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Chaotic dynamics of reversible and dissipative systems

Dissertation summary
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1 Introduction

Relevance. Reversible dynamical systems are mathematical models of many physical processes and phenomena. For example, problems of rigid body dynamics, celestial mechanics, hydrodynamics, quantum mechanics, and many others are effectively described using reversible systems. Recall that a dynamical system is reversible if it is invariant under T-symmetry ($t \to -t$) and some coordinate change ($x \to G(x)$) such that $G \circ G = id$.

Reversibility as a certain type of symmetry is most often observed in Hamiltonian dynamical systems. Moreover, for a long time the terms "reversible" and "Hamiltonian" were considered almost synonymous, since in practice the vast majority of reversible dynamical systems turned out to be Hamiltonian. Many elements of the theory of Hamiltonian systems were successfully transferred to the case of reversible systems. It led to the creation of the reversible Kolmogorov-Arnold-Moser (KAM) theory, and some results of the theory of local bifurcations were extended to the case of reversible systems, and so on.

However, the dynamics of reversible systems can fundamentally differ from the dynamics of Hamiltonian systems. This became clear after the work of Politi, Oppo, Badia (1986) and Quispel, Roberts (1989), where such dissipative elements of dynamics as periodic sources and sinks, as well as strange attractors and repellers, were found in reversible systems. In the work of Pikovsky and Topaj (2002), using the example of systems describing chains of coupled rotators, for reversible systems it was shown that a chaotic attractor and a repeller can overlap, but not coincide. Later, in the work of Gonchenko, Kazakov, Turaev (2017), this phenomenon was mathematically described. As a consequence, it led to the creation (in the works of Gonchenko and Turaev) of the concept of mixed dynamics - the third type of chaos, characterized by the irremovable inseparability of a chaotic attractor from a chaotic repeller.

Thus, it became obvious that reversible systems can demonstrate a unique type of chaotic behavior which is completely different from conservative and dissipative ones.This is a reason why the study of reversible dynamics is relevant both from the point of view of the development of the theory of dynamical systems, and for possible applications of this theory to the physical problems. The 1st and the 2nd Chapters of this dissertation work are devoted to the study of the features of the chaotic dynamics of reversible non-conservative systems. A set of new results in this part of dissertation work includes the description of bifurcations of the 1:3 resonance for the reversible nonconservative case. For reversible three-dimensional diffeomorphisms, a new scenario for the (explosive) emergence of mixed dynamics is proposed. The features of chaotic dynamics in the nonholonomic model of the Celtic stone are described. Author if the dissertation contributed to the development of the theory of reversible and nonholonomic systems, as well as made it possible to demonstrate the effectiveness of the developed methods.

As noted above, reversible systems can exhibit such dissipative elements of dynamics as strange attractors. Moreover, these attractors can be exactly the same as in dissipative systems. Only in this case, in a reversible system, each attractor corresponds to the same type of repeller - an attractor for a system with reverse time. Thus, the results of the study of strange attractors of dissipative systems can be applied to reversible systems (just as some results of the study of Hamiltonian systems can also be applied to reversible models).

Recall that according to the “P or Q” hypothesis formulated by Gonchenko, Kazakov, Turaev (2021), all strange attractors can be divided into two types: pseudohyperbolic attractors and quasiattractors. One can expect that attractors of the first type are “real” chaotic attractors, because due to the property of pseudohyperbolicity (which generalizes the concept of hyperbolicity), any of their trajectories has a positive maximum Lyapunov exponent, and this property is preserved under small perturbations of the system (for example, parameter changing). In the second case, on the contrary, one can never be sure what exactly he is dealing with, really chaotic behavior or just a long transition process, after which the trajectories of the system run away to a simple attractor (for example, a limit cycle). The problem of how to distinguish a “real” (pseudohyperbolic) attractor from a quasiattractor is one of the main problems in the theory of dynamical systems.

The 3rd Chapter of this dissertation is devoted to the solution of this problem for some classes of systems. Here, using the numerical methods developed in the work of Gonchenko, Kazakov, Turaev (2021), a study of the pseudohyperbolicity of some attractors of four-dimensional flows and three-dimensional maps was carried out. On the example of a periodically perturbed Shimizu-Morioka system, it is demonstrated that the pseudohyperbolicity of the attractor can be preserved even under considerable perturbations, numerical estimates are constructed for the magnitude of the perturbation, under which the pseudohyperbolicity of the attractor is preserved. In the class of orientation-changing three-dimensional Hénon maps, it is shown that the previously discovered non-orientable attractors of Lorenz and figure-of-eight types are not pseudohyperbolic, thereby disproving the hypothesis of their pseudohyperbolicity. Contrary, the Lorenz attractor at a point of period 2, also previously found in the nonorientable Hénon map is pseudohyperbolic.

During the work on the dissertation, author developed numerical methods that allow to identify areas with strange attractors and mixed dynamics in the space of system parameters, to carry out the continuation of bifurcation curves in the space of parameters, and also to construct invariant manifolds of saddle periodic trajectories. The developed methods are implemented within the framework of the program complex.

State-of-the-art. The property of reversibility has played and still plays an important role in various fields of natural sciences. Many systems of differential equations that are of an interest in the simulation of physical processes are reversible in time. Apparently, this was revealed for the first time in 1877 by Loschmidt for particles moving in a force field independent of velocity. Boltzmann soon noticed that Maxwell’s equations are also reversible (under the additional condition of field reversibility). In 1904, Painlevé established the property of reversibility for a freely falling body in the Newtonian equations of motion. Penrose (1979, 1989) showed that the Einstein equations of classical general relativity are reversible. In 1959 Wigner showed the importance of time reversal symmetry in quantum mechanics.

In the 70s-80s, the reversing property in problems of classical mechanics was used as a very useful tool for the study of symmetric periodic trajectories, as well as trajectories of homoclinic and heteroclinic tangency (Devaney, Churchill, Rod). Moreover, in Hamiltonian systems it often turned out that to prove certain statements it is more convenient to use the reversibility property than the Hamiltonian property (Arnold, Devaney). In the 80s-90s, many statements proved for Hamiltonian systems were adapted to the case of reversible systems (Sevryuk, Chow, Pei).

In the late 80s, after the works of Politi, Oppo, Badii (1986), Quispel, Roberts (1989), Tsang, Miroyo, Strogatz, Weisenfeld (1991), it became clear that reversible systems can fundamentally differ from Hamiltonian systems. It was shown in the corresponding works that periodic sources and sinks can appear in reversible systems. At the same time, the possibility of coexistence of such dissipative elements with regions with conservative (Hamiltonian) dynamics was pointed out. Later on, in the theoretical works of Gonchenko, Delshams, Lazaro, Lamb, Stenkin, Turaev, it was pointed out that the dissipative elements of dynamics are fundamentally inseparable from the conservative ones in general reversible systems. Numerically, such a phenomenon was discussed in the works of Pikovsky, Topaj (2002), A. Gonchenko, S. Gonchenko, Kazakov (2013), where the intersection of a strange attractor and a repeller was found. All this formed the basis for the creation by Gonchenko, Turaev (2016, 2017) of the theory of mixed dynamics as the third type of chaos, for which the chaotic attractor intersects with the chaotic repeller, but does not coincide with it. In this dissertation work, some issues related to the study of bifurcation mechanisms for the emergence of mixed dynamics, as well as the presence of this phenomenon in nonholonomic mechanics, play an important role (1st and 2nd Chapters).

