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Faculty of Mathematics

as a manuscript

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Meander diagrams of knots, links and spatial graphs

Summary of the PhD thesis
for the purpose of obtaining academic degree
Doctor of Philosophy in Mathematics

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Moscow – 2023

Contents

Introduction

The present paper refers to the classical theory of knots, links and tangles in Euclidean 3-space $\mathbb{R}^3$. This is a branch of low-dimensional topology that studies embeddings of one-dimensional manifolds into three-dimensional ones. A knot is a pair (S^3, S^1), where S^3 is a three-dimensional sphere and S^1 is the image of a circle smoothly embedded in S^3; the knot is considered up to a homeomorphism of the pair. Knots arise as the first example of interesting pair of manifolds. Indeed, the previous non-trivial pair — (S^2, S^1) — is classified by Schoenflies theorem (all such pairs are equivalent).

Knot theory is a rich area of research with lots of beautiful and deep results, and many fundamental questions of this theory are still open (for example, the problem of efficient knot recognition). At the same time knot theory has extensive connections with other areas of mathematics. For example, any connected closed orientable 3-manifold can be uniquely defined using a link (the Lickorish-–Wallace theorem). Kirby calculus is an example of the application of knot theory to the topology of 3- and 4-manifolds.

The classic way to define knots is to use knot diagrams (see 1 for detailed definitions). The theorem of Reidemeister states that two diagrams represent the same knot if and only if they are connected by a sequence of local moves (Reidemeister moves)[1]. Moreover, in many problems (for example, when constructing knot invariants) it is convenient to consider not all diagrams but only diagrams of a certain type (for example, diagrams that are closures of braid diagrams, or the so-called rectangular diagrams). An interesting question is the study of universal classes of diagrams (a class of knot diagrams is called *universal* if it contains diagrams of all knots). Note that diagrams that are closures of braid diagrams and rectangular diagrams give examples of universal classes.

The starting point of this study was the work [1], where three conjectures about meander and semi-meander diagrams were put forward (see definitions in section 2). The first two conjectures stated the universality of the classes of semimeander and meander diagrams respectively. The third conjecture stated that each two-bridge knot has a semimeander diagram which

[1] And possibly taking a mirror image.

is the minimum diagram of this knot (in terms of the number of crossings, see section 2 for details).

In this paper, we present proofs of generalized versions of the first and the second conjectures (in a non-generalized form they are derived from not widely known results of several papers) and we also prove the third conjecture, which remained open until recently. In addition, using the universality property of semi-meander diagrams, the author defines a family of new knot invariants (k-arc crossing numbers) and investigates their connection with the classical crossing number.

The main results of the paper are Theorems 1–6 published in papers [BM17, B18, BM20, BK+21].

Publications containing the main results of the thesis

[BM17] Belousov Yury, Malyutin Andrei. Simple arcs in plane curves and knot diagrams // Trudy Instituta Matematiki i Mekhaniki UrO RAN. — 2017. — Vol. 23, no. 4. — P. 63–76.

[B18] Belousov Yury. The semimeander crossing number of knots and related invariants // Zapiski Nauchnykh Seminarov POMI. — 2018. — Vol. 476. — P. 20–33.

[BM20] Belousov Yury, Malyutin Andrei. Meander diagrams of knots and spatial graphs: Proofs of generalized Jablan–Radović conjectures // Topology and its Applications. — 2020. — Vol. 274. — P. 107–122.

[BK+21] Lernaean knots and band surgery / Belousov Yu, Karev M, Malyutin A, Miller A, and Fominykh E // St. Petersburg Mathematical Journal. — 2022. — Vol. 33, no. 1. — P. 23–46.

1 Definitions

Let us first introduce the necessary definitions.

Definitions (plane curves)**.** By a *closed plane curve* we mean a smooth immersion (and its image) of a circle S^1 into the plane $\mathbb{R}^2$ (or the sphere S^2). By an *open plane curve* we mean a smooth immersion (and its image) of a closed interval into the plane $\mathbb{R}^2$ (or the sphere S^2). By an *arc* in the curve we mean a restriction of the immersion (and its image) to some interval. A curve is said to be *simple* if it is embedding.
It is assumed throughout that all curves are in *general position*, i. e. there are only finitely many self-intersections and these are transverse double points. For the case of open curves we also assume that no endpoints of the curve are double points.

Let us recall some basic definitions from knot theory. A more complete system of definitions can be found in [2, 3, 4].

Definitions (knots, links and tangles)**.** A *knot* is an embedding (and its image) of a circle S^1 into Euclidean 3-space $\mathbb{R}^3$ (or the 3-sphere S^3). We consider knots up to autodiffeomorphism (not necessarily orientation-preserving) of the ambient space. A knot $K \subset S^3$ is said to be *trivial* it bounds a disk in the ambient space.
By a *tangle* we mean a pair (B^3, T), where B^3 is a 3-disk, and T is compact proper one-dimentional submanifold (with or without boundary) of B^3. By a *link* we mean a tangle (B^3, T) where T is a manifold without boundary. A component $I \subset T$ of a tangle (B^3, T) is said to be *unknotted* if there exists a proper embedding F of a 2-disk B^2 into B^3 such that $I \subset F(B^2)$ ($F(B^2)$ can intersect $T \setminus I$).

