Skolkovo Institute of Science and Technology

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Elena Nikolaevna Gryazina

RANDOMIZED ALGORITHMS IN OPTIMIZATION AND CONTROL WITH APPLICATIONS IN ENERGY SECTOR

Dissertation summary

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Contents

Chapter 1

Introduction

Until recently, randomized algorithms in optimization were mainly focused on discrete and NP-hard problems. On the contrary, in control problems, the main efforts were aimed at obtaining a convex structure; in particular, this is why control problems deal with quadratic stability instead of stability, quadratic robust stability instead of robust stability, etc. However, in practical problems, even when working with a convex domain, the objective functional usually remains non-convex. Thus, random walk methods for generating asymptotically uniform samples are relevant both for optimization on convex domains described by linear matrix inequalities, and on non-convex (generally disconnected) feasibility domains. In some cases, convex relaxations open the way for computationally efficient algorithms for solving such problems. One such example is the power system feasibility domain, described by quadratic equations.

The development of the theory, models and methods for calculating the optimal operating modes of the energy system does not lose relevance due to the widespread use of distributed renewable energy sources, changing patterns of electricity consumption and digital transformation of the energy industry. First, the management of modern energy systems requires fast and reliable methods for assessing stability margins. In addition, it is important to make informed decisions on the installation and operation of energy storage systems, taking into account their degradation. Finally, the widespread adoption of distributed renewables is appealing to launch a peer-to-peer electricity market. All these tasks require a fundamentally different look at the power systems feasiblity domain, as well as the development of realizable optimization models.

In this summary we describe recently developed approaches to computationally demanding problems optimization and control with particular applications in energy sector. The obtained results contain randomized approaches for control and optimization and energy systems applications. We focus on the development the methodology for sampling, which showed itself as a powerful tool to solve control and optimization problems over domains with complicated geometry [1], [2], investigate convex relaxations for optimal power flow problem [3] and investigate the convexity of the quadratic image [4]. We also propose optimization models to address recently appeared problems in smart grids. There are: voltage stability margin assessment [5], optimal siting, sizing, and technology selection of Energy Storage Systems [6], energy efficient indoor microclimate control in buildings [7], centralized and distributed power systems state estimation and anomaly detection [8, 9] and peer-to-peer energy market with engineering constraints [10].

The results obtained form the basis for a comprehensive analysis, modeling and optimization of electrical distribution networks, including storage application strategies, peer-to-peer electricity trading, load identification, smart charging of electrical vehicles and other network services.

Object and goals of the dissertation.

The purpose of the dissertation is twofold. The first goal is the development of random walk methods for optimization and control problems. This includes both the development of new methods and the extension of the class of control and optimization problems that can be solved by generating asymptotically uniform samples in regions with complex geometry. The second goal is the development of the theory, models and methods for calculating the optimal operating modes of the power system, as well as other optimization models facilitating efficient and reliable operation of smart grid.

The obtained results:

1. We propose a new random walk method Billiard Walk for generating asymptotically uniformly distributed samples in a domain with an available boundary oracle.

2. We develop algorithms for constructing a boundary oracle for domains in the parameter space of stable and robustly stable polynomials, stable matrices, and domains described by linear matrix inequalities.

3. We analyse convex relaxations for the optimal power flow problem and propose an approach to substantiate their accuracy (zero duality gap) based on the analysis of the geometry of the feasibility region, which is the image of a quadratic operator.

4. We investigate the image convexity for quadratic maps. In particular, we propose randomized algorithm to obtain convexity/nonconvexity certificates for the individual quadratic transformation.

5. We provide numerically robust and fast algorithm for online voltage stability assessment estimating the static stability of a power system of several thousand nodes.

6. We propose a transformation of an optimization model for a non-convex problem of optimal placement and choice of parameters of an energy storage system, taking into account degradation, into a problem of integer convex programming.

7. We develop a distributed algorithm for clearing of peer-to-peer electricity market, taking into account network restrictions, user preferences and network fees.

Author's contribution includes the mathematical problem formulations, the development of theoretical statements, mathematical models and methods, analysis and generalization of the results.

The novelty of the proposed research lies in the development of new methods and the study of optimization models. In particular, in the dissertation the author proposes:

- Random walk method for generating asymptotically uniform samples;
- Randomized algorithm for checking convexity (or certifying non-convexity) of the image of a quadratic mapping;
- Optimization model for the problem of tracing the stability boundary of power system;
- New models of operation for the peer-to-peer electricity market.

The scope of dissertation is covered in 30 publications, among those we specifically mention papers [1], [5], [6], [7], [8], [9], [10], [11] in Q1-journals; papers [2], [3] Q2-journals; and papers [4], [12], [13], [14], [15], [16], [17] in Scopus conference proceedings.

According to regulations of the Dissertation Council in Computer Sciences of Higher School of Economics 10 papers are listed below. The defense is performed based on 7 of them (namely, first 6 from the list of first-tier publication and one mentioned second-tier publication).

First-tier publications:

1. E. Gryazina and B. Polyak, "Random sampling: Billiard walk algorithm," *European Journal of Operational Research*, vol. 238, no. 2, pp. 497–504, 2014. Scopus Q1 (main co-author; the author of this thesis has proved statements about the asymptotic uniformity of samples generated by the proposed method (Theorems 1,2), she also has carried out numerical simulation and analyzed its results)

2. B. T. Polyak and E. N. Gryazina, "Randomized methods based on new monte carlo schemes for control and optimization," *Annals of Operations Research*,

vol. 189, no. 1, pp. 343–356, 2011. Scopus Q2 (main co-author; the author of this thesis has proposed to use and compare various methods for generating samples asymptotically uniformly distributed in a given domain, has developed boundary oracle procedures for various classes of optimization and control problems, carried out numerical experiments and analyzed their results)

3. A. Zorin and E. N. Gryazina, "An overview of semidefinite relaxations for optimal power flow problem," *Automation and Remote Control*, vol. 80, no. 5, pp. 813–833, 2019. Scopus Q2 (main co-author; the author of this thesis has proposed a geometric approach to the analysis of the accuracy of convex relaxations, has selected relaxations for comparison, carried out numerical experiments to compare the accuracy and scalability of the selected relaxations)

4. M. Ali, E. Gryazina, O. Khamisov, and T. Sayfutdinov, "Online assessment of voltage stability using newton-corrector algorithm," *IET Generation, Transmission Distribution*, vol. 14, no. 19, pp. 4207–4216, 2020. Scopus Q1 (the author of this thesis has formulated the problem of monitoring stability margins in real time, she also has proposed the idea of the method, as well as scenarios for numerical experiments)

5. T. Sayfutdinov, C. Patsios, P. Vorobev, E. Gryazina, D. M. Greenwood, J. W. Bialek, and P. C. Taylor, "Degradation and operation-aware framework for the optimal siting, sizing, and technology selection of battery storage," *IEEE Transactions on Sustainable Energy*, vol. 11, no. 4, pp. 2130–2140, 2019. Scopus Q1 (the author of this thesis has proposed an approach to reformulate the optimization model, making it possible it possible to reduce the computational complexity of the application)