The second part of the dissertation work is devoted to the study of the pseudohyperbolicity of strange attractors. The theory of pseudohyperbolicity, which generalizes the classical hyperbolic and singular hyperbolic theories, was founded in the work of Turaev and Shilnikov (1998), where the first geometric examples of such attractors were also given. In a specific dynamical system, the first example of a pseudohyperbolic attractor (different from hyperbolic, as well as singular hyperbolic) was discovered by Gonchenko, Ovsyannikov, Turaev, Shilnikov (2005). In this work, it was shown that there is a discrete Lorenz attractors in the class of three-dimensional Hénon maps. A hypothesis was also put forward about the pseudohyperbolic nature of the discovered attractors. This hypothesis was later confirmed in the work of Gonchenko, Kazakov, Turaev (2021), where the first example of the Turaev-Shilnikov wild spiral attractor was also given, and its pseudohyperbolicity was established. Turaev and Shilnikov (2008) showed that the pseudohyperbolicity of the attractor is preserved under small perturbations of the system that are periodic in time. In the first part of 3rd Chapter of the dissertation work, using the example of a periodically perturbed Shimizu-Morioka system, the author showed that pseudohyperbolicity is also preserved under not even small perturbations.

As for systems from applications, the first example of a model demonstrating a pseudohyperbolic attractor was given in the work of A. Gonchenko, S. Gonchenko, where a discrete Lorentz attractor was found in the nonholonomic model of a Celtic stone. Its pseudohyperbolicity was established quite recently in the work of A. Gonchenko, S. Gonchenko, Kazakov, Kozlov. The second part of 2nd Chapter of this dissertation work is devoted to further research of the discrete attractor of Lorenz in the nonholonomic model of the Celtic stone, where the boundaries of the region of existence of this attractor are constructed by the author.

In recent works by A. Gonchenko, A. Kozlov (2016, 2017, 2021), examples (geometric constructions) of pseudohyperbolic attractors for the reversible three-dimensional maps were proposed; nonorientable Lorenz-like and eight-dimensional attractors, as well as the period-2 Lorenz-like attractor, were discovered in the class of three-dimensional Hénon maps with a negative Jacobian. Also in these works, a hypothesis was put forward about the pseudohyperbolic nature of the discovered attractors. In the second part of 3rd Chapter of the dissertation work, the hypothesis of the pseudohyperbolicity of nonorientable Lorenz-like and figure-eight attractors was rejected. At the same time, it was shown that the period-2 Lorenz-like attractor is pseudohyperbolic.

Aims and tasks of the study. Aims of the study is to study some features of the chaotic dynamics of two-dimensional and three-dimensional reversible nonconservative maps. At the same time, to develop the necessary approaches and methods. To achieve the goals set, the following tasks were considered:

- Description of symmetry-loss bifurcations in two-dimensional cubic Hénon maps under the action of reversible nonconservative perturbations.
- Development of new scenarios for the emergence of mixed dynamics in two-dimensional and three-dimensional reversible diffeomorphisms.
- Study of the features of the chaotic dynamics of the nonholonomic model of the Celtic stone, construction of the boundaries of the regions of existence of various types of strange attractors, as well as mixed dynamics in this model.
- Estimation of the magnitude of a periodic perturbation under which the pseudohyperbolicity of the Lorenz-like attractors is preserved.

Research methods. To achieve the stated goals and solve the tasks set, the author used analytical, qualitative and numerical methods of the theory of dynamical systems. In the study of specific models (nonholonomic model of a Celtic stone, two-dimensional and three-dimensional Hénon maps, perturbed Shimizu-Morioka system), numerical methods of applied bifurcation theory, methods of Lyapunov diagrams and charts of dynamic regimes, as well as numerical methods of continuation with respect to a parameter and construction of invariant manifolds for saddle periodic trajectories. Numerical methods are combined within a programming complex.

Theoretical and practical significance. The theoretical significance is due to the fact that new results have been obtained in the theory of dynamical systems. The rearrangements of bifurcation diagrams in the neighborhood of the 1:3 resonance, which arise as a result of nonconservative reversible perturbations, are described; a new «explosive» scenario for the emergence of mixed dynamics for one-parameter families of three-dimensional reversible diffeomorphisms is proposed; the first example of a system with coexisting discrete attractors and Lorenz repeller is given; it is shown that the pseudohyperbolicity of attractors can be preserved under even not small periodic perturbations of the system.

The practical significance of the dissertation work is in the development of a programming complex for the study of reversible and dissipative systems, as well as in obtaining new results in the study of an applied model describing the dynamics of a Celtic stone on a plane. With the help of the developed methods for this model, the author constructed the boundaries of the region of existence of discrete attractors and Lorenz repellers, mixed dynamics, and new type spiral attractors. As a result , it turned out that the dynamics of nonholonomic systems is extremely rich. Moreover, most of the discovered dynamic phenomena and effects have received an adequate mathematical description.

Thus, the results obtained in the dissertation make a significant contribution both to the development of the theory of dynamical systems and its applications to the study of specific reversible and dissipative models.

Provisions for the defense

1. Bifurcations of 1:3 resonance in the reversible nonconservative case.

   On the example of modified cubic Hénon maps $H_3^{\pm} : \bar{x} = y, \bar{y} = -x + M_1 + M_2 y \pm y^3$ it was shown how reversible nonconservative perturbations influence the structure of bifurcation diagrams in the case of a 1:3 resonance, i.e. bifurcations of maps near a fixed point with multipliers $e^{\pm i2\pi/3}$. Two main types of symmetry breaking bifurcations are described, as a result of which nonconservative points of period 3 are born at the resonant level. It was shown that in the case of the map H_3^- these bifurcations lead to the appearance of mixed dynamics, and in the case of the map H_3^+ the hypothesis is formulated about the occurrence of an isolating resonance.

2. Degenerate resonances $p : q$ in conservative cubic Hénon maps $H_3^{\pm}$.

   It was shown that for $M_1 = 0$ all $p : q$ resonances in conservative cubic Hénon maps $H_3^{\pm}$ are degenerate for odd q. As a result of their bifurcations, four orbits of period q can be born at the resonant level, while in the general case there should be only two orbits of such type.

3. Strange attractors in the nonholonomic model of the Celtic stone.

   In the two-parameter family of the Celtic stone model, the boundaries of the region of existence of the Lorenz attractor and the Lorenz repeller symmetric to it are constructed. The main bifurcations leading to their occurrence and destruction are described. A new type of Shilnikov attractor was also found in this model. The scenario of its occurrence and destruction is described.

4. Mixed dynamics in a nonholonomic model of a Celtic stone.

   For three-dimensional reversible maps, a new scenario for the emergence of mixed dynamics as a result of a reversible bifurcation of the merging of stable and unstable limit cycles is proposed. The implementation of the scenario is demonstrated on the example of a nonholonomic model of a Celtic stone, for which the corresponding bifurcation diagram is constructed, and the region of existence of mixed dynamics is identified.

5. Lorenz pseudohyperbolic discrete attractor in a periodically perturbed Shimizu-Morioka system.

   According to the work of Turaev-Shilnikov, the pseudohyperbolicity of the attractor is preserved for small time-periodic perturbations of the system. Using the example of a periodically perturbed Shimizu-Morioka system, the author showed that the pseudohyperbolicity is also preserved under considerable perturbations. It was found that the corresponding discrete attractor is wild, i.e. admits the existence of homoclinic tangency.

6. Verification of pseudohyperbolicity of attractors of nonorientable three-dimensional Hénon maps.

   It is shown that previously discovered nonorientable Lorenz-like discrete attractors are not pseudohyperbolic, except for one type of attractors containing a point of period 2 (as well as the previously known heteroclinic attractor).

7. Software complex for the study of reversible and dissipative systems.

   A software complex has been developed to make it possible to identify the areas of existence of strange attractors and mixed dynamics in the parameter space of the system, as well as to carry out the continuation of bifurcation curves on the parameter plane.

Novelty and reliability. All results presented in the dissertation work are new. They are in good agreement with the existing theoretical concepts and provisions. Numerical experiments are described in detail, most of them were reproduced by other researchers.

The results submitted for defense are published in the leading peer-reviewed physical and mathematical journals indexed in the scientific databases Web of Science and Scopus with quartiles Q1 - 2 papers, Q2 - 1 paper and Q3 - 2 papers.

