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Hurwitz theory in the real case and for root systems of type B and D

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1. The classical theory

Hurwitz numbers were first introduced in the end of the 19th century in the work of A. Hurwitz [**16**]. In the century to follow, Hurwitz numbers fell into considerable neglect but regained attractivity in the 1990s due to the interest from the mathematical physics community as well as from the group theorists and from the algebraic geometers.

Originally, Hurwitz numbers count isomorphism classes of ramified coverings of the sphere, where all the critical values are simple except possibly one; this one has a fixed ramification profile. Hurwitz realized that this algebro-geometric counting problem can be translated into combinatorial language by using monodromy: it is equivalent to counting the number of strings of transpositions such that their product belongs to a prescribed conjugacy class of the permutation group. Thus, Hurwitz numbers formed a bridge between two presumably distant areas of mathematics.

A formula proved in a seminal 2001 paper [**9**] by T.Ekedahl, S.Lando, M.Shapiro and A.Vainshtein (and called the ELSV formula since then) built another, somewhat unexpected, bridge by revealing deep connections of Hurwitz numbers with the geometry of moduli space of complex curves. (Namely, the Hurwitz numbers are the intersection constants of certain characteristic classes.) Hurwitz numbers that are by nature purely discrete, combinatorial object, yet have a structure rooted in the topology of moduli space of curves. One more topological incarnation of Hurwitz numbers has been found by Yu. Burman and the author and in [**2**]; see Chapter 1 for details.

Another group of results involving Hurwitz numbers, obtained during the last 30 years, was motivated directly or indirectly by the so called Witten conjecture [**30**]. The conjecture (now given several proofs, see [**20**], [**19**]) asserts that the generating function of the Hurwitz numbers satisfies a second order parabolic PDE called the cut-and-join equation and hence, is a one-parametric family of τ function of the KP integrable hierarchy. Combinatorially, the cut-and-join operator describes the behavior of a permutation multiplied by a transposition: either two cycles of the cycle decomposition are glued, or one cycle is split into two [**12**]. The operator has Schur polynomials as eigenvectors, so the Hurwitz numbers can be expressed via them [**19**].

In the following table we sum up the properties of the classical Hurwitz numbers $h_{m,\lambda}$ mentioned above:

- Algebraic definition:
  $\#(\sigma_1, \dots, \sigma_m) \ \mid \forall i \ \sigma_i \in C_2$ and $\sigma_1\sigma_2\dots\sigma_m \in C_\lambda$
- Algebro-geometric definition:
  The number of isomorphism classes
  of ramified coverings of the sphere where m finite critical
  values are simple, and the critical value ∞
  has ramification profile λ
- Geometric definition:
  The intersection constants in cohomology
  of the moduli space of stable curves: the ELSV formula
- Topological definition:
  The set of decomposition of surfaces into m
  non-twisted ribbons and n disks,

surface boundary having s components
containing $\lambda_1, \dots, \lambda_s$ vertices (endpoints of ribbon diagonals)

- Expression of the generating function $H(\beta, p_1, p_2, \dots)$ of disconnected Hurwitz numbers in terms of Schur polynomials s_λ :
  $H(\beta, p_1, p_2, \dots) = \sum_\lambda s_\lambda(1, 0, 0, \dots) s_\lambda(p_1, p_2, \dots) \exp^{\sum_{i=1}^{\infty} \lambda_i(\lambda_i - 2i + 1)\beta}$
- Solution of a differential equation Cut-and-join
  $\frac{\partial H}{\partial \beta} = \frac{1}{2} \sum_{i,j=1}^{\infty} \left((i+j) p_i p_j \frac{\partial H}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2 H}{\partial p_i \partial p_j} \right)$
- Integrable system :
  $H(\beta, p_1, p_2, \dots)$ is a τ-function of the KP hierarchy

In the present thesis we study new types of Hurwitz numbers: the twisted/real Hurwitz numbers and the Hurwitz numbers for the reflection groups B and D. Below the main results of the thesis are presented.

2. The real (twisted) theory

In Chapter 1 of the thesis we introduce a new topological model for Hurwitz numbers:

Definition. *Decorated-boundary surface* (DBS) is a triple $(M, (a_1, \dots, a_n), (o_1, \dots, o_n))$ where M is a compact surface (2-manifold) with boundary, $a_1, \dots, a_n \in \partial M$ are marked points and every o_i is a local orientation of ∂M (hence, of M itself, too) in the vicinity of the point a_i, such that

- every connected component of M has nonempty boundary, and
- every connected component of ∂M contains at least one point a_i.

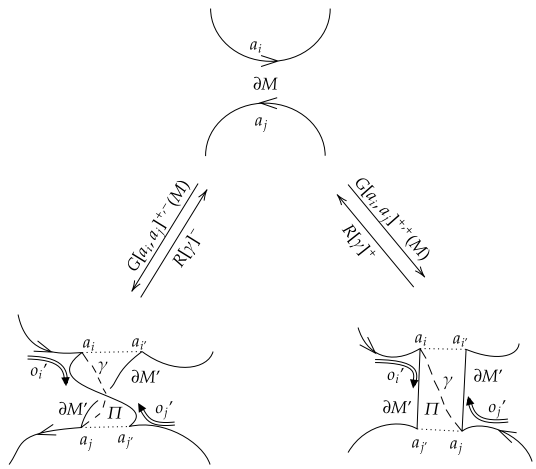
The DBS M and M' with the same number n of marked points are called equivalent if there exists a homeomorphism $h : M \to M'$ such that $h(a_i) = a'_i$ and $h_*(o_i) = o'_i$ for all $i = 1, \dots, n$. The set of equivalence classes of DBS with n marked points will be denoted $\mathcal{DBS}_n$.