6. T. Chernova and E. Gryazina, "Peer-to-peer market with network constraints, user preferences and network charges," *International Journal of Electrical Power*

& Energy Systems, vol. 131, p. 106981, 2021. Scopus Q1 (main co-author; the author of this thesis has proposed the formulation of an optimization problem for a peer-to-peer electricity market, where network restrictions are present explicitly in the form of restrictions, and also has developed a decentralized algorithm for this problem)

7. A. Ryzhov, H. Ouerdane, E. Gryazina, A. Bischi, and K. Turitsyn, "Model predictive control of indoor microclimate: Existing building stock comfort improvement," *Energy conversion and management*, vol. 179, pp. 219–228, 2019. Scopus Q1 (the author of this thesis reviewed approaches to solving the problem of indoor microclimate control, and also proposed a formal statement of the problem of model predictive control for this problem)

8. S. Asefi, Y. Madhwal, Y. Yanovich, and E. Gryazina, "Application of blockchain for secure data transmission in distributed state estimation," *IEEE Transactions on Control of Network Systems*, 2021. Scopus Q1 (the author of this thesis has proposed the formulation of the problem of distributed state estimation and also the selection of methods for its solution)

9. Z. Jin, J. Zhao, L. Ding, S. Chakrabarti, E. Gryazina, and V. Terzija, "Power system anomaly detection using innovation reduction properties of iterated extended kalman filter," *International Journal of Electrical Power & Energy Systems*, vol. 136, p. 107613, 2022. Scopus Q1 (the author of this thesis has carried out a critical analysis of the problem statement, validation, analysis and interpretation of the results obtained)

Second-tier publications:

1. B. Polyak and E. Gryazina, "Convexity/nonconvexity certificates for power flow analysis," in *Trends in Mathematics*, pp. 221–230, Springer, 2017. Scopus Q3 (main co-author; the author of this thesis has proposed the concept of using randomized algorithms to analyze the convexity/ non-convexity of the image of a quadratic mapping, has proved the statements (Theorems 1, 2), she also has

performed numerical experiments to test the operation of the algorithm and has analysed its results)

Reports at conferences and seminars:

1. EURO Mini-Conference "Continuous Optimization and knowledge-Based Technologies" EurOPT-2008, Neringa, Lithuania, 20-23.05.2008, "Randomized methods based on new Monte Carlo schemes for convex optimization".

2. The 17th World Congress on International Federation of Automatic Control (IFAC 2008), South Korea, Seoul, 6-11.07.2008, "Hit-and-Run: new design technique for stabilization, robustness and optimization of linear systems".

3. IEEE Multi-conference on Systems and Control, St-Petersburg, 8-10.07.2009, "Robust Stabilization via Hit-and-Run Techniques".

4. VII School-seminar for young researches "Upravlenie boljshimi sistemami", Perm, 26-29.05.2010, "Efficient random walk".

5. IEEE American Control Conference, Baltimore, USA, 30.06-01.07.2010, "Mixed LMI/Randomized Methods for Static Output Feedback Control Design".

6. IEEE Multi-Conference on Systems and Control, Yokohama, Japan, 8-10.09.2010, "Markov Chain Monte Carlo method exploiting barrier functions with applications to control and optimization".

7. XIII-th Conference of Young Scientists "Navigation and Motion Control", Saint-Petersburg, 15-18.03.2011, "Randomized Hit-and-Run-based methods in control problems".

8. Seminar at Apatity: Kola Branch of Petrozavodsk State University, 25-30.04.2011, "Randomized sampling algorithm for center of gravity method".

9. The 18th World Congress on International Federation of Automatic Control (IFAC 2011), Milan, Italy, 28.08-2.09.2011, "Hit-and-Run: new randomized technique for control problems recasted as concave programming".

10. 20th International Conference MATHEMATICS. COMPUTER. EDUCATION. Pushino, 28.01-2.02.2013, "Billiard walk – new sampling algorithm".

11. IEEE European Control Conference (ECC), Zurich, Switzerland, 17-19.07.2013, "Robust control of magnetic guidance lightweight AGVs path tracking using randomization methods".

12. 19th World Congress on International Federation of Automatic Control (IFAC 2014), Cape Town, South Africa, 24-29.08.2014, "Billiard walk - a new sampling algorithm for control and optimization".

13. XI School-seminar for young researches "Upravlenie boljshimi sistemami", Arzamas, 6-9.09.2014, "On the comparison of random walks".

14. XII School-seminar for young researches "Upravlenie boljshimi sistemami", Volgograd, 7-11.09.2015, "Trusted region for stability analysis of power system's operating regimes".

15. International Symposium on Energy System Optimization (ISESO 2015), Heidelberg, Germany, 9-10.11.2015, "Convexity/nonconvexity certificates for power flow analysis".

16. IEEE International Conference on the Science of Electrical Engineering (ICSEE), 16-18.11.2016, Eilat, Israel, "Fragility of the semidefinite relaxation for the optimal power flow problem".

17. International conference "Relay protection and automation for electric power systems", Saint-Petersburg, 25-28.05.2017, "Analysis of dynamic stability using adaptive quadratic Lyapunov functions".

18. XVII Baikal International School-Seminar "Methods of optimization and their applications", Maksimikha, Buryatia, 31.07-06.08.2017, "Semidefinite relaxations for the optimal power flow: robust or fragile?"

19. All-Moscow regular scientific seminar "Control Theory and Optimization" in Institute for Control Sciences, Moscow, 24.10.2017, "A few optimization problems in energy sector".

20. IEEE International Conference on Environment and Electrical Engineering and IEEE Industrial and Commercial Power Systems Europe (EEEIC / ICPS Europe), Palermo, Italy, 12-15.06.2018, "Methodology for Computation of Online Voltage Stability Assessment".

21. 1st IEEE International Youth Conference on Radio Electronics, Electrical and Power Engineering (REEPE 2019), Moscow, 14-15.03.2019, "Decentralized Optimal Power Flow Under Security Constraints", "Experimental Study of Control Strategies for HVAC Systems".

22. Russian National Committee of CIGRE, Subcommitte C5 Seminar "Energy markets and their regulation", Moscow, 8.04.2019, "Convex relaxations for OPF problem".

23. IEEE PES Innovative Smart Grid Technologies Europe (ISGT-Europe 2019), Bucharest, Romania, 29.09-02.10.2019, "Optimal Energy Management for Off-Grid Hybrid System using Hybrid Optimization Technique".

24. IEEE PowerTech, Milan, Italy, 23-27.06.2019, "Suboptimality of decentralized methods for OPF", "Fast calculation of the transfer capability margins".