Approbation of the obtained results. The main results of the dissertation work were reported at the following international conferences and seminars:

1. Poster report *"Nonconservative reversible perturbation of conservative reversible maps on the plane"*, international conference "Shilnikov WorkShop 2016", December 2016, Nizhny Novgorod, Russia.

2. Talk *"Nonconservative reversible perturbations of reversible maps with unit Jacobian"*, international conference "Dynamics, Bifurcation and Chaos 2017", July 2017, Nizhny Novgorod, Russia.

3. Poster report *"Chaotic dynamics and multistability in the nonholonomic model of Celtic stone"*, international conference "Dynamics, Bifurcations and Chaos 2018", July 2018, Nizhny Novgorod, Russia.

4. Poster report *"On the domain of existence of Lorenz-like attractors in a nonholonomic model of Celtic stone"*, international conference "Shilnikov WorkShop 2018", December 2018, Nizhny Novgorod, Russia.

5. Poster report *"Examples of Discrete Lorenz Attractors in Three-dimensional Maps"* International conference "Topological methods in dynamics and related topics", January 2019, Nizhny Novgorod, Russia.

6. Poster report *"On the region of existence of discrete Lorenz attractor in a nonholonomic Celtic stone model"*, international conference "7th Bremen Summer SchoolSymposium on Dynamical Systems, pure and applied", August 2019, Bremen, Germany.

7. Poster report *"On discrete Lorenz attractors in a Celtic stone model"*, II International Conference "Topological methods in dynamics and related topics. Shilnikov workshop", December 2019 Nizhny Novgorod, Russia.

8. Talk *"On bifurcations of nonorientable three-dimensional maps leading to chaos"*, International Conference KROMSH, September 2021, Alushta, Crimea.

9. Talk *"On discrete homoclinic attractors of three-dimensional maps"*, international conference "Shilnikov WorkShop 2021", December 2021, Nizhny Novgorod, Russia.

10. Talk *"Chaotic dynamics in the nonholonomic models of Celtic stone"*,international workshop on Computing Technologies and Applied Mathematics, July 2022, Vladivostok, Russia.

11. Poster report *"On 1:3 Resonance Under Reversible Perturbations of Conservative Cubic Hénon Maps"*, 27th International Conference on Difference Equations and Applications (ICDEA 2022), July 2022, Gif-sur-Yvette, France.

List of articles submitted for defense on the topic of the dissertation (with indication of personal contribution).

[1*] Gonchenko M.S., Kazakov A.O., Samylina E.A., Shykhmamedov A.I. *On 1:3 Resonance Under Reversible Perturbations of Conservative Cubic Hénon Maps*//Regular and Chaotic Dynamics **27(6)** (2022), pp. 198 - 216

https://link.springer.com/article/10.1134/S1560354722020058

(Principal co-author. One of two methods for constructing reversible nonconservative perturbations is proposed, and the corresponding numerical experiments are carried out.)

[2*] Kazakov A.O., Gonchenko A.S., Gonchenko S.V., Samylina E.A. *Chaotic dynamics and multistability in a nonholonomic model of a Celtic stone*//Izvestiya Vysshikh Uchebnykh Zavedenii. Radiophysics **61(10)** (2018) pp. 867 - 882

https://radiophysics.unn.ru/sites/default/files/papers/2018_10_867.pdf

(Principal co-author. A scenario for the emergence of mixed dynamics is proposed, as a result of the merging of stable and completely unstable points, all numerical experiments are carried out, one-parameter and two-parameter bifurcation diagrams are constructed.)

[3*] Gonchenko A.S., Samylina E.A. *On the region of existence of the discrete Lorenz attractor in the nonholonomic model of the Celtic stone*//Izvestiya Vysshikh Uchebnykh Zavedenii. Radiophysics **62(5)** (2019) pp. 412 - 428

https://radiophysics.unn.ru/sites/default/files/papers/2019_5_412.pdf

(Principal co-author. Boundaries of the region of existence of the Lorenz attractor are constructed and the corresponding bifurcation scenarios are described.)

[4*] Gonchenko A. S., Gonchenko M. S., Kozlov A.D., Samylina E.A. *On scenarios of the onset of homoclinic attractors in three-dimensional non-orientable maps*//Chaos **31(4)** (2021), 043122 (19 p.)

https://doi.org/10.1063/5.0039870

(Discrete Lorenz-like and eight-dimensional attractors are found in the three-dimensional Hénon map with a negative Jacobian.)

[5*] Gonchenko S., Gonchenko A., Kazakov A., Samylina E. *On discrete Lorenz-like attractors* //Chaos: An Interdisciplinary Journal of Nonlinear Science **31(2)** (2021), 023117 (20 p.)

https://aip.scitation.org/doi/abs/10.1063/5.0037621

(It is shown that the Lorenz attractor remains pseudohyperbolic under the influence of considerable periodic perturbations of the Shimizu-Morioka system.)

2 A summary of the work: main results

The main results of the dissertation work are divided into three parts:

- bifurcation of 1:3 resonance in reversible nonconservative Hénon map;

- chaotic dynamics of the nonholonomic model of the Celtic stone;
- pseudohyperbolic attractors of three-dimensional maps.

2.1 Bifurcation of 1:3 resonance in reversible nonconservative Hénon map

In 1st Chapter of the dissertation work, the problem of the influence of reversible nonconservative perturbations on the structure of bifurcation diagrams in the neighborhood of a 1:3 resonance is considered, i.e. fixed point bifurcations with multipliers $e^{\pm i2\pi/3}$. This problem was solved using the example of modified cubic maps

$$H_3^{\pm}:\ (x,y)\to(\bar{x},\bar{y})\ :\bar{x}=y,\quad \bar{y}=-x+M_1+M_2y\pm y^3, \tag{1}$$

dependent on parameters M_1 and M_2.

Both maps H_3^+ and H_3^- are conservative (their Jacobian is identically equal to one), and also invertible with respect to involution $h:(x,y)\to(y,x)$. The following modification of maps is proposed(1):

$$\tilde{H}_3^{\pm}(\varepsilon)\ :\quad \bar{x}+\varepsilon\bar{x}\bar{y}=y+\varepsilon xy,\quad \bar{y}=-x+M_1+M_2(y+\varepsilon xy)\pm(y+\varepsilon xy)^3. \tag{2}$$

For $\varepsilon>0$ maps (2) remain invertible under involution $h:(x,y)\to(y,x)$, but are no longer conservative, since their Jacobian is no longer identically equal to one.

For the obtained maps $\tilde{H}_3^{\pm}(\varepsilon)$ a detailed bifurcation analysis was carried out. Particular attention is paid to the study of local symmetry loss bifurcations associated with the 1:3 resonance. Such bifurcations lead to the appearance of pairs of nonconservative periodic trajectories. In the case under consideration, these include two types of reversible pitchfork bifurcation. For both cases, the ranges of parameters corresponding to the existence of nonconservative periodic trajectories of period-3 are established.

In the case of the map $\tilde{H}_3^+(\varepsilon)$ it is shown that the reversible pitchfork bifurcation is supercritical. As a result of this bifurcation, stable and unstable points of period 3 are born from the elliptic point E_3 of period-3, and the point E_3 itself becomes a saddle point, its Jacobian is equal to one. In the case of the map $\tilde{H}_3^-(\varepsilon)$ the pitchfork bifurcation is subcritical: a saddle point of period-3 is transformed into an elliptic point of the same period, and two saddle points O_1^3 and O_2^3 of period-3 are born next to it, and one of them has a Jacobian greater than 1 ($J(O_1^3)>1$), while the other has a Jacobian less than 1 ($J(O_2^3)<1$).

In addition, for map $\tilde{H}_3^-(\varepsilon)$ numerical confirmation of the existence of mixed dynamics is given. Here, in the neighborhood of an elliptical fixed point, chaotic dynamics is observed that is visually indistinguishable from conservative (Hamiltonian) chaos, see Fig. 1. However, numerical studies show that in this case, based on the nonconservative saddle points of period 3 O_1^3 and O_2^3, a non-rough heteroclinic contour is formed, see Fig. 2. According to [1], see also [2, 3], bifurcations of such contours lead to the coexistence of a countable number of stable and completely unstable periodic points that have a non-empty closure at the intersection, which means that the chaos observed in Fig. 1 is a mixed dynamic.

