National Research University Higher School of Economics

Faculty of Mathematics

as a manuscript

Nikita Shulga

On irrationality measure functions and distribution of rational numbers

Summary of the PhD thesis
for the purpose of obtaining academic degree
Doctor of Philosophy in Mathematics

Academic supervisor
Doctor of sciences, Professor
Nikolay Moshchevitin

Moscow - 2023

Contents

Introduction

The present thesis is devoted to two objects in Number Theory - Minkowski question mark function and the irrationality measure function.

In 1904 on the International Congress of Mathematicians, H. Minkowski [10] introduced a function $?(x)$, which was later called Minkowski question mark function. This function was defined in a following manner. Let $?(0) = 0$ and $?(1) = 1$. Then for each pair of two consecutive rational numbers a/b and a'/b', on which the function is already defined, we define $?(x)$ at a point $x = (a+a')/(b+b')$ (which currently known as a mediant of the fractions a/b and a'/b') as an arithmetic mean of the values on the endpoints, i.e.

$$?\left(\frac{a+a'}{b+b'}\right) = \frac{1}{2}\left(?\left(\frac{a}{b}\right) + ?\left(\frac{a'}{b'}\right)\right).$$

Continuing this process, we will define the function on all rational numbers on the interval $[0,1]$ is we consider starting points 0 and 1 as $0/1$ and $1/1$ respectively. It can be extended on irrational numbers by continuity. According to Minkowski, this function has several great properties, namely it maps all rational numbers to dyadic rationals, i.e. rational numbers with denominator of the form $2^n, n \in \mathbb{N} \cup \{0\}$. Moreover, it maps all quadratic irrational numbers, i.e. irrational solutions of quadratic equations, to rational numbers.

In this thesis we mainly focus on two problems concerning the Minwkoski question mark function. First, we consider a folklore conjecture about fixed points of $?(x)$ function, i.e. the solutions of the equation $?(x) = x$. Conjecture states that there exist exactly 5 fixed points - three trivial ones: $0, 1/2, 1$ and two irrational ones, located symmetrically around $1/2$. To this date it is not known even if there are only finitely many fixed points. We managed to show that if the consider the smallest or the greatest fixed point x of the Minkowski question mark function on the interval $(0, 1/2)$ and expand it into a continued fraction $x = [a_1, \ldots, a_n, \ldots]$, then

$$a_1 = 2, \quad a_{n+1} \leq \sum_{i=1}^{n} a_i \quad \text{for all } n \in \mathbb{N}.$$

We also prove that some statement about values of $?(x)$ function in rational points will imply the conjecture about (irrational) fixed points of $?(x)$.

Secondly, we consider iterations of the Minkowski question mark function, i.e. the function

$$f_n(x) := \underbrace{?(?(...?}_{\text{n times}}(x))).$$

We provide an explicit set M of real numbers given by their continued fraction expansions, such that for every element $x_0 \in M$ and any $n \in \mathbb{N}$, one has

$$[f_n(x_0)]' = 0.$$

We note that for $n = 1$, the question of the derivative $[f_1(x)]' = [?(x)]'$ is well studied, especially in recent years, see [4, 5, 6, 8, 14] and others.

Second direction of the present thesis is improving results on the difference of two irrationality measure functions. First, recall that for an irrational number $\xi \in \mathbb{R}$ we consider the irrationality measure function

$$\psi_\xi(t) = \min_{1 \le q \le t,\, q \in \mathbb{Z}} \|q\xi\|,$$

where $||.||$ denotes distance to the nearest integer.

This function contains information on how does the given irrational number ξ is approximated by rational numbers.

In [9], Kan and Moshchevitin proved that for any two irrational numbers α, β satisfying $\alpha \pm \beta \notin \mathbb{Z}$ the difference

$$\psi_\alpha(t) - \phi_\beta(t)$$

changes its sign infinitely many times as $t \to \infty$.

We will stop on one question raised by Moshchevitin in [11]. He proved the following theorem.

Theorem 1. *Let α and β be the two irrational numbers. If $\alpha \pm \beta \notin \mathbb{Z}$, then for every $T \geq 1$ there exists $t \geq T$ such that*

$$|\psi_\alpha(t) - \psi_\beta(t)| \geqslant K \cdot \min(\psi_\alpha(t), \psi_\beta(t)),$$

where $K = \sqrt{(\sqrt{5}+1)/2} - 1$. Constant K in the inequality above is optimal.