25. Cyberverse related 3D algorithms and optimization workshop, Huawei, Moscow, 15-16.09.2020, "Multi-agent distributed cooperation in control and optimization".

26. 3rd IFAC Workshop on Cyber-Physical Human Systems CPHS 2020: Beijing,

China, 3-5.12.2020, "ADMM-based Distributed State Estimation for Power Systems: Evaluation of Performance".

27. IEEE PowerTech, Madrid, Spain, 28.06-02.07.2021, "Peer-to-Peer Market with Energy Storage Systems", "Evaluation of power flow models for smart distribution grids".

28. 4th International Conference on Smart Energy Systems and Technologies (SEST), Vaasa, Finland, 6-8.09.2021, "Optimal partitioning in distributed state estimation considering a modified convergence criterion".

29. The 53rd North American Power Symposium (NAPS), 14-16.11.2021, "A Novel Open Source Power Systems Computational Toolbox".

30. All-Moscow regular scientific seminar "Control Theory and Optimization" in Institute for Control Sciences, Moscow, 27.09.2022, "Randomized algorithms and optimization problems in energy sector".

Chapter 2

Overview of the obtained results

We start with general methodology for sampling in the domains with complicated geometry [1] and we address randomized methods for control and optimization based on sampling [2]. We also provide randomized algorithm to certify convexity/nonconvexity for particular quadratic map [4]. Finally, we address other applications driven optimization problems in energy sector, where feasibility domain of power system is present in constraints set.

2.1 Random walks

Generating points uniformly distributed in an arbitrary bounded region $Q \subset \mathbb{R}^n$ (sampling) finds applications in many computational problems [18]. Straightforward sampling techniques are usually based on one of three approaches: rejection, transformation and composition. In rejection approach the region of interest Q is enclosed within the region with available uniform sampler B (usually a box or a ball). At the next step non belonging to Q samples are rejected. Suppose Q is an unit ball while bounded region B is a box $[-1,1]^n$. Then for $n = 2k$ we obtain the ratio of volumes of the box and the ball $q = \dfrac{\mathrm{Vol}(Q)}{\mathrm{Vol}(B)} = \dfrac{\pi^k}{k!2^k}$, thus $q =\sim 10^{-8}$ for $n = 20$ and we should generate $\sim 10^8$ samples to have just a few in Q. For polytopes this ratio can be much smaller. The other way to exploit pseudo-random number generator for simple region B is to map B onto Q via smooth deterministic function with constant Jacobian. For instance, to obtain uniform samples in $Q = \{x : x^T Ax \leq 1\}$, A being positive definite matrix, it suffices to take y uniform in the unit ball $||y||_2 \leq 1$ and transform them as $x = A^{-1/2}y$. Unfortunately, such a transformation exists just for a limited class of regions. In composition approach we partition Q for finite number of sets that can be efficiently sampled. Apart from narrow class of regions with available partition, Q is partitioned into finite union of simplices and the number of simplices makes the procedure computationally hard.

Other sampling procedures use modern versions of Monte Carlo technique, based on Markov Chain Monte Carlo (MCMC) approach [19]. For instance, recent efficient algorithms for volume computation based on random walks can be found in [20]. One of the most famous and effective algorithms of MCMC type is Hit-and-Run (HR) that was discussed in details in [21]. Unfortunately, even for simple "bad" sets, such as level sets of ill-posed functions, HR techniques fail or at least are computationally inefficient. A variety of applications and drawbacks of existing techniques propose much room for improvement new sampling algorithms. For instance, there were attempts to exploit the approach, developed for interior-point methods of convex optimization, and to combine it with MCMC algorithms. As a result Barrier Monte Carlo method [22] generates random points that are preferable in comparison with standard Hit-and-Run. But the complexity of each iteration in general is high enough (the calculation of $\left(\nabla^2 F(x)\right)^{-1/2}$, where $F(x)$ is a barrier function of the set, is needed). Moreover such approach can not accelerate convergence for sets like simplices.

Now we describe the random walk algorithm – Billiard Walk – motivated by physical phenomena of a gas diffusing in a vessel. A particle of gas moves with constant speed until it meets a boundary of the vessel, then it reflects (the angle of incidence equals the angle of reflection) and so on. When the particle hits another one, its direction and speed changes. In our simplified model we assume that direction changes randomly while speed remains the same. Thus our model combines ideas of Hit-and-Run technique with use of billiard trajectories. There exist a vast literature on mathematical billiards, and many useful facts can be extracted from there [23, 24].

In contrast to traditional theories, that addresses the behavior of one particular billiard trajectory in different billiard tables, their ergodic properties and the conditions for existence of periodic orbits, we extend billiard trajectories with random change of directions, this introduction of randomness enriches their ergodic properties.

Suppose there is a bounded closed connected set $Q \subset \mathbb{R}^n$ and a point $x^0 \in Q$. Our aim is to generate asymptotically uniform samples $x^i \in Q$, $i = 1, \dots, N$. The brief description of Hit-and-Run algorithm is as follows. At every step HR generates a random direction uniformly over the unit sphere and chooses next point uniformly from the segment of the line in given direction in Q.

New algorithm Billiard Walk generates a random direction uniformly as Hit-and-Run. But the next point is chosen as the end of the billiard trajectory of length ℓ. This length is chosen randomly: we assume that probability of collision with another particle is proportional to δt for small time instances δt, this validates the formula for ℓ in Algorithm 1 below.

Algorithm 1 Billiard Walk

Require: $x^0 \in \text{Int } Q$, τ
Ensure: Sequence x^i, $i = 1, \dots, N$
  $i \leftarrow 0$
  $x \leftarrow x^0$
  while $i \leq N$ **do**
    1: $\ell \leftarrow -\tau \log \xi$, $\xi \sim \mathcal{U}[0,1]$ ▷ Generate a length of the trajectory
    2: Pick random direction $d \in \mathbb{R}^n \sim \mathcal{U}[\|d\| = 1]$
    3: Construct billiard trajectory starting at x and with initial direction d. When the trajectory meets a boundary with internal normal s, $||s|| = 1$, the direction is changed as $d \leftarrow d - 2(d, s)s$
    if the point with nonsmooth boundary is met OR the number of reflections exceeds $10n$ **then**
      Go to 4:
    end if
    4: $i \leftarrow i + 1$
    5: $x^i \leftarrow$ the end point of the trajectory of length ℓ
  end while

We prove asymptotical uniformity of the samples produced by Billiard Walk for convex and nonconvex cases separately. The requirements on Q are different for these two cases, while the sampling algorithm remains the same. Consider the Markov Chain induced by the BW algorithm x^0, $x^1, \ldots$. For an arbitrary measurable set $A \subseteq Q$, denote by $\mathbf{P}(A|x)$ the probability of obtaining $x^{i+1} \in A$ for $x^i = x$ by the Billiard Walk algorithm. Then $\mathbf{P}_N(A|x)$ is the probability to get $x^{i+N} \in A$ for $x^i = x$. We also denote by $p(y|x)$ the probability density function for $\mathbf{P}(A|x)$, i.e. $\mathbf{P}(A|x) = \int\limits_A p(y|x)dy$.