Pick marked points $a_i, a_j \in \partial M$, and let $\varepsilon_i, \varepsilon_j \in \{+, -\}$. Consider points $a'_i, a'_j \in \partial M$ lying near a_i, a_j and such that the boundary segment $a_i a'_i$ is directed along the orientation o_i if $\varepsilon_i = +$ and opposite to it if $\varepsilon_i = -$; the same for j. Now take a long narrow rectangle ("a ribbon" henceforth) and glue its short sides to ∂M as shown in Fig. 2. The result of gluing is homeomorphic to a surface M' with the boundary $\partial M' \ni a_1, \dots, a_n$. The boundary of M' near a_i and a_j contains a segment of ∂M (the "old" part) and a segment of a long side of the ribbon glued (the "new" part); define local orientations o'_i, o'_j of $\partial M'$ near a_i, a_j so that the orientations of the "old" parts would be preserved (see bold curved arrows in Fig. 2); for $k \neq i, j$ take $o'_k = o_k$ by definition. Now $(M', (a_1, \dots, a_n), (o'_1, \dots, o'_n))$ is a DBS, so we defined a mapping $G[i, j]^{\varepsilon_i, \varepsilon_j} : \mathcal{DBS}_n \to \mathcal{DBS}_n$ called *ribbon gluing.* The ribbon gluing $G[i, j]^{\varepsilon_i, \varepsilon_j}$ will be called twisted if $\varepsilon_i \neq \varepsilon_j$, and non-twisted otherwise.

Denote by $E_n \in \mathcal{DBS}_n$ a union of n disks with one marked point on the boundary of each, and let now, $M \in \mathcal{DBS}_n$ be obtained by gluing of m ribbons to E_n:

$$M = G[i_m, j_m]^{\varepsilon_m, \delta'_m} \dots G[i_1, j_1]^{\varepsilon_1, \delta_1} E_n$$

that's what we will be calling a *ribbon decomposition.* The ribbon decomposition is called oriented if all the signs $\varepsilon_m = \delta_m = +$; in this case M is an oriented surface with boundary, and all the local orientations o_i agree with the global orientation.



Fix a positive integer m and a partition $(\lambda_1 \geq \cdots \geq \lambda_s)$ of the number $n \stackrel{\text{def}}{=} |\lambda| \stackrel{\text{def}}{=} \lambda_1 + \cdots + \lambda_s$ into s parts. Denote by $\mathfrak{S}_{m,\lambda}$ the set of decompositions into m ribbons and n disks of surfaces having boundary of s components containing $\lambda_1, \ldots, \lambda_s$ vertices (endpoints of ribbon diagonals).

Twisted Hurwitz numbers are defined as

Definition.

$$h^{\sim}_{m,\lambda} \stackrel{\text{def}}{=} \frac{1}{n!} \# \mathfrak{S}_{m,\lambda}$$

If $\mathfrak{S}^+_{m,\lambda}$ is the set of oriented ribbon decompositions with the same combinatorics then $\frac{1}{n!}\#\mathfrak{S}^+_{m,\lambda}$ is equal to the classical Hurwitz number (see the proof in Chapter 1 below). Thus, Definition 2 generalize the topological definition of the classical Hurwitz numbers by allowing the ribbons to be *twisted* literally so one cannot orient the surface obtained.

The analog of the combinatorial definition is as follows. Consider a fixed-point-free involution

$$\tau = (1, 1+n)(2, 2+n)\ldots(n, 2n)$$

in the symmetric group S_{2n}. Let $\sigma_1, \ldots, \sigma_m \in S_{2n}$ be transpositions. An analysis shows that the permutation

$$u \stackrel{\text{def}}{=} \sigma_1 \ldots \sigma_m \tau \sigma_m \ldots \sigma_1 \tau \in S_{2n} \tag{2.1}$$

is decomposed into independent cycles as $u = c_1 c'_1 \ldots c_s c'_s$ where $c'_i = \tau c_i^{-1} \tau$ for every $i = 1, \ldots, s$; call the pairs (c_i, c'_i) τ-symmetric cycles. Let $B^{\sim}_\lambda$ be the set of permutations whose decomposition into independent cycles consists of s pairs of τ-symmetric cycles of lengths $\lambda_1, \ldots, \lambda_s$. Denote by

$$\mathfrak{H}_{m,\lambda} \stackrel{\text{def}}{=} \{(\sigma_1, \ldots, \sigma_m) \mid \forall s = 1, \ldots, m\, \sigma_s = (i_s j_s), j_s \neq \tau(i_s),$$
$$\sigma_1\sigma_2 \ldots \sigma_m(\tau\sigma_m\tau) \ldots (\tau\sigma_1\tau) \in B^{\sim}_\lambda\}.$$

THEOREM (Chapter 1, Theorem 2.4).

$$h^{\sim}_{m,\lambda} = \frac{1}{n!}\#\mathfrak{H}_{m,\lambda}.$$

and there is an explicit one-to-one correspondence between $\mathfrak{H}_{m,\lambda}$ *and* $\mathfrak{S}_{m,\lambda}$

Then we obtain a second order parabolic PDE satisfied by the generating function of the twisted Hurwitz numbers and similar to the cut-and-join equation in the classical case; it is a particular case of the Beltrami–Laplace equation. Its right-hand side has zonal polynomials as eigenvectors, so that twisted Hurwitz numbers can be expressed via them. In more detail, consider the generating function $\mathcal{H}^{\sim}(\beta, p)$ of the twisted Hurwitz numbers defined as follows:

$$\mathcal{H}^{\sim}(\beta, p) = \sum_{m\geq 0}\sum_{\lambda} \frac{h^{\sim}_{m,\lambda}}{m!} p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_s} \beta^m.$$

THEOREM (Chapter 1, Theorem 2.12). *$\mathcal{H}^{\sim}$ satisfies the cut-and-join equation* $\frac{\partial \mathcal{H}^{\sim}}{\partial \beta} = \mathcal{CJ}^{\sim}(\mathcal{H}^{\sim})$ *where*

$$\begin{aligned} \mathcal{CJ}^{\sim} &= \sum_{i,j\geq 1} (i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} + 2ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} + \sum_{k\geq 1} k(k-1) p_k \frac{\partial}{\partial p_k} \\ &= \sum_{i,j\geq 1} (i+j)(p_i p_j + p_{i+j}) \frac{\partial}{\partial p_{i+j}} + 2ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} \qquad (2.2) \end{aligned}$$

COROLLARY. $\mathcal{H}^{\sim}(\beta, p) = \exp(\beta \mathcal{CJ}^{\sim}) \exp(p_1)$.