This theorem combined with classical Dirichlet's theorem allowed him to also improve Dubickas's result from [3], so that one has

Corollary 1. *Let α and β be the two irrational numbers. If $\alpha \pm \beta \notin \mathbb{Z}$, then for every $T \geq 1$ there exists $t \geq T$ such that*

$$\left|\frac{1}{\psi_\alpha(t)} - \frac{1}{\psi_\beta(t)}\right| \geqslant Kt, \quad \text{with } K \text{ defined above.}$$

In this thesis we generalise Theorem 1 and improve Corollary 1 to an inequality with optimal constant.

Personal contribution

All of the main results presented above were obtained by the author. In co-authored works, the author obtained more than 50% of the results. In Theorem 5 and 11 from the papers [7] and [13], the author's contribution is equal to the contribution of the co-authors.

Approbation of the results of the dissertation research

The results of this PhD thesis were presented at the following conferences:

1. Diophantine Analysis, Dynamics and Related Topics, Technion, Haifa, Israel, 5-10 February 2023
   Talk: «Minkowski question mark and folding lemma.»

2. 66th Annual Meeting of the Australian Mathematical Society, UNSW, Sydney, Australia, 6-9 December 2022
   Talk: «Rational approximations to two irrational numbers.»

3. Number Theory Down Under 10, The University of Queensland, Brisbane, Australia, 4-7 September 2022
   Talk: «Diophantine properties of fixed points and derivative of iterations of Minkowski question mark function.»

4. The conference of World-class International Mathematical Centers in Sirius, Sochi, Russia, 9-13 August 2021
   Talk: «Rational approximations to two irrational numbers.»

5. Diophantine Analysis and Related Topics, ZOOM, Moscow, Russia, 1-4 June 2021
   Talk: «Rational approximations to two irrational numbers.»

6. Transcendence and Diophantine Problems, MIPT, Moscow, Russia, 10-14 June 2019
   Talk: «Diophantine properties of fixed points of Minkowski question mark function»

Publications

The main results of the thesis are presented in 4 papers:

- *Difference of irrationality measure functions.* Accepted in **Monatshefte für Mathematik**. (2023) (with V. Rudykh).

- *Rational approximations to two irrational numbers.* **Moscow Journal of Combinatorics and Number Theory** Vol. 11 (2022), No. 1, 1–10.
- *On the derivative of iterations of the Minkowski question mark function at special points.* **Funct. Approx. Comment. Math.** 66(2): 191-202, (2022).
- *Diophantine properties of fixed points of Minkowski question mark function.* **Acta Arithmetica**. 195 (2020): 367-382. (with D. Gayfulin).

Chapter 1

Minkowski question mark function and its fixed points

For $x \in [0,1]$ we consider its continued fraction expansion

$$x = [a_1, a_2, \ldots, a_n, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}, \quad a_j \in \mathbb{Z}_+$$

which is unique and infinite when $x \notin \mathbb{Q}$ and finite for rational x. Each rational x has just two representations

$$x = [a_1, a_2, \ldots, a_{n-1}, a_n] \quad \text{and} \quad x = [a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1], \quad \text{where} \quad a_n \geq 2.$$

One of the equivalent definitions of $?(x)$ is the following formula due to Denjoy [1, 2]

Definition 1. *For irrational $x = [a_1, a_2, \ldots, a_n, \ldots]$ we define*

$$?(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^{a_1 + \ldots + a_k - 1}}$$

A folklore conjecture states that

Conjecture 1. *The Minkowski question mark function $?(x)$ has exactly five fixed points. There is only one irrational fixed point of $?(x)$ in the interval $(0, \frac{1}{2})$.*

Our computations show that if there is more then one fixed point in the interval $(0, \frac{1}{2})$, then the first 5400 partial quotients in the continued fraction expansion of these numbers coincide.

In the present thesis I establish some properties of the continued fraction expansion of certain fixed points of $?(x)$. The next four theorems are the main results about the fixed points of Minkowski question mark function.

Theorem 2. *Let $x = [a_1, \ldots, a_n, \ldots]$ be the smallest or the greatest fixed point of the Minkowski question mark function on the interval $(0, \frac{1}{2})$. Then $a_1 = 2$ and*

$$a_{n+1} \leq \sum_{i=1}^{n} a_i$$

for all $n \in \mathbb{N}$.

The following theorem is a stronger version of Theorem 2. It uses some new geometrical considerations.