For the map $\tilde{H}_3^+(\varepsilon)$, a conjecture is formulated about the possibility of the existence

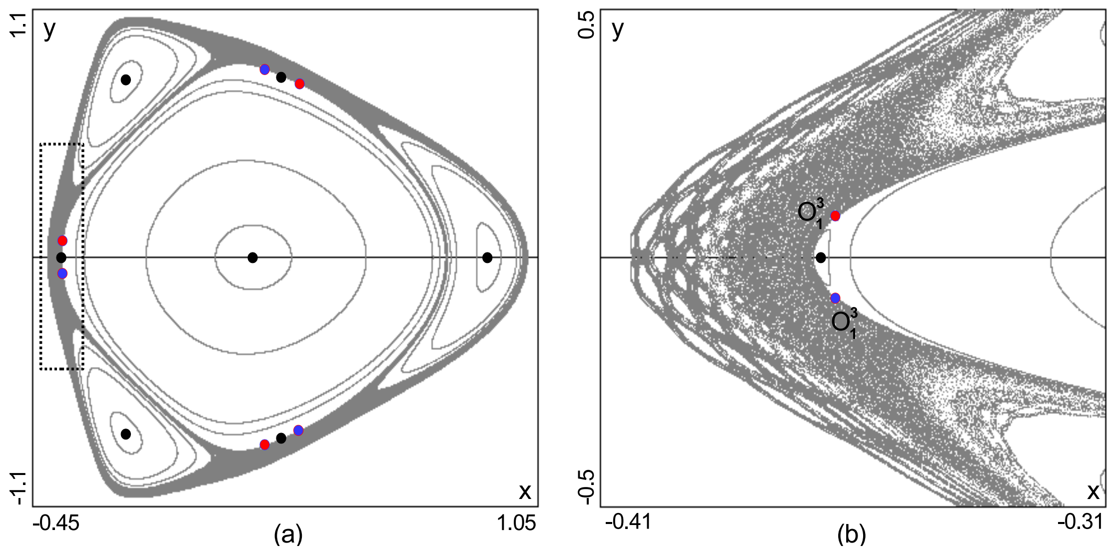


Fig. 1: Phase portrait and the zoomed fragment for map $\tilde{H}_3^-(\varepsilon)$ for $M_1 = -0.364, M_2 = -0.5$ and $\varepsilon = 0.3$. For convenience, the phase portrait is rotated by $\pi/4$. In this representation, the horizontal axis becomes $Fix(h)$. The chaotic dynamics (in the gray region) seems conservative (the phase portrait is self-symmetric with respect to the horizontal axis). The orbits O_1^3 and O_2^3 are the pair of nonconservative saddle 3-periodic orbits with the Jacobians $J(O_2^3) = 0.995 < 1$ and $J(O_1^3) = 1.005 > 1$.

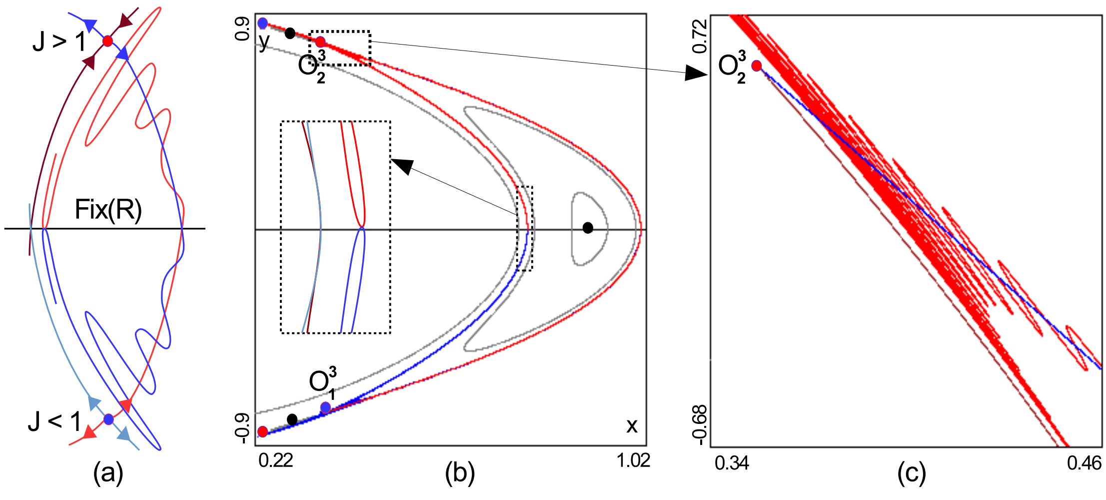
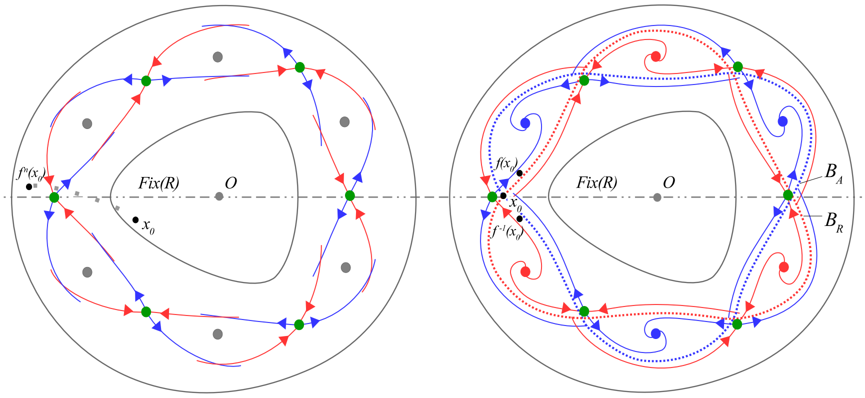


Fig. 2: (a) A schematic representation of the nontransversal heteroclinic cycle. (b), (c) Such a contour connecting orbits O_1^3 and O_2^3 in map $\tilde{H}_3^-(\varepsilon)$ for $M_1 = -0.364, M_2 = -0.5$ and $\varepsilon = 0.3$. A pair of manifolds W_1^s и W_1^u intersects transversally, while the other pair W_2^s and W_2^u has a quadratic tangency.



(a) (b)

Fig. 3: Different types of behavior in resonant zones of a symmetric elliptic point in the conservative (a) and reversible (b) cases. Periodic elliptic orbits are marked by gray bold points, while periodic sinks and sources are colored in blue and red, respectively. In plot (b) the absorbing domain B_A of a sink orbit (bounded by the blue dashed curves) intersects with the repelling domain of the source orbit (bounded by the red dashed curves). Thus, ε orbits of any point belonging to this intersection cannot leave the resonant zone, with either forward or backward iterations (isolated resonance). This means that an elliptic point of a typical two-dimensional reversible diffeomorphism is stable under permanently acting perturbations (Lyapunov stability by ε orbits).

of isolating resonances. It is well known that the phase portrait near the elliptic point of a two-dimensional invertible diffeomorphism is largely organized as in the conservative case. There is also a continuum of KAM curves surrounding an elliptical point. KAM curves are separated by resonance zones [4]. However, the behavior in resonant zones for reversible maps is fundamentally different from conservative ones, compare, for example, the figures 3a and 3b. In the conservative case, ε-trajectories can escape from any neighborhood of an elliptic point, i.e. such a point is unstable under permanent disturbances (Lyapunov instability along ε-trajectories) [5], see figure 3a. In the reversible nonconservative case, as it follows from [5, 6], it is typical, when periodic saddle points alternate with symmetrical pairs of sinks and sources in resonant zones, see figure 3b. Thus, a situation is possible in which there are intersecting absorbing regions B_A and repelling regions B_R around an elliptic point, such that direct and backward ε-trajectories of any point that belongs to the intersection $B_A\, capB_R$ cannot leave any neighborhood of this elliptical point [5, 7]. Such resonances are called *impassable* or *isolating*. In the dissertation work, a hypothesis about the existence of such resonances in the map $\tilde{H}_3^+(\varepsilon)$ was stated.