The expression of twisted Hurwitz numbers in terms of zonal polynomials is given by:

THEOREM (Chapter 1, Theorem 2.15).

$$\mathcal{H}^{\sim}(\beta, p) = \sum_{\lambda} \exp\bigl(2\beta \sum_{i} \lambda_i(\lambda_i - i)\bigr) \frac{2^{|\lambda|} Z_\lambda(p)}{H_\lambda(2) H'_\lambda(2)}.$$

where $H_\lambda(\alpha) \overset{def}{=} \prod_{(i,j)\in Y(\lambda)} (\alpha a(i,j) + \ell(i,j) + 1)$ *and* $H'_\lambda(\alpha) \overset{def}{=} \prod_{(i,j)\in Y(\lambda)} (\alpha a(i,j) + \ell(i,j) + \alpha)$. *Here $Y(\lambda)$ is the Young diagram of the partition λ, and $a(i,j)$ and $\ell(i,j)$ are the arm and the leg, respectively, of the cell $(i,j) \in Y(\lambda)$ and Z_λ are zonal polynomials.*

To give an analogue of the algebro-geometric definition of Hurwitz number we use a notion of a twisted branched covering introduced by Chapuy and Dołęga in **[7]**. In Chapter 1 we show that the generating function for twisted Hurwitz numbers satisfies the same PDE (the Beltrami–Laplace equation) with the same initial data as the generating function for twisted ramified covering defined below.

Let N denote a closed surface (compact 2-manifold without boundary, not necessarily orientable), and $p : \widehat{N} \to N$, its orientation cover. Denote by $\mathcal{T} : \widehat{N} \to \widehat{N}$ an orientation-reversing involution without fixed points such that $p \circ \mathcal{T} = p$. Also denote by $\mathcal{J} : \mathbb{C}P^1 \to \mathbb{C}P^1$ the complex conjugation, and let $\overline{\mathbb{H}} \overset{\text{def}}{=} \mathbb{C}P^1/(z \sim \mathcal{J}(z)) = \mathbb{H} \cup \{\infty\}$ where $\mathbb{H} \subset \mathbb{C}$ is the upper half-plane; $\overline{\mathbb{H}}$ is homeomorphic to a disk. Denote by $\pi : \mathbb{C}P^1 \to \overline{\mathbb{H}}$ the quotient map.

A continuous map $f : N \to \overline{\mathbb{H}}$ is called a twisted branched covering if there exists a branched covering $\widehat{f} : \widehat{N} \to \mathbb{C}P^1$ such that

(1) $\pi \circ \widehat{f} = f \circ p$, and
(2) all the critical values of $\widehat{f}$ are real.

Property (1) is equivalent to saying that $\widehat{f}$ is a real map with respect to $\mathcal{T}$, that is, $\widehat{f} \circ \mathcal{T} = \mathcal{J} \circ \widehat{f}$. The involution $\mathcal{T}$ has no fixed points, so the critical points of $\widehat{f}$ come in pairs $(a, \mathcal{T}(a))$, the ramification profile of every critical value $c \in \mathbb{R}P^1 \subset \mathbb{C}P^1$ of $\widehat{f}$ has every part repeated twice: $(\lambda_1, \lambda_1, \dots, \lambda_s, \lambda_s)$, and $\deg \widehat{f} = 2n$ is even. In this case we say that the ramification profile of the critical value $\pi(c) \in \partial\overline{\mathbb{H}}$ of the map $f : N \to \overline{\mathbb{H}}$ is $\lambda = (\lambda_1, \dots, \lambda_s)$.

The twisted branched covering f is called simple if all its critical values, except possibly $\infty \in \overline{\mathbb{H}}$, have the ramification profile $2^1 1^{n-2}$. (Equivalently, each critical value of $\widehat{f}$ has 2 simple critical points and $2n-4$ regular points as preimages.) Denote by $\#\mathfrak{D}_{m,\lambda}$ the set of isomorphism classes of simple twisted branched covering with m critical values and the critical value over ∞ has ramification profile λ.

The analog of algebro-geometric definition for twisted Hurwitz numbers is

Theorem (Chapter 1, Theorem 3.2). $\#\mathfrak{D}_{m,\lambda} = \#\mathfrak{S}_{m,\lambda} = \#\mathfrak{H}_{m,\lambda} = n! h^{\sim}_{m,\lambda}$.

3. The theory for reflection groups B and D

The Chapter 2 of the thesis deals with Hurwitz numbers for the reflection groups B and D.

The reflection group B_n has a well-known embedding (see [**15**]) into the permutation group S_{2n} as a normalizer $\mathrm{Norm}(\tau)$ of the element $\tau = (1, n+1)(2, n+2) \dots (n, 2n)$ (cf. Chapter 1 and Section 2 of the Introduction). Reflections in B_n correspond to permutations $r_{ij} = (ij)(\tau(i), \tau(j))$ and $\ell_i = (i, \tau(i))$, here $1 \le i, j \le 2n$. The group D_n is the intersection of B_n with the subgroup of even permutations; its reflections are r_{ij} only.

If $x \in \mathrm{Norm}(\tau)$ and $x = c_1 \dots c_k$ is its cycle decomposition then for every c_i, $i = 1, \dots, k$, one of the following is true: either there is another cycle $c_j = \tau c_i \tau$ of the same length, or c_i has even length and is τ-invariant: $c_i = \tau c_i \tau$.

Fix two partitions, $\lambda = (\lambda_1 \ge \dots \ge \lambda_s)$ and $\mu = (\mu_1 \ge \dots \ge \mu_t)$ such that $|\lambda| + |\mu| = n$, and consider a set $C_{\lambda|\mu}$ of elements $x \in B_n \subset S_{2n}$ such that their cycle decomposition contains pairs of cycles c_i and $\tau c_i \tau$ of lengths (each cycle) $\lambda_1, , \dots, \lambda_s$; and τ-invariant cycles $c_i = \tau c_i \tau$ of lengths $2\mu_1, \dots, 2\mu_t$ (recall that the length should be even).

Theorem ([**6**, Proposition 25]). *The set $C_{\lambda|\mu} \subset B_n$ is a conjugacy class. Every conjugacy class in B_n is $C_{\lambda|\mu}$ for some λ and μ such that $|\lambda| + |\mu| = n$.*

For D_n the description of conjugacy classes is slightly more complicated:

Theorem ([**6**, Proposition 25]).