Theorem 3. *Denote $\kappa_1 = 2\log_2(\frac{\sqrt{5}+1}{2}) - 1 \approx 0.38848383\ldots$. Let x be fixed point from Theorem 2, then*

$$a_{n+1} < \kappa_1 \sum_{i=1}^{n} a_i + 2\log_2\left(\sum_{i=1}^{n} a_i\right) \tag{1.1}$$

for all $n \geq 1$.

Formula (1.1) gives an explicit irrationality measure lower estimate for the fixed points under considerations.

Theorem 4. *Let x be fixed point from Theorem 2, then there exists $q_0 \in \mathbb{Z}_+$ such that*

$$\left| x - \frac{p}{q} \right| > \frac{1}{\left(\frac{2\kappa_1}{\log 2} \log q + O(\log\log q)\right) q^2}$$

for all $q > q_0 \in \mathbb{N}, p \in \mathbb{N}$.

The following statement reduces the problem of fixed points of $?(x)$ to the properties of values of $?(x)$ at rational points only.

Theorem 5. *Conjecture 1 follows from the inequality*

$$\left| ?\left(\frac{p}{q}\right) - \frac{p}{q} \right| > \frac{1}{2q^2}$$

for all $p, q \in \mathbb{Z}_+$ with $q \geq q_0$ for some $q_0 \in \mathbb{Z}_+$.

The key ingredient for some proofs is the following statement, which in different formulation is known as "Folding lemma" (see [12]).

Lemma 1. *Let s be an arbitrary nonnegative integer and*

$$?([a_1, a_2, \ldots, a_{n-1}]) = [b_1, b_2, \ldots, b_k], \quad b_k \neq 1.$$

Consider the number

$$\theta = [a_1, a_2, \ldots, a_{n-1}, a_n], \text{ where } a_n = \sum_{i=1}^{n-1} a_i + s, \quad s \geq 0.$$

Then

1. *If $n \equiv k (mod\ 2)$, then $?(\theta) = [b_1, b_2, \ldots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \ldots, b_1]$.*
2. *If $n \equiv k + 1 (mod\ 2)$ then $?(\theta) = [b_1, b_2, \ldots, b_k, 2^{s+1} - 1, 1, b_k - 1, b_{k-1}, \ldots, b_1]$.*

Chapter 2

Derivative of iterations of $?(x)$

In this manuscript we will also consider nth iteration of the $?(x)$ function, i.e. the function

Definition 2.

$$f_n(x) := \underbrace{?(?(...?}_{\text{n times}}(x))).$$

Iterations of Minkowski question mark function turned out to be important for studying fixed points of $?(x)$, that is the solutions of the equation

$$?(x) = x.$$

Proposition 1. *If for every rational number $p/q \in [0,1]$ one has*

$$\lim_{n\to\infty} f_n\left(\frac{p}{q}\right) = A, \text{ where } A \in \{0, \frac{1}{2}, 1\}, \tag{2.1}$$

then there exist exactly five fixed points of $?(x)$ in the interval $[0,1]$.

In fact, the opposite is also true: if the conjecture about fixed points is true, then (2.1) holds for every rational p/q.
We will study iterations of $?(x)$ from a different perspective. Notice that for almost all $x_0 \in [0,1]$ with respect to Lebesgue measure, one has

$$f_n'(x_0) = ?'(x_0) \cdot ?'(x)|_{x=?(x_0)} \cdot ?'(x)|_{x=?(?(x_0))} \cdot \ldots \cdot ?'(x)|_{x=\underbrace{?(?(...?}_{\text{n-1 times}}(x_0)))}. \tag{2.2}$$

Apart from obvious cases, it seems to be non-trivial to find explicit examples of x_0 such that $f_n'(x_0) = 0$. By obvious examples we mean all rational numbers (since (2.2) because of $?'(z) = 0$ and $?(z) \in \mathbb{Q}$ for any $z \in \mathbb{Q}$), as well as some quadratic irrationals $x_0 = [a_1, \ldots, a_t, \ldots]$ satisfying

$$\liminf_{t\to\infty} \frac{a_1 + ... + a_t}{t} > \kappa_2 = 4.401^+, \tag{2.3}$$

where for κ_2 the exact formula

$$\kappa_2 = \frac{4L_5 - 5L_4}{L_5 - L_4}, \quad L_j = \log \frac{j + \sqrt{j^2+4}}{2} - j \cdot \frac{\log 2}{2}, \quad j = 4, 5$$

was discovered in [5]. The explanation is that in [5] the following theorem was proved.

