate an additive group of the space $\mathbf{T}_p(E)$. For $p = 1$, according to (2.5), it is true that $\mathcal{K}_{1,\Delta} + \mathcal{K}_{1,\Lambda} = \mathcal{K}_{1,\Delta+\Lambda}$, i.e. the addition of tropical varieties corresponds to the Minkowski addition of polytopes. Moreover, the tropical varities $\mathcal{K}_{1,\Delta}$ are the generators of $\mathbb{R}$-algebra $\mathbf{T}(E)$.

(5) The pairing $\mathbf{T}_p(E) \times \mathbf{T}_{N-p}(E) \to \mathbf{T}_N(E)$, which is defined as the multiplication of tropical varieties, is non-degenerate.

(6) For any linear operator $s\colon V \to U$, there exists the pull back ring homomorphism $s^*\colon \mathbf{T}(U) \to \mathbf{T}(V)$. The corresondence $s \mapsto s^*$ is functorial, i.e. if $s = s_1 \cdot s_2$ then $s^* = s_2^* \cdot s_1^*$. Moreover, for any convex polytope $\Delta \subset U^*$ it is true that $s^*\mathcal{K}_{\Delta,k} = \mathcal{K}_{s'\Lambda,k}$, where $s'\colon U^* \to V^*$ is a linear operator conjugate to operator $s\colon V \to U$.

2.2. **Exponential tropical geometry** ([1,2]).

2.2.1. *Mixed Monge-Ampere operator.* The Monge–Ampere operator of degree k is the map defined by

$$(h_1, \dots, h_k) \to dd^c h_1 \wedge \dots \wedge dd^c h_k$$

(recall that, given a function g on a complex variety, the value of the 1-form $d^c g$ at a tangent vector x_t equals $dg(x_t/i)$). We regard the values of this operator as currents, i.e. linear functionals on the space of compactly supported smooth differential forms. If the h_i are continuous convex functions on the space $\mathbb{C}^n$, then the current $dd^c h_1 \wedge \dots \wedge dd^c h_k$ is well defined. This means that, if the functions h_i are locally uniformly approximated by smooth plurisubharmonic (in particular, convex) functions, then the sequence of the values of the Monge–Ampere operator weakly converges to a current not depending on the choice of an approximation. The limit current can be extended to a functional on the space of forms with continuous coefficients, i.e., is a current of measure type. For example, the values of the Monge-Ampere operator of degree n are measures in the space $\mathbb{C}^n$.

We say that a continuous function $h\colon \mathbb{C}^n \to \mathbb{R}$ is piecewise linear if there is a finite set of convex polyhedra Δ, such that $\mathbb{C}^n = \bigcup \Delta$, and $\forall \Delta\colon\ h\colon \Delta \to \mathbb{R}$ is a real first-degree polynomial. Any piecewise linear function is the difference of two convex piecewise linear functions. Therefore, the Monge–Ampere operator

$$(h_1, \cdots, h_k) \mapsto dd^c h_1 \wedge \cdots \wedge dd^c h_k$$

is well defined for piecewise linear functions h_i. By definition, *Monge-Ampere currents* are the images of the mixed complex Monge-Ampere operator on tuples of piecewise linear functions. We define the product of Monge-Ampere currents as

$$(dd^c h_1 \wedge \dots \wedge dd^c h_p) \cdot (dd^c h_{p+1} \wedge \dots \wedge dd^c h_{p+q}) = dd^c h_1 \wedge \dots \wedge dd^c h_{p+q}.$$

In what follows, we assume that all polyhedra Δ_i are cones. The ring generated by the corresponding Monge-Ampere currents, denote by $\mathcal{A}$.

Example 2.4. Let h_i be the support function of the convex polytope $\Delta_i \subset (\mathbb{C}^n)^*$ (recall that $h_i(z) = \max_{w \in \Delta_i} \operatorname{Re} w(z)$. Then the following is true.

(1) If $\Delta_i \subset \mathrm{Re}\,(\mathbb{C}^n)^*$, then the support of the measure $dd^c h_1 \wedge \ldots \wedge dd^c h_n$ is the subspace $\mathrm{Im}\,\mathbb{C}^n$, and this measure is the measure of the n-dimensional Euclidean area of the space $\mathrm{Im}\,\mathbb{C}^n$, multiplied by $n!\,\mathrm{vol}(\Delta_1, \ldots, \Delta_n)$, where $\mathrm{vol}(\Delta_1, \ldots, \Delta_n)$ is the mixed volume of polytopes Δ_i. In this way, according to the Kushnirenko-Bernstein theorem, this measure contains information about the number of common zeros of n Laurent polynomials with Newton polytopes Δ_i. Note that the measure $dd^c h_1 \wedge \ldots \wedge dd^c h_n$ does not depend on the choice of the Hermitian metric in the space $\mathbb{C}^n$.