(1) *If the partition μ contains an even number of parts then the conjugacy class $C_{\lambda|\mu} \subset B_n$ lies in D_n; if the number of parts is odd then $C_{\lambda|\mu}$ does not intersect D_n.*
(2) *If $\mu \ne \emptyset$ (and the number of parts of μ is even) then $C_{\lambda|\mu}$ is a conjugacy class in D_n.*
(3) *If at least one of the λ_i is odd then $C_{\lambda|\emptyset}$ is a conjugacy class in D_n.*
(4) *If all λ_i are even then $C_{\lambda|\emptyset}$ splits into two conjugacy classes in D_n, $C^+_{\lambda|\emptyset}$ and $C^-_{\lambda|\emptyset}$.*

Any conjugacy class in D_n is one of the classes listed above.

In particular, the reflections r_{ij} form a conjugacy class $C_{2^1 1^{n-2}|\emptyset} \subset D_n \subset B_n$ and the reflections ℓ_i, the conjugacy class $C_{1^{n-1}|1^1} \subset B_n$. We denote these classes $\mathcal{R}$ and $\mathcal{L}$, respectively.

The definition of Hurwitz numbers for the groups B_n and D_n is similar to the classical one with reflections instead of transpositions. For the B series, we count reflections of two classes (transpositions and pairs of transpositions) separately. In more detail,

Definition. A sequence of reflections $(\sigma_1, \dots, \sigma_{m+\ell})$ of the group B_n is said to have *profile* (λ, μ, m, ℓ) if $\#\{p \mid \sigma_p \in \mathcal{R}\} = m$, $\#\{p \mid \sigma_p \in \mathcal{L}\} = \ell$ and $\sigma_1 \dots \sigma_{m+\ell} \in C_{\lambda|\mu}$.
The *Hurwitz numbers for the group B_n* are

$$h^B_{m,\ell,\lambda,\mu} = \frac{1}{n!}\#\{(\sigma_1, \dots, \sigma_{m+\ell}) \text{ is a sequence of profile } (\lambda, \mu, m, \ell)\}$$

Definition. Let m be a positive integer, and λ and μ, partitions where the number of parts $\#\mu$ is even. The Hurwitz number for the group D_n is defined as $h^D_{m,\lambda,\mu} = h^B_{m,0,\lambda,\mu}$.

Denote by $\mathcal{C}_{\lambda|\mu} \stackrel{\text{def}}{=} \frac{1}{\#C_{\lambda|\mu}} \sum_{x \in C_{\lambda|\mu}} x \in \mathbb{C}[B_n]$ the normalized class sum in the group algebra of B_n. The elements $\mathcal{C}_{\lambda|\mu}$ belong to the center $Z[B_n]$ of the group algebra, and form a basis there. Consider now a ring of polynomials $\mathbb{C}[p, q]$ where $p = (p_1, p_2, \dots)$ and $q = (q_1, q_2, \dots)$ are two infinite sets of variables. The ring is graded by the total degree with $\deg p_k = \deg q_k = k$ assumed for all $k = 1, 2, \dots$. The map sending $\mathcal{C}_{\lambda|\mu}$ to $p_\lambda q_\mu \stackrel{\text{def}}{=} p_{\lambda_1} \dots p_{\lambda_s} q_{\mu_1} \dots q_{\mu_t}$ is an isomorphism between $Z[B_n]$ and the homogeneous component $\mathbb{C}[p, q]_n$ of total degree n.

The picture for the group D_n is similar (see **[6]** for the details): class sums $\mathcal{C}_{\lambda|\mu}$ for $\mu \neq \emptyset$ containing an even number of parts and the sums $\mathcal{C}_{\lambda|\emptyset} \stackrel{\text{def}}{=} \frac{1}{\#C_{\lambda|\emptyset}} \sum_{x \in C^+_{\lambda|\emptyset} \cup C^-_{\lambda|\emptyset}} x \in \mathbb{C}[D_n]$ form a basis in a space $V_n^+ \subset Z[D_n]$ isomorphic to the subspace $Q_n \subset \mathbb{C}[p, q]_n$ of polynomials of even degree by q. One has $Z[D_n] = V_n^+$ for n odd, and for n even, $Z[D_n] = V_n^+ \oplus V_n^-$ where V_n^- is spanned by $\mathcal{B}_\lambda \stackrel{\text{def}}{=} \frac{1}{\#C_{\lambda|\emptyset}} \left(\sum_{x \in C^+_{\lambda|\emptyset}} x - \sum_{x \in C^-_{\lambda|\emptyset}} x\right)$. The map sending $\mathcal{B}_\lambda$ to $p_{\lambda_1/2} \dots p_{\lambda_s/2}$ (recall that all the parts λ_i of λ should be even) is an isomorphism between V_n^- and $\mathbb{C}[p]_{n/2}$.

Consider the following generating function for Hurwitz numbers of the group B_n:

$$\mathcal{H}^B(\beta, \gamma, p, q) = \sum_{m,\ell} \sum_{\lambda,\mu} \frac{h^B_{m,\ell,\lambda,\mu}}{m!\ell!} p_\lambda q_\mu \beta^m \gamma^\ell.$$

$\mathcal{H}^B$ satisfies the *cut-and-join equations*

$$\frac{\partial \mathcal{H}^B}{\partial \beta} = \mathcal{CJ}_1(\mathcal{H}^B) \quad \text{and} \quad \frac{\partial \mathcal{H}^B}{\partial \gamma} = \mathcal{CJ}_2(\mathcal{H}^B)$$

where

$$\mathcal{CJ}_1 = \sum_{i,j=1}^{\infty}\Big((i+j)p_iq_j\frac{\partial}{\partial q_{i+j}} + 2ijq_{i+j}\frac{\partial^2}{\partial p_i\partial q_j} + ijp_{i+j}\frac{\partial^2}{\partial q_i\partial q_j} + \frac{1}{2}(i+j)q_iq_j\frac{\partial}{\partial p_{i+j}} + \frac{1}{2}(i+j)p_ip_j\frac{\partial}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial^2}{\partial p_i\partial p_j}\Big)$$

and

$$\mathcal{CJ}_2 = \sum_{i=1}^{\infty}\left(ip_i\frac{\partial}{\partial q_i} + iq_i\frac{\partial}{\partial p_i}\right)$$

Corollary.

$$\mathcal{H}^B(\beta,\gamma,p,q) = e^{\beta\mathcal{CJ}_1+\gamma\mathcal{CJ}_2}e^{p_1}$$