Theorem 1 *Assume Q is an open bounded convex set in $\mathbb{R}^n$, the boundary of Q is piecewise smooth. Then the distribution of points x^i generated by the Billiard Walk algorithm tends to the uniform one over Q, i.e.*

$$\lim_{N\to\infty} \mathbf{P}_N(A|x) = \lambda(A)$$

for any measurable $A \subseteq Q$, $\lambda(A) = Vol(A)/Vol(Q)$ and any starting point x.

Theorem 2 *Assume Q is a bounded and open set, the boundary of Q is piecewise smooth and for all $x, y \in Q$ there exists a piecewise-linear path such that it connects x and y, lies inside Q and has no more than B linear parts (B is an arbitrary positive integer). Then the distribution of points x^i generated by the Billiard Walk algorithm tends to the uniform distribution on Q in the same sense as in Theorem 1.*

2.2 Applications in optimization and control

Recent years exhibited the growing interest to randomized algorithms in control and optimization; e.g., see [18]. Up to recent years randomized algorithms in optimization were mostly oriented on discrete optimization and NP-hard problems [19]. There are few publications related to convex case [25]. On the contrary, in control field most efforts were directed on convex structure of the problem; this is why in control problems quadratic stability is used instead of stability, quadratic robust stability instead of robust stability etc. However it remains a challenging problem to deal with basic notions (such as stability) in spite of nonconvexity of the domains under consideration. It seems that Hit-and-Run method and its improved modification

Billiard Walk algorithm provide a useful opportunity to achieve this goal. Up to our knowledge the paper [2] is the first attempt to exploit these random walks in control applications. Random walks provide a promising tool for stabilization and optimization of linear systems. It allows generating random points inside the stability domain or inside performance specification domain in the space of gain matrices for feedback. Thus we can, for instance, generate stabilizing controllers of the fixed structure and optimize some performance index. The only assumption is that one admissible controller is available.

We provide implementation details to make Markov Chain Monte Carlo schemes work for generating samples *asymptotically* uniformly distributed in a bounded closed set $X \in \mathbb{R}^n$ for specific sets valiable for applications. The key concept here is *boundary oracle* – an algorithm that provides $L = \{t \in \mathbb{R} : x^0 + td \in X\}$ for given feasible starting point $x^0 \in X$, where d is a vector specifying the direction in $\mathbb{R}^n$. In the simplest case, when X is convex, this set is the closed interval $[\underline{t}, \overline{t}]$, where $\underline{t} = \inf\{x^0 + td \in X\}$, $\overline{t} = \sup\{x^0 + td \in X\}$. In more general situations boundary oracle provides all intersections of the straight line $x^0 + td, -\infty < t < +\infty$ with X. We also denote *complete boundary oracle* a boundary oracle algorithm that provides also an internal normal vectors to X at the boundary points.

Boundary oracle is available for numerous specific sets X. We provide exact formulations for Boundary Oracle for the given list of sets of practical importance. In control applications the set X is the set of design variables (e.g. controller parameters or uncertainties). It is the admissible set with respect to some specifications (e.g. the set of stabilizing controllers) and the admissible points are most often denoted by k. We keep the notation as $k \in X$ when talking about vector of controller parameters and $P \in X$ operating in the space of symmetric matrices.

1. For the sets given by linear algebraic inequalities

$$X = \left\{x \in \mathbb{R}^n : c_i^T x \leq a_i,\ i = 1, \ldots, m\right\}. \tag{2.1}$$

the boundary oracle for $x^0 + td$ is $[\underline{t}, \bar{t}]$,

$$\underline{t} = \min_{i:\ c_i^T d > 0} \frac{a_i - c_i^T x^0}{c_i^T d}, \quad \bar{t} = \max_{i:\ c_i^T d > 0} \frac{a_i - c_i^T x^0}{c_i^T d}.$$

2. Stability set for polynomials.

   Consider the affine family of polynomials

$$P(s,k) = P_0(s) + \sum_{i=1}^{n} k_i P_i(s), \tag{2.2}$$

   where $P_i(s)$ are m-th degree polynomials. The polynomial $P(s)$ is stable (Hurwitz) when all its roots have negative real parts. Define the set X in the space of parameters $k = (k_1, \ldots, k_n)$ which correspond to stable polynomials:

$$X = \{k : P(s,k) \text{ is Hurwitz}\} \tag{2.3}$$

   The geometry of such sets and of their boundaries is pretty complicated, the set X may consist of a few disjoint parts, see [26].

   Sampling procedure looks as follows. We assume that a stable polynomial $P(s,k^0)$ is given. Then we generate random $d \in \mathbb{R}^n$ uniformly distributed on the unit sphere and take $P(s,k^0 + td) = A(s) + tB(s)$, $A(s) = P(s,k^0)$, $B(s) = \sum_{i=1}^{n} d_i P_i(s)$. The explicit algorithm for finding $L = \{t \in \mathbb{R} : A(s) + tB(s) \text{ is Hurwitz}\}$ is available, see Theorem 2 and Algorithm 1 in [26]. In general L consists of not more than $m/2 + 1$ intervals.

3. Stability set for matrices. For a family of matrices $A + BKC$, where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{l\times n}$ are given and $K \in \mathbb{R}^{m\times l}$ is a variable (which represents either uncertainty or control gain) we can distinguish the set of stabilizing gains:

$$X = \{K : A + BKC \text{ is Hurwitz}\}, \tag{2.4}$$

i.e. all eigen values of $A + BKC$ have negative real parts.

The structure of this set is also analyzed in [26]. It can be nonconvex and can consist of many disjoint domains. To construct the boundary oracle we generate matrix $D = Y/||Y||, Y =$ `randn(m,l)` which is uniformly distributed on the unit sphere in the space of matrices equipped with Frobenius norm. Then we get straight line $A + B(K^0 + tD)C = F + tG, F = A + BK^0C, G = BDC$ for a matrix $K^0 \in X$. Then $L = \{t \in \mathbb{R} : F + tG \text{ is Hurwitz}\}$. L consists of finite number of intervals. Boundary oracle is the algorithm for calculating their end points. However sometimes "brute force" approach is more simple. Introduce $f(t) = \max \Re \operatorname{eig}(F + tG)$, then the end points of the intervals are solutions of the equation $f(t) = 0$ and can be found by use of standard 1D equation solvers (such as command `fsolve` in Matlab).