(2) For arbitrary polytopes Δ_i the integral of this measure over the ball of radius 1 centered at 0 is called the mixed pseudovolume of the polytopes Δ_i. According to [K3,K4], this pseudovolume is equal to the density of the set of common zeros of n exponential sums with Newton polytopes Δ_i. For mixed pseudovolumes of polyhedra, the vanishing criterion is valid, similar to the criterion for the vanishing of mixed volumes of convex bodies (see [8] and [2]). In particular, the pseudovolume of the polyhedron is 0, if and only if the polytope is contained in some proper complex affine subspace of $\mathbb{C}^n$.

We denote by $\mathcal{A}_{\mathbb{R}}$ the subring of the ring $\mathcal{A}$, consisting of Monge-Ampere currents $dd^c h_1 \wedge \cdots \wedge dd^c h_k$, such that

$$\forall (i \leq k, z \in \mathbb{C}^n, x \in \mathrm{Re}\,\mathbb{C}^n)\colon\; h_i(z + ix) = h_i(z).$$

According to [E1] and [1], the ring $\mathcal{A}_{\mathbb{R}}$ is isomorphic to the ring of tropical varieties in the space $\mathrm{Re}\,\mathbb{C}^n$.

2.2.2. *Exponential tropic varieties.* Below we define the notion of an exponential tropical variety (ETV). ETVs form a ring. This ring is isomorphic to the ring of Monge-Ampere currents. Below (see subsection 3.1) it is explained that the ring of tropical varieties in the space $\mathrm{Re}\,\mathbb{C}^n$ is isomorphic to the quasi-algebraic ring of conditions of the space $\mathbb{C}^n$. It turns out that the ring of conditions of the theory of intersections of arbitrary (not necessarily quasi-algebraic) EASs is isomorphic to the ring ETVs. The text containing the proof of this assertion is in preparation.

Let $2N$-dimensional space E be a reification of the complex vector space $\mathbb{C}^N$. Using the pairing $(z, z^*) = \mathrm{Re}\langle z, z^*\rangle$, we identify the dual space E^* with the reification of the space of complex linear functionals $\mathbb{C}^{N^*}$ in $\mathbb{C}^N$. Let $\mathcal{K}$ be a fan of cones of dimension $N+k$ in the space E. An odd function $W\colon K \mapsto W(K) \in \bigwedge_{\mathbb{C}}^m \mathbb{C}^{N^*}$ on the set of p-dimensional oriented cones $K \in \mathcal{K}$ is said to be *a complex p-chain of degree m*. As usual, we define a complex $(p-1)$-chain dW of the same degree, called the boundary of the complex p-chain W. A complex chain W is called closed if $dW = 0$.

We denote by E_K the real subspace of E generated by the cone K. We call a complex $(N+k)$-chain W of degree $N-k$ *the weight chain*, if for any $K \in \mathcal{K}$, $e_1 \in E_K, \ldots, e_{N-k} \in E_K$ is true, that

$$W(K)(e_1 \wedge \ldots \wedge e_{N-k}) \in \mathbb{R}. \tag{2.6}$$

In this case, $W(K)$ is called the *complex weight of the cone K*.

Definition 2.5. Fan of cones of dimension $N+k$ with closed complex weight chain is called the k-dimensional exponential tropical fan (hereinafter ETF). The degree of a k-dimensional ETF is, by definition, $N-k$.

From the (2.6) it follows, that the $(N-k)$-form $W(K)$ vanishes on any set of vectors, containing a vector from any complex subspace of $\mathbb{C}^N$ contained in E_K.

Definition 2.6. Any partition of ETF $\mathcal{K}$ with complex weights inherited from $\mathcal{K}$ also is ETF. Two ETFs, having a common partition are called equivalent. The equivalence class of k-dimensional ETFs is called *exponential tropical variety* (ETV) of dimension k. The union of $(N+k)$-dimensional cones with nonzero weights will be called the support of ETV.

Next, we define the linear operator $\Pi\colon \mathbf{T}(E) \to \mathbf{E}(E)$ of the space of tropical varieties to the space of ETVs in E. To do this, we consider the homomorphism of Grassmann algebras $\varrho\colon \bigwedge_{\mathbb{R}} E^* \to \bigwedge_{\mathbb{C}} \mathbb{C}^{N*}$, such that

$$\forall a \in E^*\colon \varrho(a) = -ia.$$

If $\mathcal{K}$ is a tropical fan in E with a weight chain Φ, then the fan $\mathcal{K}$ with complex weight chain $\rho(\Phi)$ is ETF.

Example 2.5. We denote by $\mathcal{K}$ the one-dimensional tropical fan of cones on the plane $\mathbb{C}$, consisting of one cone $K = \operatorname{Re}\mathbb{C}$ with a weight chain $\Phi(z) = \operatorname{Im}(z)$. Then the restriction of $\rho(\Phi)$ to the cone K is real. Really,

$$\rho(\Phi)(x) = \Phi(\varrho(x)) = \Phi(-ix) = \operatorname{Im}(-ix) = -x.$$

Therefore, $\rho(\Phi)$ is a complex weight chain and the fan $\mathcal{K}$ is ETF.

The above defined mapping, which takes the tropical fan $\mathcal{K}$ with the weight chain Φ to the ETF $\mathcal{K}$ with complex weight chain $\varrho(\Phi)$, denote by Π.