Let now

$$\mathcal{CJ} \stackrel{\text{def}}{=} \sum_{i,j=1}^{\infty}\left(ijp_{i+j}\frac{\partial^2}{\partial p_i\partial p_j} + (i+j)p_ip_j\frac{\partial}{\partial p_{i+j}}\right) \tag{3.1}$$

and

$$E \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} ip_i\frac{\partial}{\partial p_i}, \tag{3.2}$$

(the Euler field), and change the variables:

Proposition (Chapter 2, Proposition 3.1). *Let* $u_\ell = \frac{p_\ell+q_\ell}{2}$ *and* $v_\ell = \frac{p_\ell-q_\ell}{2}$. *Then*

$$\mathcal{CJ}_1 = \sum_{i,j=1}^{\infty}\Big(iju_{i+j}\frac{\partial^2}{\partial u_i\partial u_j} + (i+j)u_iu_j\frac{\partial}{\partial u_{i+j}} + ijv_{i+j}\frac{\partial^2}{\partial v_i\partial v_j} + (i+j)v_iv_j\frac{\partial}{\partial v_{i+j}}\Big) = \mathcal{CJ}_u^{\sim} + \mathcal{CJ}_v^{\sim},$$

$$\mathcal{CJ}_2 = \sum_{\ell=1}^{\infty}\ell\left(u_\ell\frac{\partial}{\partial u_\ell} - v_\ell\frac{\partial}{\partial v_\ell}\right) = E_u - E_v.$$

where by $\mathcal{CJ}_u$ and $\mathcal{CJ}_v$ we denote the operator (3.1) with u_i (resp. v_i) substituted for p_i, and similarly, E_u and E_v.

Finally one gets

Corollary (Chapter 2, corollary 3.3).

$$\mathcal{H}^B(\beta,\gamma,p_1,p_2,\dots,q_1,q_2,\dots) = \sum_{\lambda,\mu}\exp\big(\beta\sum_{i=1}^{\infty}(\lambda_i(\lambda_i-2i+1)+\mu_i(\mu_i-2i+1)+\gamma\sum_{i=1}^{\infty}(\lambda_i-\mu_i)\big) \times s_\lambda(1,0,0\dots)s_\mu(1,0,0\dots)s_\lambda((p_1+q_1)/2,\dots)s_\mu((p_1-q_1)/2,\dots)$$

A similar result for the group D_n:

Theorem (Chapter 2, Theorem 4.1). *The operator* $\mathcal{CJ}_1^D$ *is a restriction of the operator* $\mathcal{CJ}_1^B$ *on the space* $Q_n \subset \mathbb{C}[p,q]_n$ *of polynomials of even degree in* q. *The change of variables* $p_i \mapsto p_i/2$ *converts the operator* $\mathcal{CJ}_2^D$ *(defined for even* n *only) into the cut-and-join operator* (3.1) *with* $n \mapsto n/2$ *and multiplied by* 4.

The cut-and-join $\mathcal{CJ}_2$ being an Euler field, we can reduce the indice ℓ to 0. There exists an explicit formula expressing B-Hurwitz numbers $h^B_{m,0,\lambda,\mu}$ via classical ones, $h_{m,\lambda}$. For a sequence of integers $c_1, \dots, c_n$ denote by $\xi(c)$ the partition $1^{c_1} \dots n^{c_n}$; thus, $|\xi(c)| = c_1 + 2c_2 + \dots + nc_n$. Also, for positive integers p, q, r denote by f^r_{pq} the coefficient at x^r at the polynomial $(1+x)^p(1-x)^q$.

Let $\lambda \stackrel{\text{def}}{=} \xi(\gamma)$ and $\mu \stackrel{\text{def}}{=} \xi(\delta)$. Then

$$h^B_{m,0,\lambda,\mu} = \sum_{\substack{\alpha_i+\beta_i=\gamma_i+\delta_i\ \forall i\\ m_1+m_2=m}} \frac{h_{m_1,\xi(\alpha)}h_{m_2,\xi(\beta)}}{2^{\#\lambda+\#\mu}} \binom{m}{m_1} \binom{|\lambda|+|\mu|}{|\xi(\alpha)|} f^{\gamma_1}_{\alpha_1\beta_1} f^{\gamma_2}_{\alpha_2\beta_2} \cdots$$

Theorem 3.1 (Chapter 2, Theorem 5.10). *The generating function $\mathcal{H}^B(\beta, \gamma, u+v, u-v)$ is a 2-parameter family of τ-functions, independently in the u and the v variables.*

and similarly

Corollary (Chapter 2, Corollary 5.11). *The generating function $\mathcal{H}^D(\beta, u+v, u-v)$ is a 1-parameter family of τ-functions independently in the u and the v variables.*

An analog of the topological definition of Hurwitz numbers for reflection groups B consists of an oriented DBS (see the definition above) M together with an orientation-preserving involution τ leaving the ribbon decomposition invariant. Such DBS is obtained by gluing $2m+\ell$ non-twisted ribbons to $2n$ disks and has the boundary of $2s+k$ components containing $\lambda_1, \lambda_1, \dots, \lambda_s, \lambda_s, 2\mu_1, \dots, 2\mu_k$ vertices. The involution leaves ℓ ribbons invariant and has one fixed point in each of them; other ribbons form m pairs, the involution τ exchanging the members of each one. The data (λ, μ, m, ℓ) is called the profile of the B-ribbon decomposition.

Theorem 3.2 (Chapter 2, Theorem 5.2). *B-ribbon decomposition with profile (λ, μ, m, ℓ) of decorated boundary surface are in one-to-one correspondence with sequence of reflections having profile (λ, μ, m, ℓ).*

4. Relation to ramified coverings

Chapter 3 of the thesis contains a description of a direct correspondence between twisted ramified coverings and the combinatorial definition of twisted Hurwitz numbers.

Let, again, $\lambda = (\lambda_1 \ge \dots \ge \lambda_s)$ be a partition of $n \stackrel{\text{def}}{=} |\lambda| \stackrel{\text{def}}{=} \lambda_1 + \dots + \lambda_s$, and $m > 0$ be an integer. Denote by $\mathcal{H}_{m,\lambda}$ the main stratum of the standard Hurwitz space; its elements are equivalence classes of pairs (M, f) where M is a compact smooth complex curve and $f : M \to \mathbb{C}P^1$, a meromorphic function having s poles $u_1, \dots, u_s$ of multiplicities $\lambda_1, \dots, \lambda_s$ and m simple critical points.

