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2-dimensional cell complexes, real Hurwitz numbers and integrable systems

Summary of the thesis for the purpose of obtaining academic degree Doctor of Science in Mathematics.

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Introduction

The thesis covers several topics in geometry and combinatorics concentrated around the idea of counting geometric objects with presecribed singularities. The most typical of them is the problem, going back to A. Hurwitz [1], of counting equivalence classes of meromorphic functions having critical points of prescribed types and defined on a curve of a given genus. Combinatorial machinery involved in studying this question in many cases involves numerous extensions of another classical result, the matrix-tree theorem by G. Kirchhoff [2].

The main results of the thesis fall into three categories:

1. (joint with R. Fesler) Theory of real Hurwitz numbers largely parallel to the classical ("complex") theory. Several models are developped and compared; explicit formulas for the numbers are provided and the generating function thereof is shown to satisfy a PDF similar to the classical cut-and-join.
2. (some of the results joint with several authors including V. Kulishov, B. Shapiro, D. Zvonkine, A. Ploskonosov and A. Trofimova) Generalizations of the matrix-tree theorem. Special elements of group algebra of a reflection group are used instead of Laplace matrices, and determinants are replaced by a number of functions including their higher-degree analogs.
3. (joint with several authors including B. Shapiro and S. Lvovski) Geometric and combinatorial problems related to the Hurwitz problem. These includes counting functions with presrcibed singularities in special algegraic families (e.g. Severi varieties), description of maps of algebraic surfaces ramified over a given curve, and some more.

This is a brief summary of the research done; see the introductory part of the thesis for more details.

Acknowledgements

Most of my research was done in close collaboration with various authors, and could not be done in its present form without it; I wish to express my deep gratitude to all of them. Of those who were not my co-authors, I wish to deliver special thanks to Maxim Kazarian and Sergey Lando who for many years have been organizing a scientific seminar which served as an ideal media for me to develop and present my ideas.

Sergey Natanzon (1948–2020) was always an expert and an enthusiast of transferring complex geometry into real setting; his works formed, to a large extent, my ideas on this subject. Sergey's untimely death of the COVID-19 pandemic bereaved me of pleasure of having him among my collaborators.

1. Real Hurwitz numbers

Let m be an integer, and $\lambda = (\lambda_1 \geq \dots \geq \lambda_s)$, a partition. Classical (one-part) Hurwtitz numbers $h_{m,\lambda}$ dating back to [1] are defined as numbers of equivalence classes of meromorphic functions on a compact complex curve having m simple critical values and one more critical value with the profile λ (i.e. its preimage consists of a point of multiplicity λ_1, a point of multiplicity λ_2, etc.) The monodromy construction (see e.g. [3] and the references therein) proves that $n!h_{m,\lambda}$ (where $n = |\lambda|$) is equal to the number of sequences $\sigma_1, \dots, \sigma_m$ of transpositions in the group S_n such that their product has the cyclic structure λ. It was proved in [4] and later in [5] that $n!h_{m,\lambda}$ is equal to the number of *ribbon decompositions* — cell complexes with special properties on the source curve.

In this section we extend the above notions to the case of real curves without real points, that is, complex curves with fixed-point-free anti-holomorphic involution. Here we consider real meromorphic functions with real critical values, sequences of transpositions of the length $2m$ enjoying some sort of symmetry, and involution-invariant ribbon decompositions.

Generating function of the classical Hurwitz numbers

$$H(\beta, p) = \sum_{m,\lambda} h_{m,\lambda} \frac{\beta^m}{m!\,|\lambda|\,!} p_{\lambda_1} p_{\lambda_2} \dots$$

satisfies (see e.g. [3]) a parabolic PDE called cut-and-join equation. The corresponding generating function for real Hurwitz numbers is proved to satisfy a similar equation. (Both equations are members of the Laplace–Beltrami family, see [6], corresponding to the parameters $\alpha = 1$ and $\alpha = 2$, respectively.)

2. Generalization of the matrix-tree theorem

The Laplace matrix is a matrix L with nondiagonal entries w_{ij} $(1 \le i \ne j \le n)$ and diagonal entries $-\sum_{j \ne i} w_{ij}$ $(1 \le i \le n)$. The classical matrix-tree theorem discovered by G. Kirchhoff in 1847 expresses the principal minor of L as a sum of monomials of w_{ij} indexed by directed trees with n vertices and a single source. The theorem was extended later, including a formula for the arbitrary minor; see [7] for details.