Theorem 6. *For any irrational $x_0 = [a_1, \ldots, a_n, \ldots]$ that satisfies inequality (2.3), the derivative $?'(x_0)$ exists and $?'(x_0) = 0$.*

From Theorem 6 we see that if x_0 is a quadratic irrationality and formula (2.3) is satisfied then simultaneously $?(x_0) \in \mathbb{Q}$ and $?'(x_0) = 0$ and so by (2.2) we have $f_n'(x_0) = 0$.

In the present thesis we introduce a set M of irrational numbers, such that for all $x_0 \in M$ and for all $n \in \mathbb{Z}_+$ one has $f_n'(x_0) = 0$.

In order to properly formulate our result, we need to introduce some notation first.

Let $A = (a_1, a_2, \ldots, a_k)$ be a sequence of positive integers of arbitrary length $k \geq 0$. When $k = 0$ we suppose that A is an empty sequence. For a non-empty sequence A we consider the corresponding continued fraction

$$[A] = [a_1, a_2, \ldots, a_k] = \cfrac{1}{a_1 + \frac{1}{a_2 + \ldots + \frac{1}{a_k}}}.$$

For the sequence $A = (a_1, a_2, \ldots, a_k)$ we denote by

$$S_A = a_1 + a_2 + \ldots + a_k$$

the sum of all its elements.
Given a sequence $A = (a_1, a_2, \ldots, a_k)$ we write $d(A) = d(a_1, a_2, \ldots, a_k) = k$ for its length.
We denote by

$$(A, b) = (a_1, \ldots, a_k, b)$$

a sequence, obtained by concatenating sequence A with an element $b \in \mathbb{Z}_+$.
Now let us consider a set of sequences $A_i = (a_1^i, a_2^i, \ldots, a_{d(A_i)}^i)$.
By

$$(A_1, A_2) = (a_1^1, a_2^1, \ldots, a_{d(A_1)}^1, a_1^2, a_2^2, \ldots, a_{d(A_2)}^2)$$

we mean the sequence obtained by concatenating sequences A_1 and A_2.
For the continued fraction of the form $x_0 = [A_1, \tau_1, A_2, \tau_2, \ldots]$, where $\tau_i \in \mathbb{Z}_+$, we use the following notation for the sum of all partial quotients up to A_k.

$$\sigma_{A_k} = S_{A_1} + \tau_1 + S_{A_2} + \ldots + \tau_{k-1} + S_{A_k}.$$

In the present thesis we consider a special set M of irrational numbers. We take arbitrary sequences $A_i = (a_1^i, a_2^i, \dots, a_{d(A_i)}^i)$ and consider continued fraction

$$x = [A_1, \tau_1, A_2, \tau_2, \dots] \tag{2.4}$$

such that

$$\text{for all } k \in \mathbb{Z}_+ \quad \tau_k = \sigma_{A_k} + s_k, \ \text{ where } s_k > (\kappa_2 - 1)S_{A_{k+1}} + \sigma_{A_k} + 2, \ s_k \in \mathbb{Z}_+. \tag{2.5}$$

Our set M consists of all x of the form (2.4) constructed by the procedure described, that is

$$M = \Big\{ x \text{ of the form (2.4) with } \tau_k \text{ satisfying (2.5)} \Big\}.$$

Our main result in regards to the iterations of $?(x)$ is the following statement.

Theorem 7. *The set M has the properties:*
(i) for all $x \in M$, we have

$$?'(x) = 0.$$

(ii) for all $x \in M$, we have

$$?(x) \in M.$$

One of the key observations for our argument is once again Lemma 1 and the next corollary of it together with basic facts from the theory of continued fractions.

Corollary 2. *Let*

$$?([a_1, a_2, \dots, a_{n-1}]) = [b_1, b_2, \dots, b_k], \ \ b_k \neq 1.$$

and $a_n = \sum\limits_{i=1}^{n-1} a_i + s$, where $s \in \mathbb{Z}_+$.
Consider the number $\gamma = [a_1, \dots, a_n, c_1, \dots, c_s]$. Then there exists a sequence of positive integers $(b_{k+2}, \dots, b_p)$ for which

1. *If $n \equiv k (mod\ 2)$, then $?(\gamma) = [b_1, \dots, b_{k-1}, b_k - 1, 1, z, b_{k+2}, \dots, b_p]$.*

2. *If $n \equiv k + 1 (mod\ 2)$, then $?(\gamma) = [b_1, \dots, b_k, z, b_{k+2}, \dots, b_p]$.*

where

$$2^{s+1} - 1 \leq z \leq 2^{s+2} - 1.$$

Chapter 3

Irrationality measure functions

Definition 3. *For an irrational number $\xi \in \mathbb{R}$ we define the irrationality measure function*

$$\psi_\xi(t) = \min_{1 \le q \le t,\, q \in \mathbb{Z}} \|q\xi\|,$$

where $||.||$ denotes distance to the nearest integer.