For conservative maps H_3^+ and H_3^- a theorem on the degeneracy of odd resonances was formulated and proved for parameters that ensure central symmetry.

Theorem 1 *Any resonance $p:q$ of fixed point $O(0,0)$ of maps $H_3^{\pm}$ with $M_1 = 0$ is at least thrice degenerate for any odd $q > 3$.*

An illustration of this result is shown in Fig. 4. On Fig. 4a,b phase portraits are shown near degenerate resonances 1:5 and 1:7 for display H_3^+. These resonances occur for $M_2 \approx 0.575$ and $M_2 \approx 1.15$, respectively. This results in four periodic trajectories: a pair of symmetrical saddles (colored in light and dark green, respectively) and a pair of asymmetric periodic elliptical

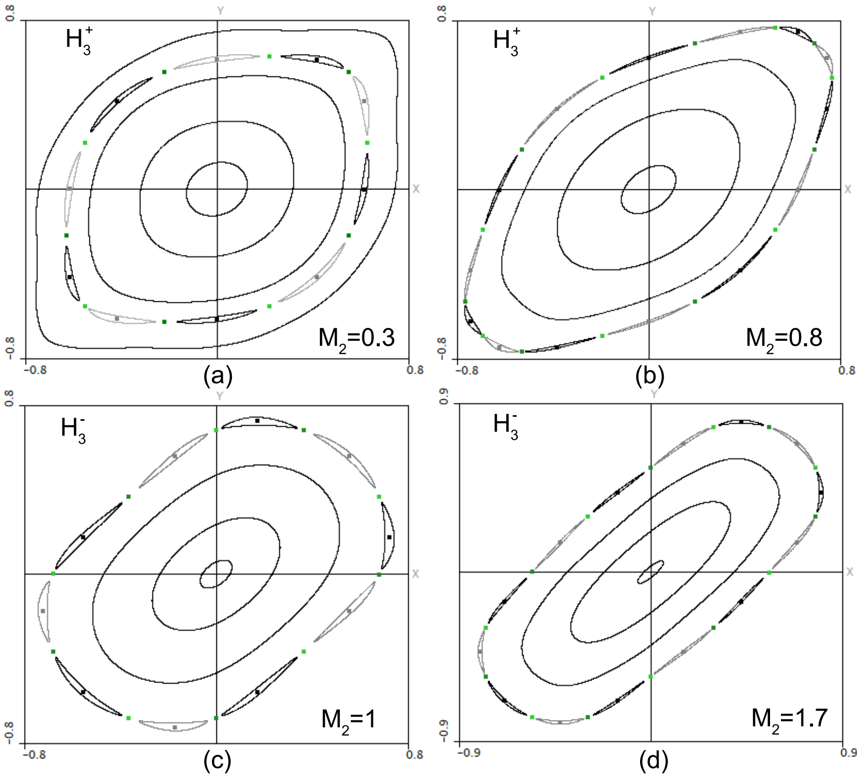


Fig. 4: Phase portraits near degenerate resonances 1:5 (left column) and 1:7 (right column) in conservative cubic Hénon maps H_3^+ (top row) и H_3^- (down row).

trajectories (colored in gray and black, respectively). On Fig. 4c,d phase portraits for map H_3^- are shown. Unlike the previous case, here the periodic elliptic points are symmetric, while the periodic saddles are not. Here the 1:5 resonance occurs at $M_2 \approx 0.66$, and the 1:7 resonance occurs at $M_2 \approx 1.36$.

2.2 Chaotic dynamics in a nonholonomic model of a Celtic stone

Celtic stone is a rigid body, the lower part of which looks like a truncated paraboloid, and the upper one looks like a plane. If you put such a stone on a horizontal surface, then it will rest on it with only one point of the convex section. The peculiarity of the Celtic stone is as follows: if you twist it counterclockwise, then it will continue its movement as an ordinary round body. However, if you twist it clockwise, then after a while the stone will slow down its movement, begin to sway, and eventually completely change the direction of its rotation to the opposite, that is, it will begin to rotate counterclockwise. This phenomenon is called the reverse effect.

The most popular mathematical model of the Celtic stone is the nonholonomic model. In addition, this model is one of the most interesting and rich in terms of possible dynamic phenomena among the models of rigid body dynamics. For the first time, a number of new phenomena were discovered in it, which previously seemed possible only in specific mathematical systems. For example, the reverse effect [8, 9, 10], strange attractors [11] and strange repellers (because the system is reversible), mixed dynamics and the Lorenz discrete attractor [12] were first discovered in the Celtic stone model , spiral chaos, etc.

The nonholonomic model of the Celtic stone is based on the principle of the absence of slippage during movement. This condition (the speed of the point of contact of the stone with the plane is zero) is expressed as $\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r} = 0$, where $\boldsymbol{\omega}$ – angular velocity, $\boldsymbol{v}$ – the speed of center of mass, $\boldsymbol{r}$ – a vector connecting the center of mass of the stone with the point of its contact with the plane.

To compile a nonholonomic model, well-known laws of mechanics are used: the law of conservation of momentum and the law of conservation of angular momentum, and the final system consists of six differential equations [11]

$$\begin{aligned} &\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\omega} + m\dot{\boldsymbol{r}} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + mg\boldsymbol{r} \times \boldsymbol{\gamma}, \\ &\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \end{aligned} \tag{3}$$

characterizing the change in the moments of inertia $\boldsymbol{M} = (M_1, M_2, M_3)$ and orientation $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ of Celtic stone.

Vectors $\boldsymbol{M}$ and $\boldsymbol{\omega}$ are related to

$$\boldsymbol{M} = \mathbf{I}\,\boldsymbol{\omega} + m\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}), \tag{4}$$

where $\mathbf{I} = diag(I_1, I_2, I_3)$ – inertia tensor. Vector $\boldsymbol{r}$ is related to δ and shape of the stone surface. For example, in the case of a Celtic stone in the form of a truncated paraboloid, it is assumed that $\boldsymbol{r} = \mathbf{Q}\boldsymbol{r}^*$, where

$$\mathbf{Q} = \begin{pmatrix} \cos\delta & \sin\delta & 0 \\ -\sin\delta & \cos\delta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

rotation matrix between the main horizontal axes of inertia and the geometric axes of the paraboloid, and the vector $\boldsymbol{r}^*$ is related to the vector $\boldsymbol{\gamma}$ using the formulas

$$r_1^* = -a_1\frac{\gamma_1}{\gamma_3}, \quad r_2^* = -a_2\frac{\gamma_2}{\gamma_3}, \quad r_3^* = -h + \frac{a_1\gamma_1^2 + a_2\gamma_2^2}{2\gamma_3^2}, \tag{5}$$

where a_1 and a_2 – principal radii of a paraboloid curvature, and h – the height of the center of mass of the stone. From these relations, the vector is also expressed $\dot{\boldsymbol{r}}$ through $\boldsymbol{M}$ and $\boldsymbol{\gamma}$.

An important property of the system (3) is its invertibility with respect to involution $h : \boldsymbol{\omega} \to -\omega$, $\boldsymbol{\gamma} \to \boldsymbol{\gamma}$ and time reversing $t \to -t$. It is also important to note that this system has two integrals of motion

$$\mathcal{H} = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\omega}) - m\mathbf{g}(\boldsymbol{r}, \boldsymbol{\gamma}) \;\text{ и }\; (\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1, \tag{6}$$

which are called the energy integral and geometric integral, respectively. Condition $(\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1$ means that the system (3) is actually five-dimensional; its phase space is $R^3 \times S^2$, such that $\boldsymbol{M} \in R^3$ and $\boldsymbol{\gamma} \in S^2 = \{(\gamma_1, \gamma_2, \gamma_3) \big| \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1\}$. Existence of the energy integral $\mathcal{H}$ means that the phase space is foliated into flow-invariant surfaces $\mathcal{H} = \text{const}$. Accordingly, the system in the restriction to each such surface is four-dimensional with a phase space homeomorphic to $S^2 \times S^2$.