We generalize the theorem in several directions. First, we prove a three-parameter family of identities between formal linear combination of graphs where the coefficients are directed versions of the Tutte polynomials defined by J. Awan and O. Bernardi [17]. Specialization of parameters in them yields analogs of the matrix-tree theorems where the determinants (minors) are replaced with specially chosen polynomials of arbitrary degree of matrix elements; we call these polynomials *higher minors*. Second, in the symmetrix case ($w_{ij} = w_{ji}$) the Laplace matrix is the matrix of the element $W \stackrel{\text{def}}{=} \sum_{1 \le i \ne j \le n} w_{ij}(1 - (ij))$ of the group algebra of the symmetric group S_n acting in its n-dimensional permutation representation. We prove that one can replace S_n with any reflection group, and W, with any element of a Lie subalgebra of the group algebra, called the algebra of Lie elements. The matrices obtained exhibit nice combinatorial properties, and we prove analogs of the matrix-tree theorem for them.

Finally, we prove a topological result counting, for an arbitrary connected graph, the number of its embeddings into the surface of minimal genus. This requires a refinement of the generating function for Hurwitz number; the formula obtained has the matrix-tree theorem as one of its ingredients.

3. Miscellaneous

The thesis contains some results not related directly to the Hurwitz problem but similar in formulation and/or using similar techniques. One of them is the matrix-subgraph theorem involving, instead of the summation over the set of spanning trees, the summation over the set of spanning subgraphs with a given 2-core.

Such sums allow to compute analogs of minors of the matrix, called f-minors; generating functions of various classes of subgraphs (like subgraphs with zero Euler characteristics) are conveniently expressed via them.

The classical Hurwitz problem counts functions with prescribed ramification among all meromorphic functions on curves of a given genus. It is possible to consider the same counting problem for functions from another family, say, for projections of plane curves. For such families we obtain interesting partial result in case where parameters of the problem (genus of the curve, its degree and degeneration parameter) satisfy a specific inequality.

Also we consider one problem in a higher dimension: to find conditions when a given curve is the ramification divisor for a map between algebraic surfaces. Here our results generalize classical theorems by V. Kulikov [8].

4. Structure of the thesis

The thesis consists of an introduction, a list of references, and of the following articles:

Numbers in brackets refer to the general bibliography at the end of this summary.

1. [16] Yu. Burman. Triangulation of surfaces with boundary and the homotopy principle for functions without critical points // Annals of Global Analysis and Geometry, Vol. 17 (1999), No. 3, pp. 221–238.
2. [12] Yu. Burman, B. Shapiro, Around matrix-tree theorem // Mathematical Research Letters, Vol. 13 (2003), No. 5. p. 7611–774.
3. [15] A. Berenstein, Yu. Burman, Quasiharmonic polynomials for Coxeter groups and representations of Cherednik algebras// Transactions of the American Mathematical Society, Vol. 362 (2010). No. 1, pp. 229–260.
4. [4] Yu. Burman, D. Zvonkine. Cycle factorizations and 1-faced graph embeddings // European Journal of Combinatorics, Vol. 31 (2010), No. 1, pp. 129–144.
5. [9] Yu. Burman, A. Ploskonosov, A. Trofimova, Matrix-tree theorems and discrete path integration // Linear Algebra and its Applications, Vol. 466 (2015), pp. 64–82.
6. [13] Yu. Burman, S. Lvovski, On projections of smooth and nodal plane curves // Moscow Mathematical Journal, Vol. 15 (2015), No. 1, pp. 31–48.
7. [11] Yu. Burman, Higher matrix-tree theorems and Bernardi polynomial // Journal of Algebraic Combinatorics, Vol. 50 (2019), No. 4, pp. 427–446.
8. [14] Yu. Burman. B. Shapiro, On Hurwitz–Severi numbers // Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, Vol. XIX (2019), No. 1, pp. 155–167.
9. [18] Yu. Burman, R. Froeberg, B. Shapiro, Algebraic relations between harmonic and anti-harmonic moments of plane polygons // International Mathematics Research Notices. Vol. 14 (2021), pp. 11140–11168.
10. [10] Yu. Burman, V. Kulishov, Lie elements and the Matrix-tree theorem // Moscow Mathematical Journal, Vol. 23 (2023), No. 1, pp. 47–58.