Definitions (diagrams)**.** Let K be a knot. The projection of K onto the plane is said to be *regular* if its image is a plane curve in general position. The *diagram* of K is its regular projections with additional information about under-/over-crossings in its double points (double poits are called *crossings*). The diagram of K is said to be *minimal* if there are no diagrams of K with less number of crossings. The *crossing number* of the knot K (designation is $\mathrm{cr}(K)$) is the number of crossings in its minimal diagram.

Definitions (spatial graphs)**.** By a *graph* we mean a finite 1-dimensional CW complex: 0-cells are its *vertices* and 1-cells are *edges*. A *spatial graph* is a subset of $\mathbb{R}^3$ that is (i) ambient isotopic to the union of a finite number of straight line segments and (ii) endowed with the structure of a graph. Two spatial graphs are said to be *equivalent* if they are related by an ambient isotopy preserving the graph structure. A *loop* in a (spatial) graph is an edge whose closure is a circle. A *knotted loop* in a spatial graph is an edge whose closure is a non-trivial knot.
We define diagrams of spatial graph as a natural generalization of diagrams of knots. A *diagram* of a spatial graph G is the plane image of a regular projection of a spatial graph G' equivalent to G with additional information of under- and over-crossings in all double points and with a set of marked points that is the image of the set of vertices of G'. The marked points in a spatial graph diagram are called the *vertices* of the diagram. The images of the graph's edges will be called the *principal arcs* (or *principal curves*) of the diagram. A principal arc is said to be *exceptional* if it represents an edge that is a knotted loop.

2 Jablan–Radović conjectures and its generalizations

Let us now move to the main subject of this paper: the Jablan–Radović conjectures (proposed in [1]) and their generalizations. We need a couple of additional definitions.

Definition 1. We say that a knot diagram D is *semimeander* if it is composed of two smooth simple arcs.

Definition 2. We say that a knot diagram D is *meander* if it is composed of two smooth simple arcs whose common endpoints lie on the boundary of the convex hull of the diagram.

Now let us generalize these definitions to the case of spatial graphs.

Definition 3. We say that a diagram D of a spatial graph G is *semi-meander* if all exceptional principal arcs of D are composed of two simple subarcs, while all other principle arcs are simple.

Definition 4. We say that a semimeander diagram D is meander if (i) all of the vertices of D lie on the boundary of the convex hull of D and (ii) each exceptional principal arc of D is cut into two simple subarcs by a point lying on the boundary of the convex hull of D.

Conjecture 1 ([1])**.** *Each knot has a semimeander diagram.*

Conjecture 2 ([1])**.** *Each knot has a meander diagram.*

Before formulating the third conjecture, let us recall the definitions.

Definitions. A simple arc of a knot diagram is called a *bridge* if it has no under-crossings. A *2-bridge* knot is a knot that has a diagram with two bridges containing all crossings.

Conjecture 3 ([1])**.** *Each 2-bridge knot has a minimal diagram that is semi-meander.*

Statements implying the truth of Conjecture 1 were independently proved in the papers [5, 6, 7, 8, 9]. The same for Conjecture 2 — in the works [10, 9]. Conjecture 3 remained open until recently.

Conjectures 1 and 2 have a natural generalization to the case of spatial graphs. In the paper [11] of A. V. Maluytin and the author the appropriate generalizations and Conjecture 3 were proved.

Theorem 1 ([11])**.** *Each spatial graph has a meander diagram.*

Theorem 2 ([11])**.** *Each 2-bridge knot has a minimal diagram that is semi-meander.*

3 k-arc crossing number

The results of the previous section obviously implies that each knot possesses a diagram composed of k smooth simple arcs, where k is any integer greater than one. There arises a natural question: for a fixed knot K, how much does a diagram of K composed of k smooth simple arcs differ from a minimal diagram of K? We introduce the following definition.

Definition 5. Let K be a knot. Then the minimum number of crossings among all diagrams of K composed of at most k smooth simple arcs is called the *k-arc crossing number of K*. (The designation is $\mathrm{cr}_k(K)$.) Furthermore, the 2-arc crossing number is also called the*semimeander crossing number*.

In the author's paper [12] the relation between the k-arc crossing number and the classical crossing number is investigated. In particular, the following theorems have been proved:

Theorem 3 ([12])**.** *For each knot K, the following inequality holds:*

$$\mathrm{cr}_2(K) \leqslant \sqrt[4]{6}^{\,\mathrm{cr}(K)}.$$

Theorem 4 ([12])**.** *For each knot K and for any integer $k \geqslant 2$, the following inequality holds:*

$$\mathrm{cr}_k(K) \leqslant \mathrm{cr}_{k+1}(K) + \frac{2(\mathrm{cr}_{k+1}(K))^2}{(k+1)^2}.$$