4. Robust stability set. For the affine family of polynomials with uncertain parameters $q \in Q$ this set is defined as

$$X = \{k : P_0(s, q) + \sum_{i=1}^{n} k_i P_i(s, q) \text{ is Hurwitz for all } q \in Q\} \tag{2.5}$$

If Q is a finite set $\{q_1, \ldots, q_m\}$ and m is small, the set X is the intersection of m sets corresponding to m uncertainties q_i, thus the boundary oracle is the intersection of corresponding boundary oracles: $L = \bigcap L_i$. There are also some other cases, when L can be calculated explicitly, for instance $p_i(s, q)$ being interval polynomials.

5. For linear matrix inequalities (LMI) set

$$X = \{x \in \mathbb{R}^n : A_0 + \sum_{i=1}^{n} x_i A_i \preceq 0\}, \tag{2.6}$$

where A_i are symmetric matrices of a certain size for all i, $A \preceq 0$ means that A is negative semidefinite. To derive the boundary oracle we exploit the following result for $A = A_0 + \sum_{i=1}^{n} x_i^0 A_i$, $B = \sum_{i=1}^{n} d_i A_i$.

Let $A \prec 0$ and $B = B^T$. Then the matrix $A + tB$ is negative definite for $t \in (\underline{t}, \overline{t})$:

$$\underline{t} = \begin{cases} \max\limits_{t_i<0} t_i, & \\ -\infty, & \text{if all } t_i > 0, \end{cases} \qquad \overline{t} = \begin{cases} \min\limits_{t_i>0} t_i, & \\ +\infty, & \text{if all } t_i < 0. \end{cases}$$

where t_i are the generalized eigenvalues of the matrix pencil $(A, -B)$, i.e., $Ae_i = -t_i Be_i$. For $t \notin (\underline{t}, \overline{t})$ the matrix $A + tB$ loses negative definiteness.

6. Set of symmetric matrices satisfying linear matrix inequality defined by Lyapunov inequality

$$X = \{P : AP + PA^T + C \preceq 0, \ P \succeq 0\}, \tag{2.7}$$

where A is a stable matrix and $C \succ 0$. This set is always convex, and boundary oracle can be found explicitly. Indeed, take $P_0 \in X$ and generate $D = D^T$ — a matrix specifying the direction. Then $A(P_0 + tD) + (P_0 + tD)A^T + C \preceq 0 \Leftrightarrow F + tG \prec 0$, $F = AP_0 + P_0A^T + C$, $G = AD + DA^T$. For this case $L = (\underline{t}, \overline{t})$ and $\overline{t} = \min \lambda_i$, $\underline{t} = \min \mu_i$, where λ_i are positive real eigenvalues of matrix pencil $F, -G$, while μ_i are positive real eigenvalues of matrix pencil F, G.

7. Set of symmetric matrices satisfying quadratic matrix inequalities (QMI)

$$X = \left\{P : AP + PA^T + PBB^T P + C \preceq 0, \ P \succeq 0\right\}. \tag{2.8}$$

Boundary oracle is obtained similarly to the case 6 after representing the first linear matrix inequality in block form using Schur complement.

Efficient sampling in the domains described by linear matrix inequalities bring promising randomized tool to solve computationally hard problems in control including synthesis of static-output feedback controllers, H_2 and H_∞ optimal control [12].

2.3 Convex relaxations for optimal power flow

Optimal Power Flow problem is one of the most rutined tasks in power systems operation. It aims to determine an optimal power production mode for a given network. There exist different formulations of this problem but the most widespread and accurate one is the optimal power flow problem, which rests on the physical Kirchhoff and Ohm laws. Common optimality criteria are to minimize the total generation cost or loss subject to engineering constraints. Also note a separate approach focused on the stable operation of a power network, which is known as the anti-blackout approach. This problem has higher complexity due to additional constraints connected with its physical nature. As a rule, the stable mode is not optimal in the classical formulation. Thus, the integration of the two approaches seems promising for the industry. A distinctive feature of the optimal power flow (OPF) problem in the classical formulation is its nonconvexity, which makes convex optimization tools directly inapplicable. The system operators adopt the linearized formulation of the problem – so called *DC OPF*. After the linearization the problem can be solved in a fast and simple way, but at the price of the resulting accuracy. Therefore, the methods for solving the original nonconvex problem (AC OPF) are of major interest. Relaxations are a rather popular technique for managing the problem's nonconvexity. Relaxations can be used to considerably reduce the problem's complexity and to solve it in acceptable time with sufficient accuracy. Unfortunately, relaxations do not guarantee the exact solution, and still there are no general formulas or theorems that would describe the existence conditions of the exact solution in some general formulation of the problem.

The OPF problem can be reformulated as a quadratically constrained quadratic programming (QCQP) problem in which the objective function and also all associated constraints are quadratic functions. Unfortunately, in this formulation the problem still remains nonconvex but different convex relaxations can be used. This approach has been intensively developed in the recent years. However, there is no guarantee that a given method will yield the exact solution (or even any solution at all!), which forms its major drawback. For a certain class of the problems, a given method may work for some problems of the class and fail for the other. Consider the standard mathematical formulation of the OPF problem. A power network is a graph G in which the nodes N correspond to the generators and consumers while the edges E to the power lines. The edge e_{ij} exist only if nodes i and j are connected with a power line. A generator, or a consumer, or both simultaneously can be located in each node.

The optimal operating mode of a power network is determined using different approaches but the exact mode corresponds to the solution of the AC OPF problem. This is achieved by adding different engineering constraints for an accurate consideration of all specifics of a given power network. This explains the crucial importance of the problem for the industry. The major difficulty of the AC formulation is its nonconvexity, which creates obstacles on the way towards fast and exact solution. This difficulty can be eliminated using convex relaxations: the original set of admissible solutions is replaced by its convex hull, and the problem is solved on the latter. Note that the original physical structure of the problem is retained. Unfortunately, relaxations may turn out to be inexact. As of today, the exactness conditions have been established for the tree networks only. Moreover, real networks may have cycles.

In paper [3] we consider five different relaxations: semidefinite (SDP), chordal, conic (SOCP), moment-based and QC relaxations. The application of the first three relaxations has been described step-by-step on a simple example. Each of them has certain advantages and shortcomings. For instance, the semidefinite relaxation is very easy to understand and use. The chordal relaxation requires designing the chordal extension of the network graph and obtaining the maximal cliques, which is also a nontrivial problem; however, it is solved once for a given network. The transition from the complete network matrix to the submatrices of its cliques allows using the network sparsity without considerable accuracy losses in comparison with the semidefinite relaxation. The conic relaxation also utilizes the network sparsity and does not require any additional transformations of the graph but it is less accurate than the semidefinite and chordal ones. The accuracy of the moment-based is increasing with its order but a high-order relaxation introduces a huge number of new variables; in real networks, this may dramatically affect the computational time or even make the problem infeasible. The QC relaxation achieves the accuracy of the semidefinite without imposing the rank condition, i.e., the voltages can be always restored. In addition, the accuracy of this relaxation is similar to that of the SDP relaxation.