It is clear for any $\xi \in \mathbb{R}$ and for every $t \ge 1$ we have

$$\psi_\xi(t) \le \frac{1}{t}. \tag{3.1}$$

In [9], Kan and Moshchevitin proved that for any two irrational numbers α, β satisfying $\alpha \pm \beta \notin \mathbb{Z}$ the difference

$$\psi_\alpha(t) - \phi_\beta(t)$$

changes its sign infinitely many times as $t \to \infty$.

Obviously, it follows that the difference of their reciprocals

$$R(t) = R_{\alpha,\beta}(t) = \frac{1}{\psi_\beta(t)} - \frac{1}{\psi_\alpha(t)}$$

also changes sign infinitely many times as $t \to \infty$.

Dubickas in [3] proved the following result.

Theorem 8. *For any irrational numbers α, β satisfying $\alpha \pm \beta \notin \mathbb{Z}$ the sequence $R(n), n \in \mathbb{Z}_+$ is unbounded.*

To formulate further results, we need the constants $\tau = \frac{\sqrt{5}+1}{2}, \phi = \frac{\sqrt{5}-1}{2}$,

$$K = \sqrt{\tau} - 1 = 0.2720^+, \tag{3.2}$$

$$C = K(\sqrt{\tau} + \tau^{-3/2}) = \sqrt{5}(1 - \sqrt{\phi}) = 0.47818^+. \tag{3.3}$$

In 2019 Moshchevitin [11] proved the following

Theorem 9. *Let α and β be the two irrational numbers. If $\alpha \pm \beta \notin \mathbb{Z}$, then for every $T \geq 1$ there exists $t \geq T$ such that*

$$|\psi_\alpha(t) - \psi_\beta(t)| \geqslant K \cdot \min(\psi_\alpha(t), \psi_\beta(t)). \tag{3.4}$$

It was also proved that constant K in (9') is optimal.
Using Theorem 9 and (3.1) Moshchevitin deduced a stronger version of Theorem 8.

Corollary 3. *Let α and β be the two irrational numbers. If $\alpha \pm \beta \notin \mathbb{Z}$, then for every $T \geq 1$ there exists $t \geq T$ such*

$$\Big|\frac{1}{\psi_\alpha(t)} - \frac{1}{\psi_\beta(t)}\Big| \geqslant Kt, \quad \text{with } K \text{ from (3.2).}$$

In this manuscript we give an improvement of this result to one with the best possible constant.

Theorem 10. *1) Let α and β be the two irrational numbers. If $\alpha \pm \beta \notin \mathbb{Z}$, then for every $T \geq 1$ there exists $t \geq T$ such that*

$$\Big|\frac{1}{\psi_\alpha(t)} - \frac{1}{\psi_\beta(t)}\Big| \geqslant Ct, \quad \text{where } C \text{ is defined in (3.3).} \tag{3.5}$$

2) The constant C in (3.5) is optimal, that is for any $\varepsilon > 0$, there exist α and β with $\alpha \pm \beta \notin \mathbb{Z}$ such that

$$\Big|\frac{1}{\psi_\alpha(t)} - \frac{1}{\psi_\beta(t)}\Big| \leqslant (C + \varepsilon)t$$

for all sufficiently large t.

Definition 4. *For a pair of irrational numbers α, β satisfying $\alpha \pm \beta \notin \mathbb{Z}$, define $C_{\alpha,\beta}$ as*

$$C_{\alpha,\beta} = \sup\{C : |\psi_\alpha(t) - \psi_\beta(t)| \geqslant C \cdot \min(\psi_\alpha(t), \psi_\beta(t)) \ \text{ for infinitely many } t \in \mathbb{N}\}.$$

Using this notation, Theorem 9 can be reformulated as follows.