Theorem 2.1. (1) *The mapping Π is a surjective linear operator from the space of tropical fans in E onto the space of* ETFs*, and also from the space of tropical varieties to the space of* ETVs $\mathbf{E}(E)$ *in E.*

(2) *The kernel* $\ker\Pi$ *of the operator* Π *is an ideal of the ring of tropical varieties.*

(3) *The k-dimensional tropical fan $\mathcal{K}$ belongs to* $\ker\Pi$*, if and only if for any k-dimensional cone $K \in \mathcal{K}$ the dimension of the maximal complex subspace of the space E_K is greater than $k-N$. In particular, if* $\dim\mathcal{K} < N$*, then* $\Pi(\mathcal{K}) = 0$.

Further, using the mapping Π, we consider the space ETVs, as the quotient ring of the ring of tropical varieties by the ideal $\ker\Pi$. We note, that the homomorphism Π preserves degrees (but not dimensions!) of tropical varieties.

Example 2.6. The ETV $\Pi(\mathcal{K}_{\Delta,k})$ (see example 2.3) can be described as follows. Let K be a cone, dual to the k-dimensional face Λ of the polytope $\Delta \subset E^* = \mathbb{C}^{N*}$. We denote by ω *a complex symplectic form* on the space

$\mathbb{C}^{N*} \oplus \mathbb{C}^N$ (i.e. $\omega = dz \wedge dz^*$) and put $\Omega = (-i)^{2N-k}\omega^k/k!$. Consider the projection mapping $\pi_\Lambda\colon \Lambda \oplus \mathbb{C}^N \to \mathbb{C}^N$. Choose the orientation of the face Λ and denote by $W(K)$ the push forward $(\pi_\Lambda)_*\Omega$ of Ω. The orientation of the face Λ corresponds to its co-orientation and, therefore, to the orientation of the cone K. Thus, on a fan of cones, dual to the faces of Δ of dimension $\geq k$, a complex k-chain $W\colon K \mapsto W(K)$ of degree $(2N-k)$ is defined. The chain W is weighted and closed. Let's denote the defined ETF by $\mathcal{K}_{\Delta,k;\mathbb{C}}$. it is true that

$$\mathcal{K}_{\Delta,k;\mathbb{C}} = \Pi(\mathcal{K}_{\Delta,k}).$$

From this we get that

1) ETVs $\mathcal{K}_{\Delta,1;\mathbb{C}}$ span the ring $\mathbf{E}(E)$
2) $\mathcal{K}^m_{\Delta,1;\mathbb{C}} = m!\,\mathcal{K}_{\Delta,m;\mathbb{C}}$.

2.2.3. ETVs *as Monge-Ampere currents.* Consider ETF $\mathcal{K}$ of dimension k as a current $\bar{\mathcal{K}}$ of degree $2k$ (linear functional on the space of compactly supported differential forms of degree $2k$) as

$$\bar{\mathcal{K}}(\varphi) = \sum_{K\in\mathcal{K}, \dim K = N+k} \int_{\mathcal{K}} W(K)\wedge\varphi$$

(recall that the weighted form $W(K)$ is real-valued). If $k > 0$, then the current $\bar{\mathcal{K}}$ is closed (this follows from the closedness of the weight form W). For equivalent ETFs $\mathcal{K}, \mathcal{L}$ the currents $\bar{\mathcal{K}}, \bar{\mathcal{L}}$ are the same.

Theorem 2.2. *Map $\mathcal{K} \mapsto \bar{\mathcal{K}}$ is an isomorphism of the ring* ETFs *into the ring of Monge-Ampere currents $\mathcal{A}$ (see subsection 2.2.1).*

We fix the Hermitian metric of the space $\mathbb{C}^N$ and denote by χ the characteristic function of the unit ball centered at 0. For 0-dimensional ETF $\mathcal{K}$ we put $\operatorname{psv}(\mathcal{K}) = \bar{\mathcal{K}}(\chi)$ (recall that the support fan of a 0-dimensional ETF has a dimension N).

Definition 2.7. We will call $\operatorname{psv}(\mathcal{K})$ the pseudovolume of the 0-dimensional ETV $\mathcal{K}$. If $h_1, \dots, h_N$ are support functions of polytopes $\Delta_1, \dots, \Delta_N$, then $\operatorname{psv}(dd^c h_1 \wedge \dots \wedge dd^c h_N)$ is called the *mixed pseudovolume* of these polytopes.

Remark 2.3. Because the Monge-Ampère operator $h \mapsto (dd^c h)^k$ is defined for any plurisubharmonic (in particular, convex) function h, then the pseudovolumes and mixed pseudovolumes of any convex bodies are also correctly defined in a Hermitian complex space. Pseudovolume as a function on the set of convex bodies is a unitary invariant valuation; see [Al].

The geometric definition of the pseudovolume of a polyhedron is as follows. For the N-dimensional face Λ of the polytope $\Delta \subset \mathbb{C}^{N*}$ we introduce the notation:

(1) $\operatorname{vol}_N(\Lambda)$ is an N-dimensional volume Λ,
(2) $A(\Lambda)$ is an angle of the dual cone Λ (full N-dimensional angle is 1),

(3) $c(\Lambda)$ is an area distortion coefficient under the orthogonal projection $R_\Lambda^\perp \to \sqrt{-1}R_\Lambda$, where R_Λ is a tangent space of the face Λ, and $R_\Lambda^\perp$ is an orthogonal to R_Λ subspace $\mathbb{C}^{N*}$.