Following [**7**], call a real meromorphic function *simple* if all its critical points u_i, except possibly poles, are simple, and the critical values are as simple as possible: $f(u_i) \ne f(u_j)$ unless $u_j = u_i$ or $u_j = \mathcal{T}(u_i)$. A simple real meromorphic function is called *fully real* if all its critical values are real.

Note that the simplicity condition is generally not assumed for poles. The involution $\mathcal{T}$ has no fixed points, so the ramification profile of $\widehat{f}$ over ∞ has every part repeated twice: $(\lambda_1, \lambda_1, \dots, \lambda_s, \lambda_s)$, and $\deg \widehat{f} = 2n$ is even. We say then that the profile of the simple twisted real function above is $\lambda = (\lambda_1, \dots, \lambda_s)$, and also write $\deg f = n$ by a slight abuse of notation.

Denote by $\mathbb{R}\mathcal{H}_{m,\lambda}$ the set of simple real meromorphic functions of the profile λ with $2m$ simple critical points $u_1, \mathcal{T}(u_1), \dots, u_m, \mathcal{T}(u_m)$, up to equivalence (similar to the classical Hurwitz space, see **[25]** for details). The subset of fully real functions is denoted by $\mathfrak{R}\mathfrak{H}_{m,\lambda} \subset \mathbb{R}\mathcal{H}_{m,\lambda}$. For $F \in \mathfrak{R}\mathfrak{H}_{m,\lambda}$ we usually denote by $y_0 < \dots < y_m \in \mathbb{R}$ its critical values (each one assumed in two simple critical points). For $\widehat{f} \in \mathfrak{R}\mathfrak{H}_{m,\lambda}$ let $u \in \mathbb{R} \subset \mathbb{C}P^1$ be a regular (not critical) value of $\widehat{f}$ such that $u < y_0$; then the preimage $\widehat{f}^{-1}(u) \subset \hat{N}$ consists of $2n$ points and the preimage $f^{-1}(\pi(u)) \subset N$, of n points. Fix a bijection $\hat{\nu} : \widehat{f}^{-1}(\hat{u}) \to \mathcal{A}_n$ such that if $\hat{\nu}(x) = k$ then $\hat{\nu}(\mathcal{T}(x)) = \bar{k}$ for all $k = 1, \dots, n$. A simple fully real ramified covering together with the point u and the bijection $\hat{\nu}$ is called labelled. The set of labelled fully real simple ramified coverings $(\widehat{f}, \nu)$ where $\widehat{f} \in \mathfrak{R}\mathfrak{H}_{m,\lambda}$ was denoted $\mathfrak{D}_{m,\lambda}$ above.

Consider pair matchings on the set $\mathcal{A}_n = (1, \bar{1}, \dots, n, \bar{n})$; they are identified with involutions without fixed points in the permutation group S_{2n}. Given two such involutions δ_1 and δ_2, let $\sigma \stackrel{\text{def}}{=} \delta_1\delta_2 \in S_{2n}$. Since $\delta_1\sigma\delta_1 = \delta_2\delta_1 = \sigma^{-1}$, the cycle decomposition of σ looks like $c_1c_1' \dots c_sc_s'$ where $c_k' \stackrel{\text{def}}{=} \delta_1 c_k^{-1}\delta_1$; therefore, the cycles c_k and c_k' have the same length. Let λ_i be the length of c_i; denote the partition $(\lambda_1, \dots, \lambda_s)$ by $\Lambda(\delta_1, \delta_2)$; one has $|\Lambda(\delta_1, \delta_2)| = \lambda_1 + \dots + \lambda_s = n$. To a pair matching δ one can relate a graph $\Gamma(\delta)$ with the vertex set $\mathcal{A}_n$: two vertices p and q are joined by an (non-oriented) edge if $\delta(p) = q$. An edge union of the graphs $\Gamma(\delta_1)$ and $\Gamma(\delta_2)$ is a union of cycles of the lengths $2\lambda_1, \dots, 2\lambda_s$

Denote by $\mathfrak{P}_{m,\lambda}$ the set of sequences of pairs matchings $\delta_{-1}, \dots, \delta_{m-1}$ satisfying the conditions

$$\Lambda(\delta_k, \delta_{k+1}) = 2^1 1^{n-2} \quad \text{for all } k = -1, \dots, m-2 \tag{4.1}$$

$$\delta_{-1} = (1, \bar{1}) \dots (n, \bar{n}) \tag{4.2}$$

$$\Lambda(\delta_{-1}, \delta_{m-1}) = \lambda \tag{4.3}$$

In **[1]** a direct one-to-one correspondence Δ is established between the set $\mathfrak{P}_{m,\lambda}$ and $\mathfrak{D}_{m,\lambda}$.

Let now a sequence of transpositions $((i_1, j_1), \dots, (i_m, j_m)) \in \mathfrak{H}^{\mathbb{R}}_{m,\lambda}$ and the product $x_k = (i_1, j_1) \dots (i_k, j_k)$ for all $k = 1 \dots m$. Let $\delta_k = (\tau x_{k+1})\tau(\tau x_{k+1})^{-1}$ for $k = 0, \dots, m-1$, and $\delta_{-1} \stackrel{\text{def}}{=} \tau$.

Theorem A . *(Chapter 3, Theorem 2.2) The map* $\mathcal{P}((i_1, j_1), \dots, (i_m, j_m)) \stackrel{def}{=} (\tau, \delta_0, \dots, \delta_{m-1})$ *is a* $2^m : 1$ *correspondence between the sets* $\mathfrak{H}^{\mathbb{R}}_{m,\lambda}$ *and* $\mathfrak{P}_{m,\lambda}$*. The composition of* $\mathcal{P}((i_1, j_1), \dots, (i_m, j_m))$ *with* Δ *gives a* $2^m : 1$ *correspondence between* $\mathfrak{H}^{\mathbb{R}}_{m,\lambda}$ *and* $\mathfrak{D}_{m,\lambda}$

Denote also $\mathbb{R}\mathcal{H}^{\text{num}}_{m,\lambda}$ the set of 4-tuples $(M, \mathcal{T}, f, \nu)$, up to equivalence, such that

- $(M, \mathcal{T}, f) \in \mathbb{R}\mathcal{H}_{m,\lambda}$,
- ν is a bijection from the set of critical points of f to $\{1, \dots, 2m\}$ such that $\nu(\mathcal{T}(u_i)) = 2m + 1 - \nu(u_i)$,
- If $1 \le \nu(a) < \nu(b) \le m$ then $\text{Re}(f(a)) < \text{Re}(f(b))$.