We deal with the classical formulation of the OPF problem with classical generators and a given demand has been considered. Generally speaking, the real problem is far difficult due to additional engineering constraints. First, in the recent years the share of alternative generators has been significantly increased. They supply very cheap power but suffer from high instability. Therefore, the attempts to add renewable sources into the problem cause various uncertainty. Besides renewable generation, another considerable uncertain factor is the demand, which also represents a random variable. The classical formulation with added uncertainty leads to the stochastic optimal power flow problem. The solution of this problem should avoid excessive conservatism (which often occurs in stochastic optimization), since even small improvements gain important savings. Second, many different criteria of power security or redundancy have to be considered in practice, e.g., the (N1) security criterion (an optimal regime must be stable if one generator or power line fails). Nevertheless, the main issue concerns the conditions under which the relaxations preserve their exactness for mixed networks. This issue still remains open even in the simple AC formulation without stochastics and additional engineering constraints. Small data changes can make the problem infeasible or the resulting solution can have rank > 1, meaning that the optimal mode is unrestorable. In addition to the difficulties connected with nonconvexity and inexactness of the relaxations, the problem can be infeasible due to high dimension. For example, the Russian power system includes about 9000 nodes; hence, the SD relaxation will involve a symmetric variable matrix of dimensions (9000×9000). The semidefinite problems of such a dimension will be almost unsolvable or the solution time will exceed all available limits. Of course, the chordal and conic relaxations can be used to reduce the dimension owing to network sparsity. But this will be insufficient or the chordal and conic relaxations will be inexact. Thus, numerical methods to parallel the problem are required.

2.4 Nonconvexity of quadratic image

Optimal Power Flow problem is considered as minimization of quadratic performance function subject to linear and quadratic equality/inequality constraints, AC power flow equations specify the feasibility domain. Similar quadratic problems arise in discrete optimization, uncertainty analysis, physical applications. In general they are nonconvex, nevertheless, demonstrate hidden convexity structure. We investigate the "image convexity" property. That is, we consider the image of the space of variables under quadratic map defined by power flow equations (the feasibility domain). If the image is convex, then original optimization problem has nice properties, for instance, it admits zero duality gap and convex optimization tools can be applied.

Randomized methods give a tool to certify convexity/nonconvexity of the image under quadratic map [4]. There are several classes of quadratic maps representing the image convexity. We aim to discover similar structure and to obtain convexity or nonconvexity certificates for the individual quadratic transformation. We also provide the numerical algorithms exploiting convex relaxation of quadratic mappings for checking convexity.

Theorem 3 *The convex hull for the feasibility set E is*

$$G = conv(E) = \{\mathcal{H}(X) : X \succeq 0, X_{n+1,n+1} = 1\},$$

where $X = X^T \in \mathbb{R}^{(n+1)\times(n+1)}$,

$$\mathcal{H}(X) = (\langle H_1, X\rangle, \langle H_2, X\rangle, \ldots, \langle H_m, X\rangle)^T$$

$$H_i = \begin{pmatrix} A_i & b_i \\ b_i^T & 0 \end{pmatrix}.$$

Hence we can provide simple sufficient conditions for *membership oracle*, i.e. checking if a particular point $y \in \mathbb{R}^m$ is feasible (belongs to E). Indeed, it is necessary to have

$y \in G$, that is to solve corresponding linear matrix inequality. Alternatively, introduce the variable $c \in \mathbb{R}^m$ and construct matrix $A = \sum c_i A_i$, vector $b = \sum c_i b_i$ and block matrix $H(c) = \begin{pmatrix} A & b \\ b^T & -(c,y) \end{pmatrix}$, then the sufficient condition for $y \notin G$ has the form:

Theorem 4 *If there exists c such that for a specified y*

$$H(c) \succ 0,$$

then y is not feasible.

Indeed if the LMI above is solvable, there exists the separating hyperplane, defined by its normal c that strictly separates y and $G = conv(E)$, hence y does not belong to E. Now we can proceed to a nonconvexity certificate.

Theorem 5 *Let $m \geq 3$, $n \geq 3$, $b_i \neq 0$, and for some $c = (c_1, c_2, \ldots, c_m)^T$, the matrix $A = \sum c_i A_i \succeq 0$ has a simple zero eigenvalue $Ae = 0$, and for $b = \sum c_i b_i$ we have $(b, e) = 0$. Denote $d = -A^+ b$, $x_\alpha = \alpha e + d$, $f^\alpha = f(x^\alpha) = f^0 + f^1\alpha + f^2\alpha^2$. If $|(f^1, f^2)| < \|f^1\| \cdot \|f^2\|$, then E is nonconvex.*

Geometrically the condition implies that the linear function (c, f) attains its minimum on E at points f^α only. But parabola f^α is nonconvex, thus the supporting hyperplane touches E on a nonconvex set.

Now the main problem is to find c (if exists) which satisfies Theorem 5 and hence discovers nonconvexity of the feasible set. For this purpose let us construct so called *boundary oracle* for G. For given $y^0 \in E$ and the arbitrary direction $d \in \mathbb{R}^m$ the following Semidefinite Program (SDP) with variables $t \in \mathbb{R}$, $X = X^T \in \mathbb{R}^{(n+1)\times(n+1)}$ specifies the boundary point $y^0 + td$ of the convex hull:

$$\begin{aligned} &\max t \qquad (2.9)\\ &\mathcal{H}(X) = y^0 + td \\ &X \succeq 0 \\ &X_{n+1,n+1} = 1. \end{aligned}$$

If we obtain $\text{rank}(X) = 1$ for the solution of (2.9) we claim that the obtained boundary point is on the boundary of E. Otherwise, the boundary point of the convex hull does not belong to E.

On the other hand the dual problem to (2.9) gives us normal vector c for the boundary point:

$$\begin{aligned} \min \gamma + (c, y^0) \\ (c, d) = -1 \\ H = \begin{pmatrix} \sum c_i A_i & \sum c_i b_i \\ \sum c_i b_i^T & \gamma \end{pmatrix} \succeq 0 \end{aligned} \tag{2.10}$$

This is SDP problem with variables c, γ.

Equipped with boundary oracle technique (which provides both a boundary point of G and the normal vector c in this point) we are able to generate vectors c to identify nonconvexity as in Theorem 5. Thus we arrive to

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Algorithm 2 Convexity/nonconvexity certificate

Require: $x^0 \in \mathbb{R}^n$, $y^0 = f(x^0)$, N.
  $i \leftarrow 1$
  while $i \leq N$ **do**
    Generate random direction $d^i \sim \mathcal{U}[\|d\| = 1]$, $d^i \in \mathbb{R}^m$
    Solve SDP (2.10) with $d = d^i$
    if the obtained c satisfies Theorem 5 **then**
      Nonconvexity detected. ▷ Save c
    end if
  end while

At the first glance, simpler approach can be applied. Take $c \in \mathbb{R}^m$ and minimize (c, y) on G (given by lemma 1) if such minimum exists. If $A = \sum c_i A_i \succ 0$ the minimum is unique and obtained at rank-1 matrix xx^T, $x = A^{-1}b, b = \sum c_i b_i$, and x gives a boundary point of the feasibilty set E. However to identify nonconvexity we should find c such that A is singular. The probability of this event is zero if we sample c randomly. In our approach (when we generate directions d) the probability of finding a boundary point on a "flat" part of the boundary of G (which correspond to nonconvex E) is positive. In simulation results nonconvexity was identified in all examples, where it has been recognized by other methods.