The pseudovolume of Δ equals

$$\sum_{\Lambda\subset\Delta,\,\dim(\Lambda)=N} c(\Lambda)\, A(\Lambda)\, \mathrm{vol}_N(\Lambda).$$

The pseudovolume of a polygon in $\mathbb{C}^1$ is equal to its semi-perimeter, the pseudovolume of the polyhedron $\Delta \subset \mathrm{Re}\, C^{N*}$ is equal to its N-dimensional volume.

2.3. **Eigenfunctions of Laplace operator** ([4]). Let M be a compact Riemannian manifold without boundary, $n = \dim M$, and dx the Riemannian measure on M. For an eigenvalue λ of the Laplace operator Δ on M let $H(\lambda)$ denote the corresponding eigenspace, i.e.,

$$H(\lambda) = \{f \in C^\infty(M, \mathbb{R})|\, \Delta u + \lambda u = 0\}.$$

Then $H(\lambda)$ is a finite dimensional real vector subspace of $L^2(M, dx)$, considered with the induced scalar product. We note that the space $H(\lambda)$ and the scalar product are invariant under any isometry of M. Our goal is to define and, under certain assumptions, to evaluate the average number of zeros of the system of equations

$$u_1 = u_2 = \ldots = u_n = 0, \tag{2.7}$$

where $u_i \in H(\lambda)$ are linearly independent. The linear envelope of u_i is a subspace $U \subset H(\lambda)$ of dimension n. We denote by $Z(U)$ the number of isolated common zeros of functions u_i from (2.7). The average number of zeroes $\mathfrak{M}(\lambda)$ is defined as the integral of $Z(U)$ over the Grassmanian $\mathrm{Gr}_n(H)$ with respect to the normalized measure induced by the Haar measure of $SO(N, R)$ acting on $H(\lambda)$.

Theorem 2.3. *Let $M = K/V$ be a homogeneous space of a compact connected Lie group K with a K-invariant Riemannian metric. Then*

$$\mathfrak{M}(\lambda) \le \frac{2}{\sigma_n n^{n/2}} \lambda^{n/2} \mathrm{vol}\, M \tag{2.8}$$

For isotropically irreducible homogeneous spaces, the estimate (2.8) becomes an equality, which was previously proven in [Gi]. In this case, applying equality (1.3), we obtain the following result (also from [Gi])

$$\mathfrak{M}(\lambda_1, \ldots, \lambda_n) = \frac{2}{\sigma_n n^{n/2}} \mathrm{vol}\, M \sqrt{\lambda_1 \cdots \lambda_n},$$

where $\mathfrak{M}(\lambda_1, \ldots, \lambda_n)$ is an average number of zeroes of systems (2.7) with $u_i \in H(\lambda_i)$.

Remark 2.4. (1) The right side of the estimate (2.8), up to a coefficient, coincides with the first term of the asymptotics of the eigenvalue number from the celebrated Weyl's law, see [Iv]. Therefore, we obtain an estimate for the average number of common zeros of eigenfunctions in terms of the asymptotic expression for the eigenvalue number.

(2) The classical Courant's theorem [Co] says that the number of nodal domains defined by the k-th eigenfunction does not exceed k. Consider now the set Z of common zeros of $m \leq n$ eigenfunctions. In order to carry over Courant's theorem to this case, V.Arnold suggested to study the topology of the analytic set Z and to find the dependence of suitable topological invariants of Z on the number of the corresponding eigenvalue of the Laplacian, see [A] (see Problem 2003-10), p.174). We follow Arnold's suggestion for $m = n$ under certain additional assumptions of group-theoretic character.

2.4. **Smooth version of BKK theorem** ([6,7]). Let $V_1, \ldots, V_n$ be finite dimensional spaces of smooth functions on n-dimensional differentiable manifold X. In [6] we considered equations

$$f_1 = \ldots = f_n = 0;\ f_i \in V_i \tag{2.9}$$

Each function f_i is defined up to a non-zero constant multiplication factor. Therefore the space of systems (2.9) is compact. In this case we assume that spaces V_i are Euclidian. The average number of common zeroes of the functions f_i turns out to be equal to the mixed volume of Banachs bodies corresponding to chosen metrics in V_i. Each of these Banach bodies is a family of ellipsoids; see also [ZK]. This statement is also used to compute the average number of roots in various scenarios, see, for example, the previous section 2.3.

In [7] we consider systems of equations

$$f_1 - c_1 = \ldots = f_n - c_n = 0;\ f_i \in V_i,\ 0 \neq c_i \in \mathbb{R}. \tag{2.10}$$

In this case (see section 3.2), there is a wide choice of averaging methods. Each averaging method corresponds to a set of Banach bodies in X; see section 3.2. These Banach bodies can be arbitrary. We also prove that no matter how we choose the method of averaging, the average number of roots always equals to the mixed volume of Banach bodies.