Finally let the map the map $\mathcal{LL} : \mathcal{H}_{m,\lambda} \to \mathbb{C}^{(m)}$ called the *Lyashko–Looijenga map* (also the LL-map or the branch map), sending a point (M, f) in $\mathcal{H}^{\text{num}}_{m,\lambda}$ to the a set of points $(y_1, \dots, y_m) \in \mathbb{C}P^1$, the critical values of f. An important local result for the $\mathcal{LL}$ map is

Theorem B. *(Chapter 3, Theorem 3.8) The local multiplicity of the Lyashko–Looijenga map $\mathcal{LL}$ near the point $F \in \mathfrak{R}\mathfrak{H}_{m,\lambda}$ is 2^m.*

We then prove that the correspondences of theorem A and theorem B are actually mutually inverse, with a coefficient of 2^m. Indeed by monodromy we first show that

Theorem C. *(Chapter 3, Theorem 4.1) The correspondence $(F, \nu) \mapsto (\delta_{-1} \stackrel{def}{=} \tau, \delta_0, \dots, \delta_{m-1})$ is a one-to-one map from $\mathfrak{D}_{m,\lambda}$ to $\mathfrak{H}^{\mathbb{R}}_{m,\lambda}$.*

With this theorem in hand we can finally get the result:

Theorem (Chapter 3, Theorem 4.3). *The correspondences between $\mathfrak{D}_{m,\lambda}$ and $\mathfrak{H}^{\mathbb{R}}_{m,\lambda}$ obtained in Theorem C and Theorem A are mutually inverse.*

We now give few words about a conjectural algebro-geometric definition for the Hurwitz numbers of the reflection groups B.

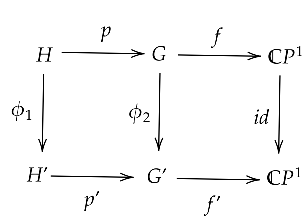
The (conjectural) algebro-geometric definition of Hurwitz numbers for reflection groups B and D looks as follows. Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_s)$ and $\mu - (\mu_1 \geq \dots \geq \mu_t)$ be two partitions such that $|\lambda| + |\mu| = n$. Consider holomorphic maps $H \xrightarrow{p} G \xrightarrow{f} \mathbb{C}P^1$ where

- G and H are complex curves;
- f is a degree n holomorphic maps with m simple critical points and a critical value $\infty \in \mathbb{C}P^1$ described below;
- p is a degree 2 holomorphic map with ℓ simple critical points;
- The preimage $f^{-1}(\infty) = \{x_1, \dots, x_s, q_1, \dots, q_t\}$ where the multiplicity of x_i is λ_i, $i = 1, \dots, s$, and the multiplicity of q_i is μ_i, $i = 1, \dots, t$. Additionally, it is supposed that $x_1, \dots, x_s$ are regular values of p, and $q_1, \dots, q_t$, its critical values.

Once the degree of p is 2, the curve H is hyperelliptic, and there is a hyperelliptic involution $\mathcal{T} : H \mapsto H$ such that $p \circ \mathcal{T} = p$ (it exchanges the two preimages of any point $x \in G$); the critical points of p are the fixed points of $\mathcal{T}$.

We call the diagram $H \xrightarrow{p} G \xrightarrow{f} \mathbb{C}P^1$ described above a B-ramified covering with the profile (λ, μ, m, ℓ). A D-ramified covering is a B-ramified covering with the profile $(\lambda, \mu, m, 0)$, so that the involution $\mathcal{T}$ has no fixed point.

The B-(or D-)ramified coverings F and F' are said to be equivalent if there is a pair of biholomorphic maps ϕ_1 and ϕ_2 such that the following diagram commutes:



Conjecture. Let λ and μ two partitions such that $|\lambda| + |\mu| = n$. There is a one-to-one correspondence between equivalence classes of B-ramified coverings having profile (λ, μ, m, ℓ) and B-ribbon decompositions with the same profile.

Corollary. *Let λ and μ two partitions like above. There is a one-to-one correspondence between equivalence classes of B-ramified coverings having the profile $(\lambda, \mu, m, 0)$ and D-ribbon decomposition with the same profile.*

Overview of the results. Once more we sum up the results with some open question in the two following tables

Real Hurwitz numbers $h^{\sim}_{m,\lambda}$

- Algebraic definition:
  $\#(\sigma_1 \dots, \sigma_m) \mid \sigma_i \in C_2$ and $\sigma_1\sigma_2 \dots \sigma_m \tau \sigma_m \dots \sigma_1 \tau \in B^{\sim}_{\lambda}$
- Algebro-geometric definition:
  The number of isomorphism classes
  of ramified coverings of the sphere where m finite real simple
  critical values (the profile $[2^2, 1^{2n-4}]$),
  and the critical value ∞
  and the critical value over ∞ has the profile (λ, λ)
- Geometric definition:
  Open question
- Topological definition:
  The set of decomposition of surfaces into m
  possibly twisted ribbons and n disks,
  surface boundary having s components
  containing $\lambda_1, \dots, \lambda_s$ vertices (endpoints of ribbon diagonals)
- Expression of the generating function $\mathcal{H}^{\sim}(\beta, p_1, p_2, \dots)$ of disconnected real Hurwitz numbers in terms of in terms of zonal polynomials Z_λ :
  $\mathcal{H}^{\sim}(\beta, p_1, p_2, \dots) = \sum_\lambda \exp\big(2\beta \sum_i \lambda_i(\lambda_i - i)\big) \frac{2^{|\lambda|} Z_\lambda(p)}{H_\lambda(2) H'_\lambda(2)}$.
- Solution of the differential equation Beltrami–Laplace:
  $\frac{\partial \mathcal{H}^{\sim}}{\partial \beta} = \sum_{i,j \geq 1} (i+j)(p_i p_j + p_{i+j}) \frac{\partial \mathcal{H}^{\sim}}{\partial p_{i+j}} + 2ij p_{i+j} \frac{\partial^2 \mathcal{H}^{\sim}}{\partial p_i \partial p_j}$
- Integrable system :
  Open question