To conclude it is the strong support of the convexity assumption if our algorithm does not meet nonconvexity after large number of iterations N for various y^0.

2.5 Online assessment of voltage stability

The study of the geometry and the boundary of the feasibility domain continued with the development of a method for fast security assessment. The real-time robust and secure operation of power systems has become a challenging task, as the operating state evolves rapidly due to uncertainties associated with increasing renewable generation, less predictable loads, and various forms of contingencies. Therefore, an online voltage stability assessment is required to avoid any undesirable system behaviours or a large-scale blackout. Such evaluation is not just difficult but also computationally intensive mainly due to the continuously changing state of a grid. This study presents a numerically robust and fast algorithm for online voltage stability assessment with ease of implementation and programming. The proposed approach updates distance of voltage collapse in real-time by incorporating base-case collapse point solution and incoming data from measurement devices [5].

Modern power systems are more vulnerable to instabilities as a result of operational proximity with their loadability limits. Factors such as heavy loading conditions, uncertainties from renewable generation, load recovery dynamics, and different types of contingencies like line tripping or generator outages have made the secure operation of a grid challenging task. In both planning and operational stages, network security is associated with voltage stability, which relates to the ability of the power system to maintain steady-voltage levels at all buses after being subjected to a disturbance [27]. Electrical grids experience voltage instability when the operating regime moves closer to voltage collapse or saddle-node bifurcation point, after which the real solution to power flow equations disappears. Therefore, information about the margin of voltage collapse is necessary for better security assessment.

Studies performed to validate the feasibility of an operating regime at any given time contain two broad categories. The first kind uses voltage stability index (VSI), a scalar parameter that can be monitored as network state changes over time [28]. Index-based methods are simple, computationally tractable, and provide a notion of instability; however, these methods are not suitable for precautionary measures. In contrast to the index-based methods, the continuation techniques (CPF) [29] and direct methods provide a quantitative bound on the distance to collapse in the parameter space (like voltage setpoints or power injections). The algorithmic procedure described in this section belongs to the latter category.

The online studies rely on real-time measurements from the phasor measurement unit (PMU) or supervisory control and data acquisition (SCADA) devices, and updates margin of voltage collapse for moving the state of a network. Besides, they have a significant computational time constraint such that all the incoming measurements can be processed for real-time assessment. The proposed approach is referred to as the Newton-Corrector (NC) algorithm. Unlike CPF solvers, it does not require an explicit tracking of the solution manifold. The mathematical structure of the NC algorithm extends the system of power flow equations using an additional equation to characterise solutions on the boundary of solvability. This auxiliary condition is denoted as the parametric equation. Three different versions of the parametric equation are formulated, which provides sufficient freedom to change the voltage on sensitive buses, do not allow any unexpected numerical updates:

1. Eigenvector-based condition $p_{\text{eig}} = z_0^T J^N y^0 = 0$
2. Singular value-based condition $p_{\text{svd}} = u_n^T J^N v_n^0 = 0$
3. Sensitivity-based condition $p_{\text{s}} = \sum_{k=1}^{n} (|V_k| - |V_k^{\text{pre}}|) \times \text{Im}(v_n^0)$

The algorithm enables a fast evaluation of the margin to collapse for each new state of a grid, with precision and ease of implementation from the computational context.

To update the margin to collapse for given ΔS_i^N, the Newton Correction algorithm was formulated with three different parametric equations (i.e. p_{eig}, p_{svd}, and p_{s}). If x^0 and λ^0 represents a base-case collapse solution, then the following steps are performed such that the algorithm can always find a feasible solution.

1. First, information about new state ΔS_i^N is evaluated from the measurement devices.

2. Then, NC iterations are initiated to compute the margin to collapse for ΔS_i^N with x^0 and λ^0 as an initial guess.

3. Once the algorithm reaches a solution, i.e. x^N and λ^N, the degeneracy condition of J^N is checked through $g(x) = 0$.

4. If the condition holds, then x^N and λ^N is the desired solution.

5. Otherwise, x^N and $\lambda^N$are set as an initial guess and iterations process is initiated by replacing $p(x, \lambda)$ equation with $g(x)$.

The numerical test for the proposed algorithm are reported in [5].

2.6 Optimization model for degradation aware siting, sizing and technology selection for energy storage

The widespread introduction of energy storage systems (ESS) brings also computationally demanding optimization problems in energy sector. We contribute to the problem of optimal placement, choice of parameters, and technology of the energy storage system (which is refered as SST problem), taking into account degradation [6], which makes the optimization problem nonconvex. A solution method is proposed based on transforming the problem into a mixed integer convex programming (MICP) problem by replacing continuous variables that cause non-convexity with discrete ones. The resulting MICP problem has been solved using the Branch-and Bound algorithm along with convex programming, which performs an efficient search and guarantees the globally optimal solution.

A widely used approach to SST problem consists of formulating it as an optimization problem which has to be efficiently solved even for large systems. Thus, careful selection of mathematical models for representing different battery characteristics is required to keep the optimization problem convex; consequently, it is challenging to incorporate processes like degradation to the overall problem formulation. The proposed approach improves solution quality via a degradation aware framework, which incorporates the effects of Depth of Discharge (DoD) and State of Charge (SoC) on battery degradation.

Nonlinearity associated with degradation-aware ESS sizing is dealt with in various ways. In [30] the whole enumeration approach is applied to find the optimal combination of site, size, and technology that gives the least operational cost of a network. In [31] a hybrid heuristic search is applied, where mixed-integer linear programming is used for unit commitment problem, and genetic algorithm is applied for ESS siting and sizing. A stationary degradation map is employed to perform degradation-aware sizing in [32]. Standard optimization problem formulations consist of only equalities and inequalities, and sequential algorithms cannot be directly employed. Furthermore, considering the degradation effects from both SoC and DoD results in the nonconvex optimization problem, for which standard solvers cannot guarantee a globally optimal solution.

The optimization problem is designed to find the optimal combination of site, size and technology of an ESS with respect to the optimal power flow, the optimal scheduling of all power generation and consumption units, the optimal battery operation schedule taking into account an accurate degradation model of the Li-ion battery storage as a function of both DoD and SoC. The stochastic objective function is formulated to find a trade-off between investment cost for ESS and benefits associated with ESS operation. The objective function allows an ESS to perform Energy Time-Shift application, reducing the average daily operational cost of the network over a set of scenarios that represent the whole lifetime horizon of

ESS.