2.5. **Real roots of systems of random Laurent polynomials** ([10]). It is known that the expectation of the fraction of real zeros of a real polynomial of increasing degree m asymptotically equal to $\frac{2}{\pi}\frac{\log m}{m} \asymp 0$; see [Kac]. This calculation assumes that the coefficients of the polynomial are normally and independently distributed. with zero means and unit variances. For more details on the distribution of the number of real solutions to systems of random polynomial equations, see the review [EK] and the references therein.

The transition from ordinary polynomials to Laurent ones leads to an unexpected result. The restriction of real Laurent polynomial of degree m (see below Definition 2.8)

$$P(z) = a_0 + \sum_{1 \leq k \leq m} a_k z^k + \overline{a_k} z^{-k}$$

to the circle $z = e^{i\theta}$ is a trigonometric polynomial

$$P(e^{i\theta}) = a_0 + \sum_{1\le k\le m} \alpha_k \cos(k\theta) + \beta_k \sin(k\theta),$$

where $\alpha_k = (a_k + \overline{a_k})/2$, $\beta_k = (a_k - \overline{a_k})/(2i)$. The restriction is an isomorphism of the space of real Laurent polynomials of degree m onto the space of real trigonometric polynomials with spectrum $(-m, \dots, -1, 0, 1, \dots, m)$. The randomness assumption is that the numbers

$$a_0/\sqrt{2\pi}, \alpha_1/\sqrt{\pi}, \beta_1/\sqrt{\pi}, \dots, \alpha_m/\sqrt{\pi}, \beta_m/\sqrt{\pi}$$

are normally and independently distributed with zero means and unit variances. It turns out that, as the degree m increases, the average fraction of zeros located on the unit circle of a random real Laurent polynomial tends not to 0, but to $1/\sqrt{3}$. This follows from the known results on the distribution zeros of random trigonometric polynomials in one variable; see, for example, [ADG] and the bibliography there. Below we describe a multidimensional analogue of this phenomenon.

Definition 2.8. If a Laurent polynomial is real-valued on a compact subtorus $T^n = \{z \in (\mathbb{C}\setminus 0)^n\colon\ z = (e^{i\theta_1}, \dots, e^{i\ theta_n})\}$ of the torus $(\mathbb{C}\setminus 0)^n$, then we call it *real Laurent polynomial.* If the Laurent polynomial vanishes at the point $z \in T^n$, then z is called the *real zero* of the Laurent polynomial.

The following statements are direct consequences of the definition of 2.8.

(A) Laurent polynomial $P(z) = \sum_{\lambda\in\Lambda} a_\lambda z^\lambda$ is real if and only if 1) its support Λ is centrally symmetric, and 2) $\forall \lambda \in \Lambda\colon\ a_{-\lambda} = \overline{a_\lambda}$.

(B) The set of zeros of the real Laurent polynomial is invariant under the mapping $(z_1, \dots, z_n) \mapsto (\bar{z}_1^{-1}, \dots, \bar{z}_n^{-1})$.

Next, we consider Laurent polynomials in n variables and use the concepts *the system of random real Laurent polinomial supported at* Λ and *the mean fraction* $\mathrm{real}_n(\Lambda)$ *of real roots* of such random systems.

Theorem 2.4. *Let B_m be a ball with radius m in $\mathbb{R}^n$ centre at the origin, $\mathbb{Z}^n$ be the integer lattice in $\mathbb{R}^n$, and let $\Lambda_m = B_m \cap \mathbb{Z}^n$. Then*

$$\lim_{m\to\infty} \mathrm{real}_n(\Lambda_m) = \left(\frac{\sigma_{n-1}}{\sigma_n}\beta_n\right)^{\frac{n}{2}},$$

where $\beta_n = \int_{-1}^{1} x^2(1-x^2)^{\frac{n-1}{2}}dx$, and σ_k is a volume of k-dimensional unit ball.

The following table shows the values β_n for $1 \le n \le 20$; note that $\sqrt{\frac{\sigma_0}{\sigma_1}\beta_1} = \frac{1}{\sqrt{3}}$ (see below Example 2.8).

Remark 2.5. The expression $x^2(1-x^2)^{\frac{n-1}{2}}dx$ is the so-called *Tchebyshev differential binomial.* Tchebyshev proved [Tch] that the binomial $x^m(a +$

TABLE 1. β_n for $n \leq 20$

n	1	2	3	4	5	6	7	8	9	10
β_n	$\frac{2}{3}$	$\frac{\pi}{8}$	$\frac{4}{15}$	$\frac{\pi}{16}$	$\frac{16}{105}$	$\frac{5\pi}{128}$	$\frac{32}{315}$	$\frac{7\pi}{256}$	$\frac{256}{3465}$	$\frac{21\pi}{1024}$
n	11	12	13	14	15	16	17	18	19	20
β_n	$\frac{512}{9009}$	$\frac{33\pi}{2048}$	$\frac{4096}{109395}$	$\frac{429\pi}{32768}$	$\frac{2048}{45045}$	$\frac{715\pi}{65536}$	$\frac{65536}{2078505}$	$\frac{2431\pi}{262144}$	$\frac{131072}{4849845}$	$\frac{4199\pi}{524288}$

$bx^n)^p dx$ is non-integrable in elementary functions outside three cases of integrability found by L. Euler. For odd n, the above expression refers to the first, and for even n, to the third case.