Hurwitz numbers for reflection group B $h^B_{(\lambda,\mu,m,\ell)}$:

- Algebraic definition:
  $\#(\sigma_1 \dots, \sigma_{m+\ell}) \mid \#\{p \mid \sigma_p = r_{ij}, 1 \leq i < j \leq 2n\} = m$,
  $\#\{p \mid \sigma_p = l_i, 1 \leq i \leq 2n\} = \ell, \sigma_1 \dots \sigma_{m+\ell} \in C_{\lambda|\mu}$
- Algebro-geometric definition (Conjecture):
  the number of isomorphism classes of diagrams of ramified
  coverings $F : H \xrightarrow{p} G \xrightarrow{f} \mathbb{C}P^1$ where G and H are complex
  curves f is a degree n meromorphic function with m simple
  critical points, p is a degree 2 holomorphic map with ℓ
  simple critical points; 2cmthe critical value of F over ∞ has
  ramification profile $(\lambda, \lambda, 2\mu)$
- Geometric definition:

Open question

- Topological definition:
  The set of decompositions into $2m+\ell$ non-twisted ribbons
  and $2n$ disks having boundary of $2s+k$ components containing
  $\lambda_1, \lambda_1, \dots, \lambda_s, \lambda_s, 2\mu_1, \dots 2\mu_k$ vertices, the surface is equipped
  with an orientation-preserving involution
  having ℓ invariant ribbons and a fixed point in each
- Generating function in terms of polynomials $\mathcal{H}^B(\beta, \gamma, p_1, p_2, \dots)$ of disconnected B-Hurwitz numbers in terms of Schur polynomials s_λ :
  $\mathcal{H}^B(\beta, \gamma, p_1, p_2, \dots, q_1, q_2, \dots) = \sum_{\lambda,\mu} \exp\big(\beta \sum_{i=1}^{\infty}(\lambda_i(\lambda_i - 2i+1) + \mu_i(\mu_i - 2i+1) + \gamma \sum_{i=1}^{\infty}(\lambda_i - \mu_i)\big)\times$
  $s_\lambda(1,0,0\dots)s_\mu(1,0,0\dots)s_\lambda((p_1+q_1)/2,\dots)s_\mu((p_1-q_1)/2,\dots)$
- Solution of a differential equation Pair of cut-and-joins:

$$\frac{\partial \mathcal{H}^B}{\partial \beta} = \sum_{i,j=1}^{\infty}\Bigg((i+j)p_iq_j\frac{\partial}{\partial q_{i+j}} + 2ijq_{i+j}\frac{\partial^2}{\partial p_i \partial q_j} + ijp_{i+j}\frac{\partial^2}{\partial q_i \partial q_j}$$
$$+\tfrac{1}{2}(i+j)q_iq_j\frac{\partial}{\partial p_{i+j}} + \tfrac{1}{2}(i+j)\Bigg)\mathcal{H}^B$$
$$\frac{\partial \mathcal{H}^B}{\partial \gamma} = \sum_{i=1}^{\infty}\left(ip_i\frac{\partial}{\partial q_i} + iq_i\frac{\partial}{\partial p_i}\right)\mathcal{H}^B$$

- Integrable system :
  The generating function $\mathcal{H}^B(\beta, \gamma, u+v, u-v)$ is a 2-parameter
  family of τ-functions

Addendum

While writing the thesis, a paper by Denis Gorodkov, Maksim Karev and the author [**11**] appeared containing the proof of the conjecture above, and provided the geometric definition for Hurwitz numbers for the reflection group B. In more details:

Usual Hurwitz numbers are known to satisfy the ELSV-formula [**9**]: for any $g \geq 0, k_1, \dots, k_n \in \mathbb{N}$ we have

$$h_{g;k_1,\dots,k_n} = (2g-2+n+\sum_{i=1}^{n} k_i)!\prod_{i=1}^{n}\frac{k_i^{k_i}}{k_i!}\int_{\bar{\mathcal{M}}_{g,n}}\frac{\Lambda_{g,n}}{(1-k_1\psi_1)\cdots(1-k_n\psi_n)},$$

where $\bar{\mathcal{M}}_{g,n}$ is the Deligne–Mumford compactification of the moduli space of genus g curves with n marked points, $\Lambda_{g,n}$ is the total Chern class of the dual to the Hodge vector bundle, and ψ_i are the corresponding ψ-classes. It is shown in [**11**] that:

Theorem. *The logarithm of the generating function $\mathcal{H}^B$ in variables u and v equals*

$$\log \mathcal{H}^B = \sum_{\substack{g\geq 0, n\geq 1\\ k_1,\dots,k_n\in\mathbb{N}}} \frac{(2\beta)^{2g-2+n+\sum_{i=1}^n k_n} e^{\sum_{i=1}^n k_n \gamma}}{n!}$$

$$\times \prod_{i=1}^n \frac{k_i^{k_i}}{k_i!} \int_{\bar{\mathcal{M}}_{g,n}} \frac{\Lambda_{g,n}}{(1-k_1\psi_1)\cdots(1-k_n\psi_n)} u_{k_1}\dots u_{k_n} + \sum_{\substack{g\geq 0, n\geq 1\\ k_1,\dots,k_n\in\mathbb{N}}} \frac{(2\beta)^{2g-2+n+\sum_{i=1}^n k_n} e^{\sum_{i=1}^n k_n - \gamma}}{n!}$$

$$\times \prod_{i=1}^n \frac{k_i^{k_i}}{k_i!} \int_{\bar{\mathcal{M}}_{g,n}} \frac{\Lambda_{g,n}}{(1-k_1\psi_1)\cdots(1-k_n\psi_n)} v_{k_1}\dots v_{k_n}.$$

The results of this dissertation are published in two articles:

- R.Fesler Hurwitz numbers for reflection groups B and D, *Mathematical Notes* vol.114:5-6.
- Y. Burman, R. Fesler, Ribbon decomposition and twisted Hurwitz numbers, *Mathematics Research Reports, Volume 5 (2024) p. 1-19*

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