In the problem under consideration, as in many other problems of rigid body dynamics, it is convenient to use the Andoyer–Deprit variables, (L, H, G, g, l), in which the condition

$(\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1$ (geometric integral) is done automatically. Everywhere below, graphical illustrations of the numerical calculation are given for the three-dimensional Poincaré map of some secant $\mathcal{S} = \{g = 0\}$, on which the coordinates $l, L/G$ and H/G are introduced.

In 2nd Chapter of the dissertation, author described bifurcations, as a result of which the type of stable regimes in the (3) model changes, and chaotic dynamics arise: strange attractors and mixed dynamics. In this case, the following parameters of the model were considered fixed

$$m = 1, \mathrm{g} \ = 100, a_1 = 9, a_2 = 4, h = 1,$$

On the (δ, E)-parameter plane bifurcation diagrams are constructed for two different cases:

1. $I_1 = 5, I_2 = 6, I_3 = 7$;
2. $I_1 = 2, I_2 = 6, I_3 = 7$.

In the first case, the following results were obtained:

- A new type of Shilnikov attractor has been discovered, see Fig. 5a. This attractor, in contrast to the classical Shilnikov attractor, contains a pair of fixed points of the saddle-focus type with a two-dimensional unstable manifold. It is shown that this attractor is born in a "hard way" as a result of the subcritical Neimark-Sacker bifurcation, and collapses as a result of a boundary crisis.

- It is shown that when the energy parameter changes, the attractor and repeller are separated, see Fig. 5b, can merge into one chaotic set corresponding to mixed dynamics, see Fig. 5c.

- A new scenario for the "explosive" emergence of mixed dynamics as a result of a reversible bifurcation of the merging of stable and unstable limit cycles is proposed. The implementation of the scenario is demonstrated on the example of the merger of the so-called. BM-cycles in a nonholonomic model. The corresponding bifurcation diagram is constructed, and the region of existence of mixed dynamics is identified.

- It is shown that the mixed dynamics in the nonholonomic model of a Celtic stone can coexist with conservative structures like KAM-tori, see Fig. 5d.

In the second case, in the two-parameter family of the model, the boundaries of the region of existence of the discrete Lorenz attractor and the Lorenz repeller symmetric to it are constructed, see Fig. (6). The Lorenz attractor contains a saddle fixed point O_a with two-dimensional stable and one-dimensional unstable manifolds, the Lorenz repeller contains a saddle fixed point O_r with one-dimensional stable and two-dimensional unstable manifolds. The main bifurcations leading to their occurrence and destruction are described.

It is shown that the Lorenz attractor in the case under consideration arises as a result of a heteroclinic bifurcation, when the one-dimensional unstable manifold of the point O_a lies on an invariant curve, which, in turn, is born from a homoclinic structure of the "figure-eight, butterfly" type. The attractor is destroyed as a result of a boundary crisis, when the unstable manifold of the point O_a begins to intersect with the stable manifold that bounds its domains of attraction. The corresponding bifurcation curves are constructed on the parameter plane (δ, E) using the programming complex developed by the author.

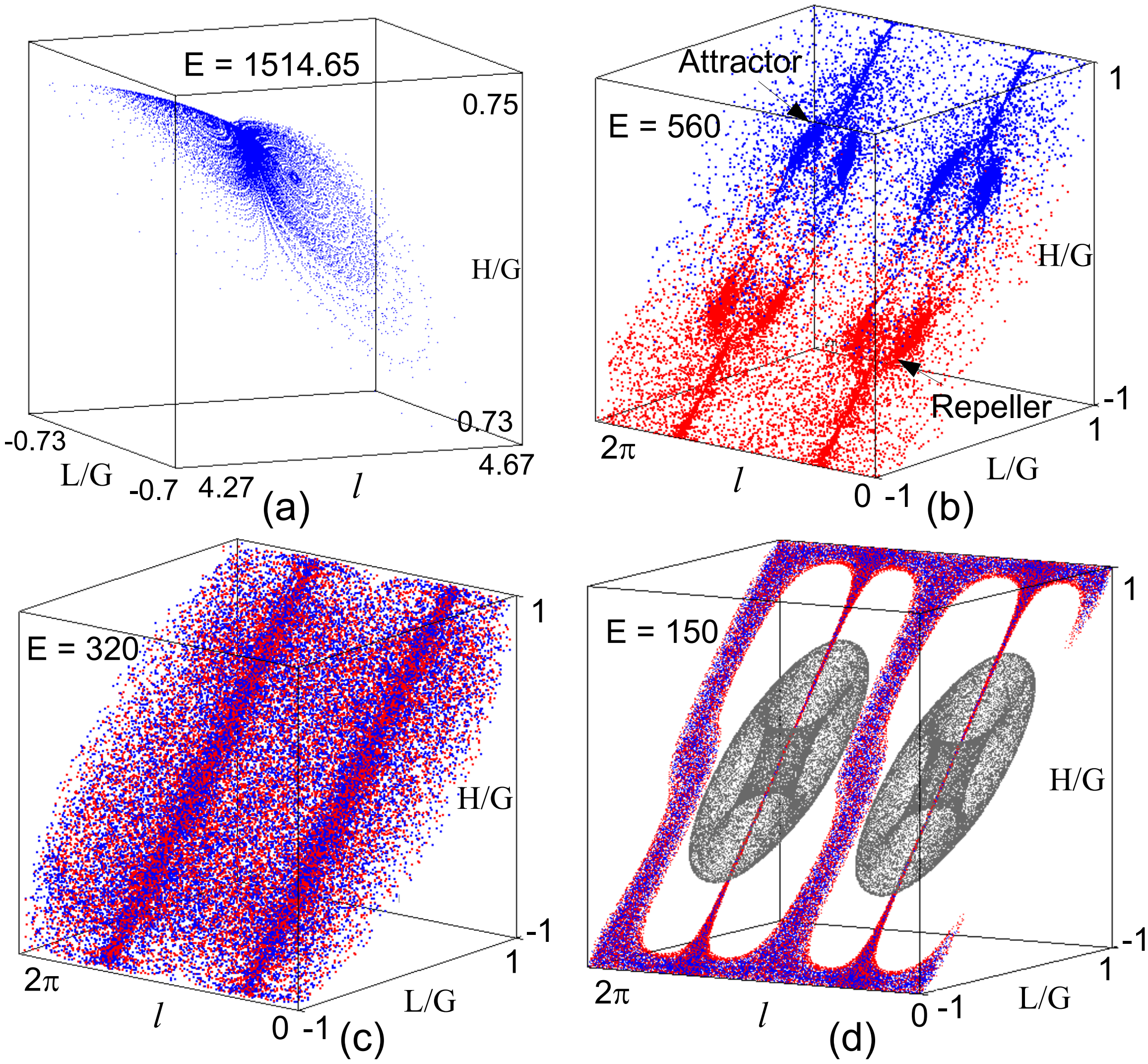


Fig. 5: Phase portraits of chaotic regimes for the Poincaré map in the nonholonomic model of Celtic stone (3) for $I_1 = 5, I_2 = 6, I_3 = 7$ (blue color denotes iterations of points on the attractor, red – on the repeller, gray – on conservative invariant sets): (a) Shilnikov's discrete attractor; (b) the chaotic attractor and repeller do not intersect; (c) mixed dynamics (attractor and repeller intersect but do not coincide); (d) mixed dynamics coexists with "KAM-tori".