Recall that the Newton polytope of the Laurent polynomial is the convex hull of $\mathrm{conv}(\Lambda)$ its support Λ. We also define the ellipsoid $\mathrm{ell}(\Lambda) \subset \mathrm{conv}(\Lambda)$, called *the Newton ellipsoid of* Λ, and prove that

$$\mathrm{real}_n(\Lambda) = \frac{\mathrm{vol}\,(\mathrm{ell}(\Lambda))}{\mathrm{vol}\,(\mathrm{conv}(\Lambda))} \tag{2.11}$$

For n = 1, the Newton ellipsoid is a line segment with ends at the points $\pm\sqrt{\frac{1}{N}\sum_{\lambda\in\Lambda}\lambda^2}$, where $N = \#\Lambda$.

Example 2.7. In the one-dimensional case, the segments $\mathrm{ell}(\Lambda)$ and $\mathrm{conv}(\Lambda)$ coincide only for supports of the form $\Lambda = \{\lambda, -\lambda\}$. In this case, (2.11) implies that $\mathrm{real}_1(\Lambda) = 1$, that is, all zeros of any polynomial of the form $az^\lambda + \bar{a}z^{-\lambda}$ lie on the unit circle, which is true, because these zeros are 2λ roots of $-\bar{a}/a$.

Example 2.8. Let $\Lambda_m = \{-m, \dots, -1, 0, 1, \dots, m\}$. Then

$$\sqrt{\frac{1}{\#\Lambda_m}\sum_{k\in\Lambda_m} k^2} = \sqrt{\frac{2(1^2+\ldots+m^2)}{2m+1}} = \sqrt{\frac{m(m+1)}{3}}$$

Hence, according to (2.11), we get that $\mathrm{real}_1(\Lambda_m) = \sqrt{\frac{m+1}{3m}}$, and hence $\lim_{m\to\infty}\mathrm{real}_1(\Lambda_m) = 1/\sqrt{3}$.

We also define the fraction of real roots $\mathrm{real}_n(\Lambda_1, \dots, \Lambda_n)$ for systems of Laurent polynomials with supports $\Lambda_1, \dots, \Lambda_n$. In this case, the volumes in the numerator and denominator of the fraction (2.11) are replaced by mixed volumes of the corresponding Newton ellipsoids and Newton polytopes. Using the geometry of the formula (2.11), we calculate the asymptotics of $\mathrm{real}_n(\Lambda_1, \dots, \Lambda_n)$ for growing supports $\Lambda_1, \dots, \Lambda_n$, and apply this calculation to prove Theorem 2.4.

3. MAIN RESULTS

3.1. **Intersections of exponential analytic sets.** First (using the notation of the subsection 1.1), we define the notions of tropicalization, weak density, and intersection index of quasi-algebraic EASs. Let G be a finitely generated subgroup of the additive group $(\operatorname{Re}\mathbb{C}^{n*})_+$ of $\operatorname{Re}\mathbb{C}^{n*}$, and let $\mathbb{T}_G$ be the torus of characters of G. Assume that G contains a basis of the space $\operatorname{Re}\mathbb{C}^{n*}$. For any $z \in \mathbb{C}^n$ we take $\omega_G(z)$ to be the character of G defined by $\omega_G(z)\colon g \mapsto \mathrm{e}^{\langle z,g\rangle}$. In this way we obtain an embedding of groups $\omega_G\colon \mathbb{C}^n \to \mathbb{T}_G$.

Definition 3.1. We call the image $\omega(\mathbb{C}^n)$ of ω the standard winding on the torus $\mathbb{T}_G$, and $\omega_G\colon \mathbb{C}^n \to \mathbb{T}_G$ the standard winding map.

Let E_G denote the ring of ESs supported in G. The image $\omega_G(\mathbb{C}^n)$ of the standard winding is everywhere Zariski dense in $\mathbb{T}_G$. Therefore, the pullback map $\omega_G^*\colon \mathbb{C}[\mathbb{T}_G] \to E_G$ is a ring isomorphism[1].

Definition 3.2. Let I be the ideal of E_G generated by the equations of an EAS X. Let $M_G \subset \mathbb{T}_G$ denote the zero locus of the ideal $\omega^*(I) \subset \mathbb{C}[\mathbb{T}_G]$. We call M_G a model of the EAS X.

Definition 3.3. (1) If the model M_G of EAS X is equidimensional (i.e., it consists of irreducible components of equal dimensions), then EAS X is called equidimensional.

(2) The codimension M_G in $\mathbb{T}_G$ is said to be the algebraic codimension of EAS X and denoted by $\operatorname{codim}_{\mathrm{a}} X$.