5. List of the main results of the thesis

1. In [5] introduce a notion of a real (twisted) Hurwitz number $h^{\sim}_{m,\lambda}$ (where m is an integer and λ, a partition of $n = |\lambda|$) and prove that three sets $\mathfrak{D}_{m,\lambda}$, $\mathfrak{S}_{m,\lambda}$ and $\#\mathfrak{H}_{m,\lambda}$ have the cardinality $n!h_{m,\lambda}$. Here

- $\mathfrak{D}_{m,\lambda}$ is the set of equivalence classes of real meromorphic functions on a real curve without real points having m real critical values with the profile $2^2 1^{n-4}$ and one critical value ∞ of the profile $1^{2\lambda_1} 2^{2\lambda_2} \dots$.
- $\mathfrak{S}_{m,\lambda}$ is the set of sequences $(\sigma_1, \dots, \sigma_m)$ of transpositions $\sigma_l \in S_{2n}$ such that the cyclic structure of the permutation $x\tau x^{-1}\tau \in S_{2n}$ where $x = \sigma_1 \dots \sigma_m$ and $\tau = (1, n+1) \dots (n, 2n)$, contains cycles $c_1, c_1', c_2, c_2', \dots$ where for all k the length of c_k and c_k' is λ_k and $c_k' = \tau c_k \tau$.
- $\#\mathfrak{H}_{m,\lambda}$ is the ribbon decompositions of a real surface with boundary, having m ribbons and a profile λ; see the thesis for exact definition.

Also we prove that the generating function

$$H^{\sim}(\beta, p) \overset{\text{def}}{=} \sum_{m \geq 0} \sum_{\lambda} \frac{h^{\sim}_{m,\lambda}}{m!\,|\lambda|\,!} p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_s} \beta^m$$

satisfies the partial differential equation $\frac{\partial H^{\sim}}{\partial \beta} = \mathrm{CJ}^{\sim}(H^{\sim})$ where $\mathrm{CJ}^{\sim}$ ("a twisted cut-and-join operator") is the Laplace–Beltrami operator with the parameter $\alpha = 2$ (the classical cut-and-join is the same with $\alpha = 1$).

2. Let V be finite-dimensional vector space; $e_1, \dots, e_N \in V$ and $\alpha_1, \dots, \alpha_N \in V^*$; denote by $M[e, \alpha] : V \to V$ the rank 1 operator $M[e, \alpha](v) = \langle \alpha, v \rangle e$. In [9] we consider a linear operator $R : V \to V$ given by $R = P(M[e_1, \alpha_1], \dots, M[e_N, \alpha_N])$ where

$$P(x_1, \dots, x_N) = \sum_{s=1}^{m} \sum_{1 \leq i_1, \dots, i_s \leq N} c_{i_1, \dots, i_s} x_{i_1} \dots x_{i_s}$$

is a noncommutative polynomial of some degree m. We obtain an explicit formula for the characteristic polynomial of R via e_i, α_h and $c_{i_1 \dots i_s}$. The formula looks like "discrete path integral" and implies Cauchy–Binet formula as well as many versions of the matrix-tree theorem and G.Kenyon's formula for the determinant of the graph Laplacian.

3. In [11] we prove a three-parametric family of identities

$$\Delta \mathcal{B}_{n,k}(q, y, z) = \widehat{\mathcal{B}}_{n,k}(q, y-1, z-1).$$

where $\mathcal{B}_{n,k}(q, y, z)$ and $\widehat{\mathcal{B}}_{n,k}(q, y, z)$ are formal linear combinations of directed graphs with n vertices and k edges. The coefficient at a graph G in $\mathcal{B}_{n,k}$ is $B_G(q, y, z)$, a directed analog of its Tutte polynomial, defined earlier in [17]; the coefficient in $\widehat{\mathcal{B}}_{n,k}$ is the degree k part of this polynomial. Δ is a special linear operator ("an abstract Laplacian"). Specialization of parameters gives several results in the flavor of the matrix-tree theorem, in particular the following:

$$\Delta \det{}^I_{n,k} = \frac{(-1)^k}{k!} \sum_{G \in \mathfrak{A}^I_{n,k}} G.$$

where $\det^I_{n,k}$ is the formal alternating sum of totally cyclic graphs having vertices from the set $I = \{i_1, \dots, i_s\} \subset \{1, \dots, n\}$, and only them, isolated, and $\mathfrak{A}^I_{n,k}$ is the set of acyclic graphs having the vertices from I, and only them, as sinks. The left-hand side is actually a higher-degree analog of the determinant (a minor).