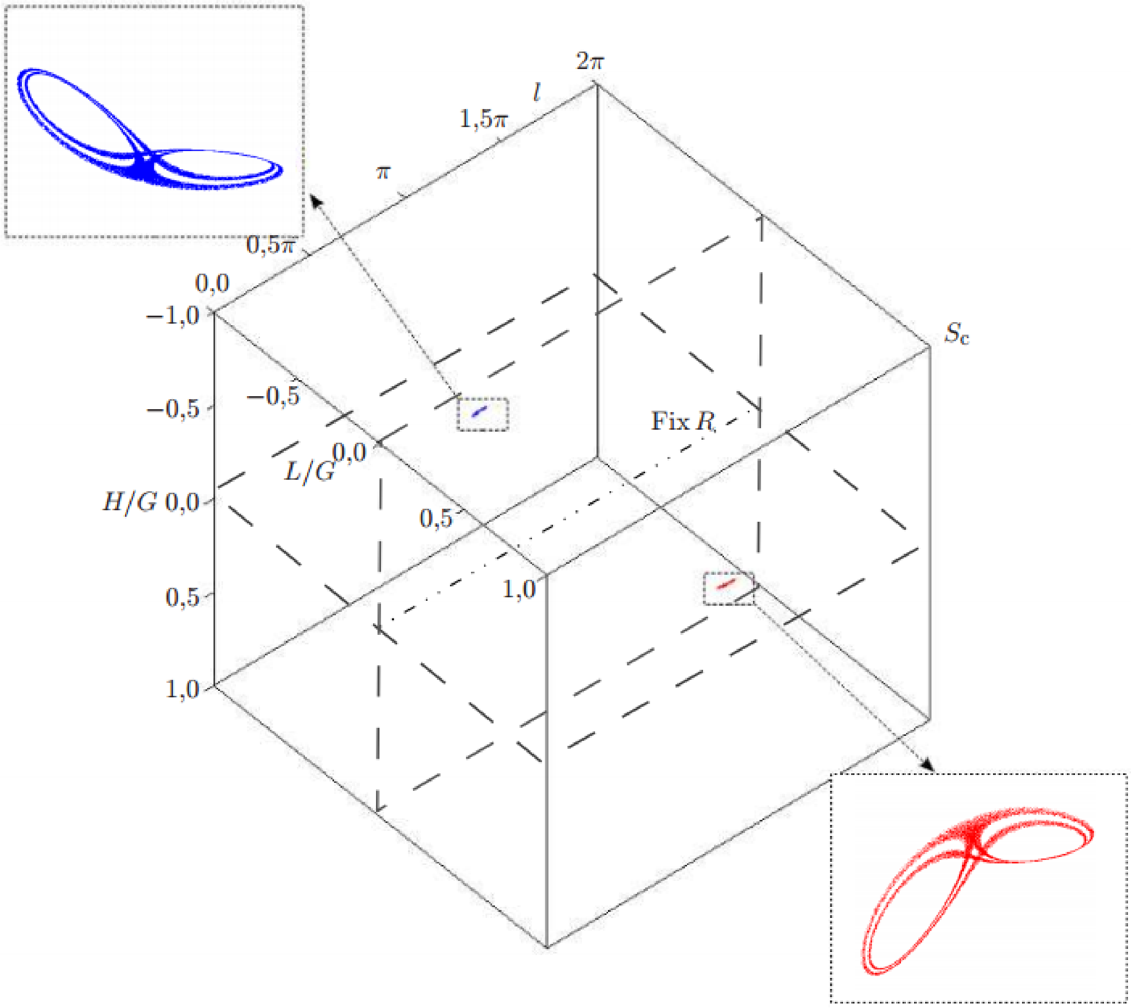


Fig. 6: Phase portraits of discrete Lorenz attractor (blue) and Lorenz repeller (red) for Poincaré map in nonholonomic model of Celtic stone (3) for $I_1 = 2, I_2 = 6, I_3 = 7$ and $(\delta, E) = (0.425, 743.3)$).

2.3 Pseudohyperbolic attractors of three-dimensional maps

Definition 1 *Recall that an attractor of a system is called pseudohyperbolic if:*

1. *in a neighborhood of the attractor there is a pair of continuous invariant linear subspaces E^{ss} (strongly contracting) and E^{cu} (centrally unstable) transversal to each other;*

2. *the differential of the system on the strongly contracting subspace E^{ss} exponentially contracts any direction, and any contraction in E^{ss} is stronger than any possible contraction in E^{cu};*

3. *the differential of the system on the centrally unstable subspace E^{cu} expands the volume exponentially.*

The 3rd chapter of this dissertation is devoted to the study of pseudohyperbolicity of some strange attractors of three-dimensional maps. In its first part, the problem of periodic perturbations of the Shimizu-Morioka system is considered

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - \lambda y - xz \\ \dot{z} = -\alpha z + x^2, \end{cases} \tag{7}$$

depending on the parameters α and λ. As is known, see for example [13, 14, 15, 16], this system has a Lorenz attractor, see also [17]. In the following, we will fix the system parameters as follows: $\alpha = 0.35, \lambda = 0.9$. At such values, the Lorenz attractor is observed in the system.

According to the Turaev-Shilnikov theorem, the pseudohyperbolicity of attractors is preserved under small periodic perturbations of systems of differential equations that demonstrate the pseudohyperbolicity of attractors [18]. Thus, if we take, for example, the Shimizu-Morioka system with the considered parameters α and λ, add a time-periodic perturbation to one of its equations, then for small values of this perturbation, this system will also have a pseudohyperbolic attractor .

In this section, on the example of the following periodic perturbation of the Shimizu-Morioka system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - \lambda y - xz \\ \dot{z} = -\alpha z + x^2 + \varepsilon z \sin t, \end{cases} \tag{8}$$

the author of the dissertation showed that the pseudohyperbolicity of the attractor is also preserved for large perturbations ε. It was found that the corresponding discrete Lorenz attractor is wild, i.e. admits the existence of homoclinic tangency.

In the dissertation work, all three conditions for determining pseudohyperbolicity for attractors of the (8) system are verified for various values of the perturbation ε. Spectrum of Lyapunov exponents for the attractor found for $\varepsilon = 0.02$:

$$\Lambda_1 = 0.04728, \Lambda_2 = -0.00111, \Lambda_3 = -1.29617.$$

Since $\Lambda_2 > \Lambda_3$ and $\Lambda_1 + \Lambda_2 > 0$, we can conclude that conditions 2 and 3 of the definition of pseudohyperbolicity are satisfied.

Diagrams for E^{ss} and E^{cu}, see Fig. 7c and 7d, show that the continuity conditions for these subspaces also hold here (condition 1 of the definition 1). Thus, we can conclude that this discrete Lorenz attractor is pseudohyperbolic.

Moreover, the attractor under consideration is wild, which can be seen from the numerical construction of long pieces of the unstable manifold [5*]. Behavior of the unstable separatrices Γ_1 and Γ_2 of the fixed point O, see Fig. 8, show the impossibility of avoiding homoclinic tangency of these separatrices with the two-dimensional unstable manifold of the point O.

The results of numerical studies show that the pseudohyperbolicity of the attractor breaks down approximately at $\varepsilon \approx 0.045$.