Theorem 9'. *Let α and β be the two irrational numbers satisfying $\alpha \pm \beta \notin \mathbb{Z}$. Then*

$$C_{\alpha,\beta} \geq C_1,$$

where $C_1 = \sqrt{\frac{\sqrt{5}+1}{2}} - 1 \approx 0.272^+$.

The main result of is

Theorem 11. *Let α and β be the two irrational numbers satisfying $\alpha \pm \beta \notin \mathbb{Z}$. If at least one of the numbers α or β is not equivalent to $\tau = \frac{\sqrt{5}+1}{2}$, then*

$$C_{\alpha,\beta} \geq C_2,$$

where $C_2 = \sqrt{\sqrt{2}+1} - 1 \approx 0.5537^+$.

We also prove that the constant C_2 in the previous theorem cannot be improved.

Theorem 12. *The constant C_2 in Theorem 11 is optimal in a sense that there exist two irrational numbers θ and ω, such that $\theta \pm \omega \notin \mathbb{Z}$, at least of them is not equivalent to τ and*

$$C_{\theta,\omega} = C_2.$$

In the proof we exploit the language of combinatorics on words. We present the main construction, necessary for the proof of all main results.

Consider a union $U = D_\alpha \cup D_\beta$ of two sequences $D_\alpha = \{q_0 = 1 \leq q_1 < q_2 < \ldots\}$ and $D_\beta = \{t_0 = 1 \leq t_1 < t_2 < \ldots\}$ of denominators of convergents of α and β. We construct an infinite word W on an alphabet $\{B_*^*, Q^*, T_*\}$, where $*$ will match some indices of elements from D_α and D_β, by the following procedure. The first element of U is $1 = q_0 = t_0$. It belongs to both sequences, so we start our infinite word with B_0^0. Then, we are looking at the next element of U. There are three potential situations:

1. If the next element of U is q_j from the set D_α and it does not belong to the set D_β, we write the letter Q^j;

2. If the next element of U is t_i from the set D_β and it does not belong to the set D_α, we write the letter T_i;

3. If the next element of U is equal to q_j from D_α and also to t_i from D_β, we write the letter B_i^j.

This construction allows us to analyse mutual arrangement of denominators of convergents for two numbers, hence giving us an opportunity to find necessary large distances between two functions.

Bibliography

[1] A. Denjoy. Sur une fonction de Minkowski. *J. Math. Pures Appl.*

[2] A. Denjoy. Sur une fonction réelle de Minkowski. *C. R. Acad. Sci. Paris*, 1932.

[3] A. Dubickas. On rational approximations to two irrational numbers. *J. Number Theory*, 177:43–59, 2017.

[4] A. A. Dushistova, I. D. Kan, and N. G. Moshchevitin. Differentiability of the Minkowski question mark function. *J. Math. Anal. Appl.*, 401(2):774–794, 2013.

[5] A. A. Dushistova and N. G. Moshchevitin. On the derivative of the Minkowski $?(x)$ function. *Fundam. Prikl. Mat.*, 16(6):33–44, 2010.

[6] D. R. Gaifulin and I. D. Kan. The derivative of the Minkowski function. *Izv. Ross. Akad. Nauk Ser. Mat.*, 85(4):5–52, 2021.

[7] D. Gayfulin and N. Shulga. Diophantine properties of fixed points of Minkowski question mark function. *Acta Arith.*, 195(4):367–382, 2020.

[8] D. R. Gayfulin and I. D. Kan. Stationary points of the Minkowski function. *Mat. Sb.*, 212(10):3–15, 2021.

[9] I. D. Kan and N. G. Moshchevitin. Approximations to two real numbers. *Unif. Distrib. Theory*, 5(2):79–86, 2010.

[10] H. Minkowski. Zur Geometrie der Zahlen. *Verhandlungen des III Internationalen Mathematiker-Kongresous*, pages 164–173, 1904.

[11] N. Moshchevitin. Über die Funktionen des Irrationalitätsmaßes für zwei irrationale Zahlen. *Arch. Math. (Basel)*, 112(2):161–168, 2019.

[12] A. J. van der Poorten and J. Shallit. Folded continued fractions. *J. Number Theory*, 40(2):237–250, 1992.

[13] V. Rudykh, N. Shulga. Difference of irrationality measure functions. *Monatsh Math* (2023).

[14] P. Viader, J. Paradís, and L. Bibiloni. A new light on Minkowski's $?(x)$ function. *J. Number Theory*, 73(2):212–227, 1998.