This definition uses the group G, containing the supports of the EAS X equations. However, it is easy to see that $\operatorname{codim}_{\mathrm{a}} X$ and the property of X to be equidimensional do not depend on the choice of G. The concept of equidimensional EAS is a replacement for the nonexistent concept of irreducibility: any EAS is uniquely represented as a union of equidimensional EASs of different algebraic codimensions.

Definition 3.4. The tropicalization M_G^{trop} of the model M_G (see [9]) of EAS X is a tropical variety in the space $\operatorname{Re}\mathfrak{T}_G$, where $\mathfrak{T}_G$ is the Lie algebra of the torus $\mathbb{T}_G$. Let $s_G\colon \operatorname{Re}\mathbb{C}^n \to \mathfrak{T}_G$ be a restriction of differential of the mapping ω_G at the point 0 to the space $\operatorname{Re}\mathbb{C}^n$. The pullback $X^{\mathrm{trop}} = s_G^* M_G^{\mathrm{trop}}$ of the tropical variety M_G^{trop} (see property (6) at the end of the section 2.1) is a tropical variety in the space $\operatorname{Re}\mathbb{C}^n$, and is called the tropicalization of EAS X.

It turns out that *the tropicalization X^{trop} of* EAS X *does not depend on a choice of group G.*

Below we use the following notation: for a 0-dimensional tropical variety $\mathcal{K}$ in the space $\operatorname{Re}\mathbb{C}^n$ its zero cone weight form (see definition 2.3) is equal to $w(\mathcal{K}) \cdot \mathrm{vol}_n$, where vol_n is a volume form, corresponding to the Euclidean metric in $\operatorname{Re}\mathbb{C}^n$, and $w(\mathcal{K}) \in \mathbb{R}$.

Definition 3.5. The dimension of the tropical variety X^{trop} equals to $n - \operatorname{codim}_{\mathrm{a}} X$. In particular, if $\dim_{\mathrm{a}} X = n$, then $\dim X^{\mathrm{trop}} = 0$. We denote the constant $w(X^{\mathrm{trop}})$ by $d_w(X)$ and call it the weak density of EAS X.

[1] The practice of viewing exponential sums as restrictions of Laurent polynomials to a dense winding on the torus goes back to Weyl's celebrated paper [W38].

Example 3.1. If EAS $X \subset \mathbb{C}^1$ is given by the equation $f(z) = 0$, then $d_w(X) = \frac{p}{2\pi}$, where p is the length of Newton's segment of ES f. The number of zeros of the function f in a circle of increasing radius r asymptotically equals to $2rd_w(X)$.

Example 3.2. (see [Z,K2]) Assume that EAS is given by equations $f = g = 0$. If f and g have no common divisor in the ring of ESs, then $\mathrm{codim_a} X = 2$; otherwise $\mathrm{codim_a} X = 1$. In particular $0 \in \mathbb{C}$ treated as the EAS given by the equations $\mathrm{e}^z - 1 = \mathrm{e}^{\sqrt{1}z} - 1$ has algebraic codimension 2. Thus, the codimension of an analytic set X can be lower than $\mathrm{codim_a} X$. Let (X, z) be the irreducible germ of an EAS X at $z \in X$. If (X, z) has lower codimension than $\mathrm{codim_a} X$, then the germ is said to be atypical. It is known that each atypical germ of an EAS lies in a proper affine subspace of $\mathbb{C}^n$. In particular, each component of anEAS of algebraic codimension 2 in $\mathbb{C}^2$ is an affine line. In addition, it is known that the set of minimal affine subspaces containing atypical components is small in a certain sense. The phenomenon of atypically large intersections of algebraic varieties with windings of tori also studied in algebraic Diophantine geometry; see [BMZ].

The following tropical definition of intersection index $I(X, Y)$ for EASs X, Y, such that $\mathrm{codim_a} X + \mathrm{codim_a} Y = n$, is equivalent to the definition 1.2.

Definition 3.6. $I(X, Y) = w\left(X^{\mathrm{trop}} \cdot Y^{\mathrm{trop}}\right)$, where $X^{\mathrm{trop}} \cdot Y^{\mathrm{trop}}$ is a product in the ring of tropical varieties.

Recall that equidimensional EASs X, Y of algebraic codimension k are called numerically equivalent, if for any EAS Z of algebraic codimension $n - k$ it is true that $I(X, Z) = I(Y, Z)$. The main results of the EASs intersection theory are as follows

(1) The algorithm De Concini and Procesi for constructing the ring of conditions $\mathcal{E}_n$ for the above defined numerical equivalence of EASs terminates successfully.

(2) The mapping $X \to X^{\mathrm{trop}}$ is constant on the numerical equivalence classes, and defines an isomorphism of the ring of conditions $\mathcal{E}_n$ onto the ring of tropical varieties $\mathbf{T}(\mathrm{Re}\,\mathbb{C}^n)$ in $\mathrm{Re}\,\mathbb{C}^n$.

Note, that the Hermitian metric of the space $\mathbb{C}^n$ is involved in the definitions of weak density and of intersection index. However, the partition of EASs into numerical equivalence classes and the structure of the ring of conditions do not depend on the choice of this metric.

In conclusion, we present several consequences of the main statements.