Remark. It is obvious that if K is a fixed knot, then the numbers $\mathrm{cr}_k(K)$ form a nondecreasing sequence. Using the designation

$$\mathrm{p}^*(K) = \min\{k \mid \mathrm{cr}_k(K) = \mathrm{cr}(K)\},$$

we obtain the following chain of inequalities:

$$\begin{aligned} \mathrm{cr}_2(K) \geqslant \mathrm{cr}_3(K) \geqslant \dots \geqslant \mathrm{cr}_{\mathrm{p}^*(K)-1}(K) > \\ > \underbrace{\mathrm{cr}_{\mathrm{p}^*(K)}(K) = \mathrm{cr}_{\mathrm{p}^*(K)+1}(K) = \dots}_{=\mathrm{cr}(K)} \end{aligned}$$

In this context, it is convenient to interpret $\mathrm{cr}(K)$ as $\mathrm{cr}_\infty(K)$.

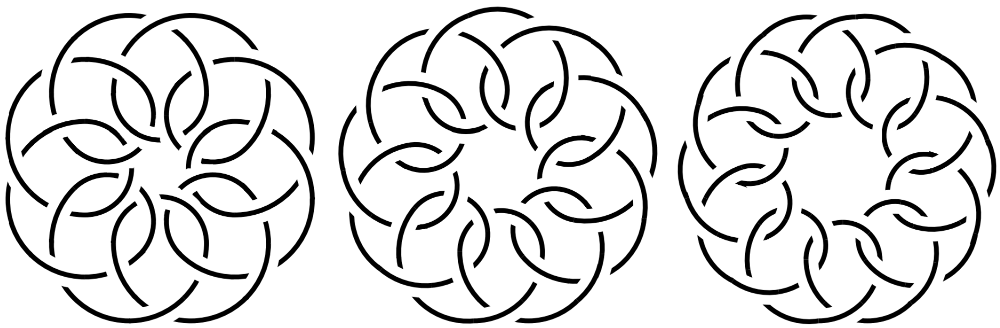

Figure 1: Examples of knots K with $\mathrm{p}^*(K) = \frac{\mathrm{cr}(K)}{3}$.

Remark. It is clear, that for each nontrivial knot K the following inequality holds: $\mathrm{p}^*(K) \leqslant 2\,\mathrm{cr}(K)$. We also note that the upper estimate for $\mathrm{p}^*(K)$ cannot grow slower than linear function of $\mathrm{cr}(K)$. This is proved by the examples shown in Fig. 1. The diagrams presented there (and similar ones) are unique (up to isotopy) minimal diagrams of the corresponding knots (by the third Tait conjecture, see [13, 14]). However, none of these diagrams can be split int less than $\frac{\mathrm{cr}(K)}{3}$ smooth simple arcs.

Remark. Using Theorem 4, we can sharpen the estimate for the semimeander crossing numbers for certain classes of knots. Namely, if we consider knots K with $\mathrm{p}^*(K) \leqslant n$, where n is a fixed number, then $\mathrm{cr}_2(K)$ can be estimated by a polynomial of degree at most 2^{n-2} in $\mathrm{cr}(K)$. However, this does not allow us to sharpen the estimate of the semimeander crossing number for all knots. (See the previous remark.)

To prove Theorems 3 and 4, we used the following result proved by A. V. Malyutin and the author in the paper [15]:

Theorem 5 ([15])**.** *For any knot K with $\mathrm{cr}(K) \geqslant 10$ there is a minimal diagram with a simple arc passing through 8 crossings.*

4 Band surgeries

The author had used a technique involving meander diagrams of knots to prove a theorem about band surgery operations. Let us recall the defenition of a band surgery.

Definition 6 (band surgery)**.** Let L be a link in $\mathbb{R}^3$, let I denote the segment $[0, 1]$, and let $f\colon I \times I \to \mathbb{R}^3$ be an embedding such that

$$f(I \times I) \cap L = f(I \times \partial I).$$

Then we say that the link

$$M = (L \setminus f(I \times \partial I)) \cup f(\partial I \times I)$$

is obtained from L by the *band surgery* along the *band* $f(I \times I)$. (See Figure 2.)

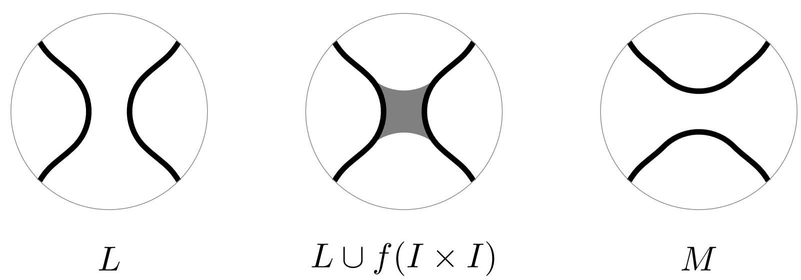

Figure 2: Band surgery.

Theorem 6 ([16])**.** *If K is a nontrivial knot, then there exists a Brunnian two-component link L that is obtained from K by a single band surgery, and for L the following inequality holds*

$$\operatorname{cr}(L) \leqslant 4\sqrt[4]{6}^{\operatorname{cr}(K)} + 2.$$

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