4. In [14] we study the following problem: given a point p on the plane, how many are there nodal plane curves of degree d and genus g having a given set of ℓ tangents at p, given set of lines passing through p and tangent to the curve elsewhere and, dimension permitting, nodes at given lines passing through p. The answer is

$$\begin{array}{ll} \binom{d}{2}^{d+\ell-g-2} d^\ell h_{g,1^d}/d! & \text{if } d+\ell \geq g+2, \\ d^{d+2\ell-g-2}\binom{2g-d-\ell-1}{g-3} h_{g,1^d}/d! & \text{if } d+\ell < g+2 \leq d+2\ell \end{array}$$

where $h_{g,1^d}$ is the classical Hurwitz number. The remaining case $g+2 > d+2\ell$ is still open.

References

[1] A. Hurwitz, Über die Entwicklungskoeffizienten der lemniskatischen Funktionen // Math. Ann. Vol. 51 (1899), pp. 196–226 (Mathematische Werke II, pp. 342–373).

[2] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung det linearen Verteilung galvanischer Ströme gefurht wird // Ann. Phys. Chem., Vol. 72 (1847), pp. 497–508.

[3] Maxim E. Kazarian and Sergey K. Lando, An algebro-geometric proof of Witten conjecture // J. Amer. Math. Soc., 20 (2007), 1079–1089.

[4] Yu. Burman, D. Zvonkine. Cycle factorizations and 1-faced graph embeddings // European Journal of Combinatorics, Vol. 31 (2010), No. 1, pp. 129–144.

[5] Yu. Burman, R. Fesler, Ribbon decomposition and twisted hurwitz numbers, submitted to Communications in Contemporary Mathematics.

[6] Ian G. Macdonald, Symmetric functions and Hall polynomials, w/contributions by A. Zelevinsky, *Oxford Mathematical Monographs* (The Clarendon Press, Oxford University Press, second edition, 1995).

[7] S. Chaiken, A combinatorial proof of the all minors matrix tree theorem // SIAM J. Alg. Disc. Math., Vol. 3 (1982), no. 3, pp. 319–329.

[8] Vik. S. Kulikov, Chisini's conjecture // Izv. Ross. Akad. Nauk Ser. Mat. Vol. 63 (1999), no. 6, pp. 83–116 (Russian), English translation: Izv. Math. Vol. 63 (1999), no. 6, pp. 1139–1170.

[9] Yu. Burman, A. Ploskonosov, A. Trofimova, Matrix-tree theorems and discrete path integration // Linear Algebra and its Applications, Vol. 466 (2015), pp. 64–82.

[10] Yu. Burman, V. Kulishov, Lie elements and the Matrix-tree theorem // Moscow Mathematical Journal, Vol. 23 (2023), No. 1, pp. 47–58.

[11] Yu. Burman, Higher matrix-tree theorems and Bernardi polynomial // Journal of Algebraic Combinatorics, Vol. 50 (2019), No. 4, pp. 427–446.

[12] Yu. Burman Y. M., B. Shapiro, Around matrix-tree theorem // Mathematical Research Letters, Vol. 13 (2003), No. 5. p. 761–774.

[13] Yu. Burman, S. Lvovski, On projections of smooth and nodal plane curves // Moscow Mathematical Journal, Vol. 15 (2015), No. 1, pp. 31–48.

[14] Yu. Burman. B. Shapiro, On Hurwitz–Severi numbers // Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, Vol. XIX (2019), No. 1, pp. 155–167.

[15] A. Berenstein, Yu. Burman, Quasiharmonic polynomials for Coxeter groups and representations of Cherednik algebras// Transactions of the American Mathematical Society, Vol. 362 (2010). No. 1, pp. 229–260.

[16] Yu. Burman. Triangulation of surfaces with boundary and the homotopy principle for functions without critical points // Annals of Global Analysis and Geometry, Vol. 17 (1999), No. 3, pp. 221–238.

[17] J. Awan, O. Bernardi, Tutte polynomials for directed graphs, arXiv:1610.01839v2.

[18] Yu. Burman, R. Froeberg, B. Shapiro, Algebraic relations between harmonic and anti-harmonic moments of plane polygons // International Mathematics Research Notices. Vol. 14 (2021), pp. 11140–11168.