Corollary 3.1. *The ring of conditions $\mathcal{E}_n$ is generated by the images of exponential hypersurfaces.*

Corollary 3.2. *The ring $\mathcal{E}_n$ is isomorphic to the ring of convex polytopes in the space* $\mathrm{Re}\,\mathbb{C}^n$ *(the definition of polytope ring in [8]).*

Corollary 3.3. *For any quasi-algebraic* ESs $f_1, \dots, f_n$ *there is an exponential hypersurface* $\Xi \subset \mathbb{C}^{n\times n}$*, such, that the following is true. If* $(w_1, \dots, w_n) \notin \Xi$*, then the weak density of* EAS *given by the equations* $f_1(z + w_i) = \ldots = f_n(z + w_n) = 0$*, is equal to* $n! \operatorname{vol}(\Delta_1, \dots, \Delta_n)$*.*

3.2. **Smooth version of BKK theorem.** Let $V_1, \dots, V_n$ be finite-dimensional spaces of smooth functions on the n-dimensional differentiable manifold X. For $f_i \in V_i$, $a_i \in \mathbb{R}$ we identify the equation $f_i - a_i = 0$ with affine hyperplane $H_i \in \operatorname{AGr}^1(V_i^*) = \{v^* \in V_i^* \colon v^*(f_i) = a_i\}$ (recall that $\operatorname{AGr}^k(V)$ is the manifold of affine subspaces in V of codimension k). Respectively, we identify the set of equations $\{f_i - a_i = 0\}$ with the set of affine hyperplanes $\{H_i \in \operatorname{AGr}^1(V_i^*)\}$.

Definition 3.7. A translation invariant signed Borel measure on $\operatorname{AGr}^k(V)$, finite on compact sets, is said to be a normal measure. The vector space of normal measures on $\operatorname{AGr}^k(V)$ is denoted by $\mathfrak{m}_k$.

Remark 3.1. Normal measures form a ring. On the one hand, this ring is a smooth version of the ring of tropical varieties (see [8,9]), on the other hand, it is isomorphic to the ring of "valuations on convex bodies" constructed by S. Alesker (see [Al2]). The operation of multiplication of normal measures, as well as the connection between the concept of a normal measure with the concept of "valuation on convex bodies" is explained in [7].

Let $\nu_1 \in \operatorname{AGr}^1(V_1^*), \dots, \nu_n \in \operatorname{AGr}^1(V_n^*)$ be smooth normal measures of degree 1. Consider the measure $\nu = \nu_1 \times \ldots \nu_n$ on the manifold $\operatorname{AGr}^1(V_1^*) \times \ldots \times \operatorname{AGr}^1(V_n^*)$. The integral of the number of roots of systems of the form

$$f_1 - a_1 = \ldots = f_n - a_n = 0. \tag{3.1}$$

with respect to measure ν is called *the average number of roots of systems* (3.1).

Let $\mu \in \operatorname{AGr}^1(E)$. We define a positively homogeneous function $H_\mu \colon E \to \mathbb{R}$ as follows: for $e \in E$, we set $H_\mu(e)$ equal to the measure of the set of affine hyperplanes, intersecting the segment $[0, e]$. If the function H_μ is convex, then it is a support function of some centrally symmetric convex body in the dual space E^*. If the measure ν is non-negative, then the function H_μ is convex. In this case, the corresponding convex body is a *zonoid*; see [S]. Recall that a zonotope is a polytope representable as the Minkowski sum of segments, and zonoid is the limit of the sequence of zonotopes converging with respect to the Hausdorff topology on the set of convex bodies in the space E^*. If the ball of Banach metric is a zonoid, then such a metric is said to be *a zonoid metric*. Thus, the nonnegative measure $\mu \in \operatorname{AGr}^1(E)$ defines a zonoidal Banach metric in the dual space E^*. On the other hand, any smooth Banach metric in the space E^* is the difference of two zonoid metrics and, hence, it comes by this way from some (perhaps not everywhere positive) measure μ on the manifold $\operatorname{AGr}^1(E)$.

In this way, the collection of measures $\nu_i \in \mathrm{AGr}^1(V_i^*)$ defines the Banach metrics on the spaces V_i. Any collection of Banach metrics of the spaces V_i can be obtained in a similar way. If the measures ν_i are non-negative, then these metrics are zonoid metrics.

We denote by D_i the unit ball of the Banach metric, corresponding to the measure ν_i. Consider the mapping $F_i\colon X \to V^*$, assigning to the point x the functional $F_i(x)\colon v \mapsto v(x)$. Let $dF^*_{i,x}\colon V \to T^*_x X$ be a linear operator, conjugate to the differential of the mapping F_i at the point x. Denote by $\mathcal{B}_i(x)$ the image $dF^*_{i,x}(D_i)$ of the unit ball D_i. Family of convex bodies $\mathcal{B}_i(x)$ in fibers of the cotangent bundle T^*X forms a Banach body on the manifold X. The main result [8] is that that *average number of solutions of systems of the form* (3.1) *is equal to the mixed symplectic volume of Banach bodies* (see subsection 1.2) $\mathcal{B}_1, \ldots, \mathcal{B}_n$.

List of publications presented for the defence.

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