In the second part of 3rd Chapter of the dissertation work, the hypothesis of the pseudohyperbolicity of previously discovered attractors in a three-dimensional nonorientable Hénon map is verified

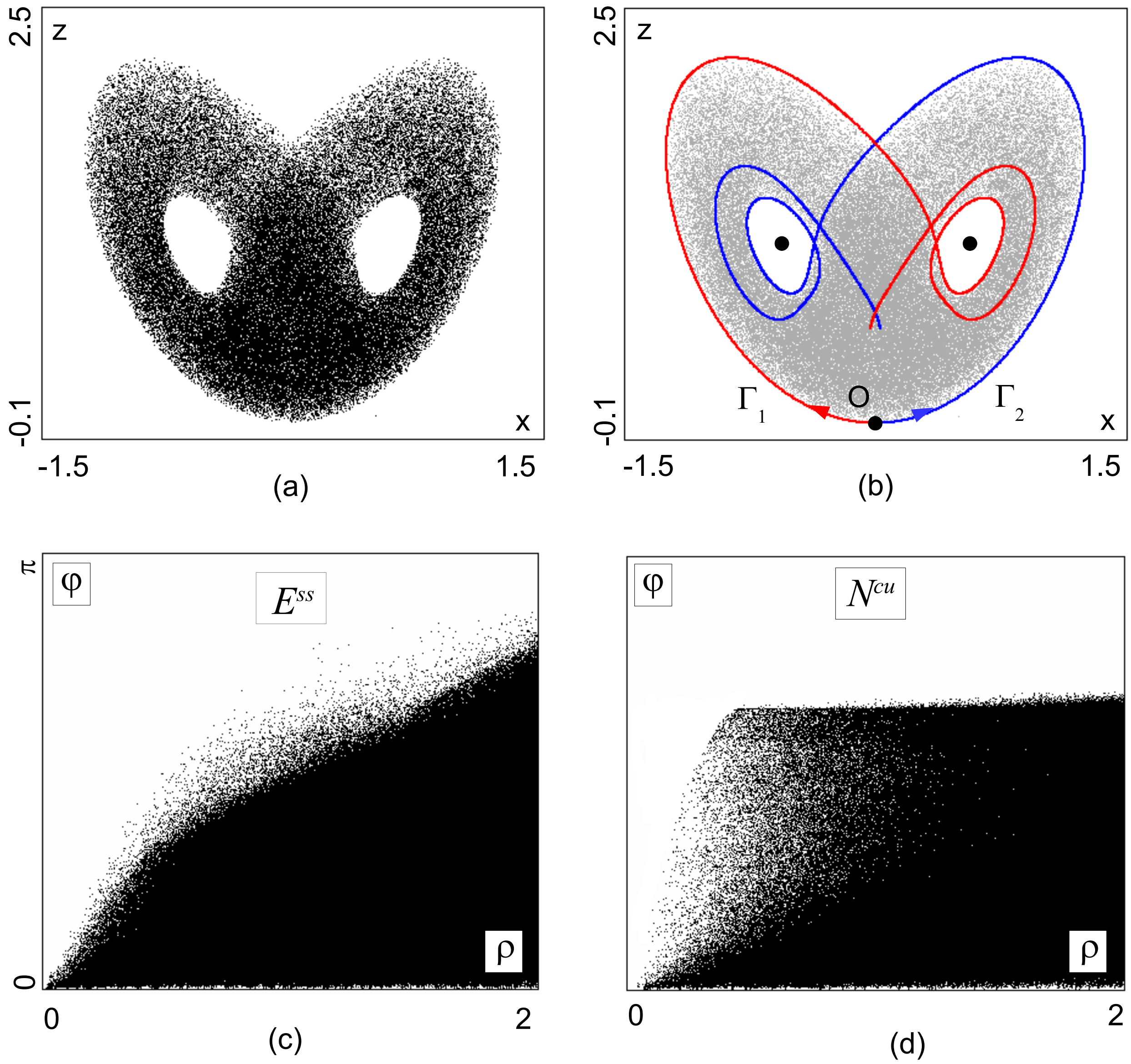


Fig. 7: Results of verification of the pseudohyperbolicity of the Lorenz attractor in the perturbed Shimizu-Morioka system (8) for parameters $\alpha = 0.35, \lambda = 0.9, \varepsilon = 0.02$: (a) Phase portraits of attractor; (b) behavior of unstable separatrices; (c) and (d) subspace continuity diagrams E^{ss} and N^{cu}.

$$\begin{cases} \bar{x} = y, \\ \bar{y} = z, \\ \bar{z} = Bx + Az + Cy + g(y, z). \end{cases} \tag{9}$$

This map depends on the parameters A, B, C (B is the Jacobian of the map), and the function $g(y, z)$ depending on the variables y and z and turning together with the first derivatives into zero for $y = z = 0$.

In the case of $B < 0$, the map (9) is reversing. Its bifurcations on the parameter plane (A, C) (for a fixed value of the Jacobian B) were studied in [19, 20] using the method of so-called chart of saddles, which is a partition of the parameter plane (A, C) by curves delimiting regions with different arrangements of multipliers of the fixed point $O(0, 0, 0)$ around which a homoclinic attractor is formed.

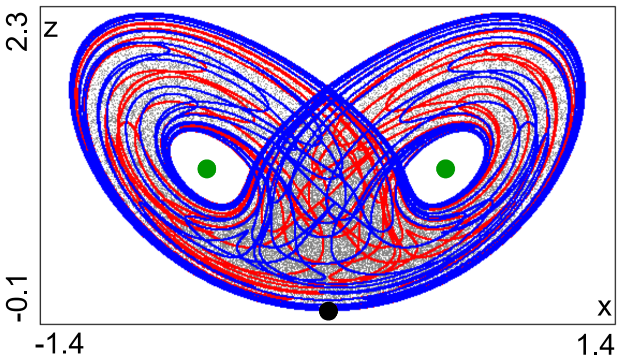


Fig. 8: Behavior of unstable separatrices Γ_1 and Γ_2, confirming the presence of homoclinic tangencies for the attractor shown in Fig. 7a.

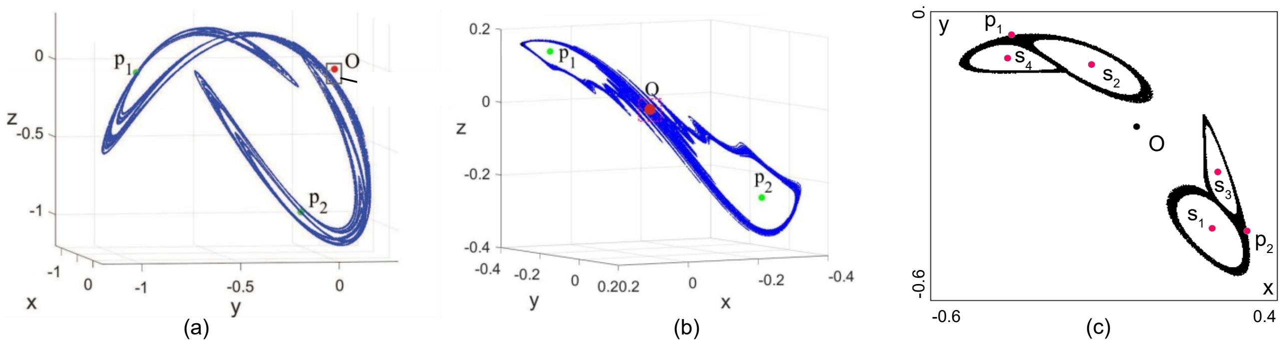


Fig. 9: Phase portraits for attractor in map (9), whose pseudohyperbolicity was verified: (a) nonorientable Lorenz attractor, $g(y,z) = y^2 + 1.5z^2 + 10yz + 2z^3$, $B = -0.5, A = -2.599$ and $C = -2.056$; (b) nonorientable figure-eight attractor, $g(y,z) = y^2 + 1.5z^2 + 10yz + 2z^3$, $B = -0.5, A = -2.599$ and $C = -2.056$; period-2 Lorenz attractor, $g(y,z) = -z^2$, $B = -0.8, A = -1.3072$ and $C = -1.05$.

In the papers [4*, 5*], in the class of nonorientable maps (9), new examples of homoclinic attractors were found: nonorientable Lorenz-like and figure-eight attractors, as well as the period-2 Lorenz attractor. Also, a hypothesis was stated that these attractors are pseudohyperbolic.

Numerical studies carried out in the dissertation work showed that the nonorientable Lorenz-like and figure-eight attractors previously found in [4*], see also Fig. 9b, c are not pseudohyperbolic, and the period-2 Lorenz attractor found in [5*], see also Fig. 9c, is pseudohyperbolic.

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