We omit the detailed formulation of the optimization problem here as it contains 18 lines of engineering constraints with extended list of variables and parameters. However, we provide the equations for the capacity fade rate characteristics have been taken from experimental papers, and reproduced from the initial nonuniform data by means of quadratic functions as:

$$\gamma^{\text{Idle}}(SoC) = A_1 SoC^2 + B_1 SoC + C_1$$
$$\gamma^{\text{Cycle}}(DoD) = A_2 DoD^2 + B_2 DoD,$$

where SoC is an average daily state of charge, DoD is a cycle depth of discharge, A_1, A_2, B_1, B_2, C_1 are fitting parameters for the corresponding capacity fade rate characteristics for idling γ^{Idle} and cycling γ^{Cycle}. When considering both sizing and degradation at the same time, the resulting optimization problem is neither linear nor convex. Specifically, the rated energy capacity variable is multiplied by the capacity fade rate characteristics γ^{Idle} and γ^{Cycle}. Standard numerical approaches do not guarantee the global optimum solution for this kind of problem. To overcome this, we propose to substitute continuous variables SoC and DoD, which are the cause of nonconvexity, with integer variables. Therefore, the nonconvex continuous problem becomes a MICP, where the optimization problem possesses the property of convexity for the fixed SoC and DoD.

Thus the inherited non-convexity of degradation-aware SST problem has been resolved with Mixed Integer Convex Programming (MICP) problem reformulation. Finally, to evaluate the performance of this formulation, the obtained results have been compared to four other approaches, as well as offline performance evaluation.

After performing numerical tests we have found that the optimal battery use does not necessarily correspond to it reaching its End of Life state at the end of the service lifetime, which is the result of nonlinear degradation mechanisms from both idling and cycling. Finally, the proposed methodology allows formulating computationally tractable stochastic optimization problem to account for future network scenarios.

2.7 Optimization model for peer-to-peer market with network constraints

In energy systems, there is a clear trend towards decentralization, the topology of networks is changing, microgrids are becoming more widespread, capable of operating both in grid-connected and in an islanded mode. This encourages us to reconsider approaches for dispatch, state assessment, and organization of energy markets. To uncover the potential of distributed generation, novel market structures besides a feed-in-tariff operation could be proposed. There are three possible market models to integrate prosumers and distributed generators: peer-to-peer (P2P) market, prosumer-to-grid operation, and utilization of organized prosumer groups. This work focuses on the design of the P2P electricity market, offering more independence and freedom of action to market participants. The P2P trading scheme enables new types of services and proposes additional value as differentiated contracts, enforced consumer preferences, and increased utilization of distributed generation.

Following the decentralization trend we investigate the architecture of fully decentralized electricity trading (peer-to-peer) market [10] that respects network restrictions. With an increase of distributed generation growing attention is paid to the possibilities of its utilization in the network. Largely driven by distributed ledger technologies, the peer-to-peer market architectures ignored network constraints for a long time, paying more attention to the organization of the financial transactions. In this section we briefly describe an optimization-based peer-to-peer market design, incorporating network constraints, user preferences, and trade-independent network fees. In this way, we ensure a meeting of three requirements critical to the practical implementation of the peer-to-peer market as secure operation, consumer-centric nature of the market, and the provision of benefits for the grid.

We were inspired by the approach [33] that enforces network constraints in an exogenous manner, supplementing objective function with the trade-dependent network charge component, which can be used to release the stress on the grid. The methodology is optimization-based with the power balance ensured through reciprocity constraints. However, introducing fees does not guarantee an absence of congestion. In order to ensure the feasibility of market outcome and the adherence of power flow limits, we apply the matrix of loading vectors (power transfer distribution factors (PTDF) approach) in a built-in form and exploit an exogenous approach to include users' preferences and network charges. We for the first time develop a distributed procedure for the P2P market with built-in congestion management.

Each agent, participating in the bilateral market, firstly solves its local optimization problem. Updated values p_{nm}^{k+1} are reported as the trading proposals to the agents from the trading partnership set of agent n. Following prescribed power limits, the market participant calculates the values ${\lambda_{nm}}^{k+1}$ based on its trading proposal and counteroffer, computes the residuals, then broadcasted to the trading community. The supervisory agent collects the total trading proposals $\mathbf{p}^{k+1}$ and updates auxiliary variables ${\mathbf{y}_1}^{k+1}$ and ${\mathbf{y}_2}^{k+1}$, and Lagrangian multipliers ${\boldsymbol{\mu}_1}^{k+1}$ and ${\boldsymbol{\mu}_2}^{k+1}$. It tests the stopping criteria

$$x_l^{k+1} = \left(\mu_{1l}^{k+1} - \mu_{1l}^{k}\right)^2 + \left(\mu_{2l}^{k+1} - \mu_{2l}^{k}\right)^2, \sum_{l \in L} x_l^{k+1} \leq {\varepsilon_x}^2.$$

The fulfillment of this criterion together with stopping criteria is related to reciprocity requirement ($p_{ij} = -p_{ji}$) indicates that the algorithm has converged to the equilibrium.

Chapter 3

Conclusion

The results of this Thesis are covered in published papers [1], [2], [3], [4], [5], [6], [10], [7], [8], [9], [11]. The defense is based on the first four publications from this list.

In papers [1], [2], [4] we develop randomized algorithms for optimization and control based on random walks.

Papers [3], [5], [6], [10] cover various applications for power systems. The results obtained form the basis for a comprehensive analysis, modeling and optimization of electrical distribution networks, including storage application strategies, peer-to-peer electricity trading, load identification, smart charging of electrical vehicles and other network services.

Let us list the main results that are obtained in this thesis and submitted for defense.

1. We propose a new random walk method Billiard Walk for generating asymptotically uniformly distributed samples in a domain with an available boundary oracle.

2. We develop algorithms for constructing a boundary oracle for domains in the parameter space of stable and robustly stable polynomials, stable matrices, and domains described by linear matrix inequalities.

3. We analyse convex relaxations for the optimal power flow problem and propose an approach to substantiate their accuracy (zero duality gap) based

on the analysis of the geometry of the feasibility region, which is the image of a quadratic operator.

4. We investigate the image convexity for quadratic maps. In particular, we propose randomized algorithm to obtain convexity/nonconvexity certificates for the individual quadratic transformation.

5. We provide numerically robust and fast algorithm for online voltage stability assessment estimating the static stability of a power system of several thousand nodes.

6. We propose a transformation of an optimization model for a non-convex problem of optimal placement and choice of parameters of an energy storage system, taking into account degradation, into a problem of integer convex programming.

7. We develop a distributed algorithm for clearing of peer-to-peer electricity market, taking into account network restrictions, user preferences and network fees.

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