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Topological and Homotopy Classification of Morse-Smale Diffeomorphisms on Surfaces

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Introduction

In 1937 A.A. Andronov and L.S. Pontryagin [1] introduced the concept of a *rough system* of differential equations defined in a limited part of the plane, as a system that does not change its qualitative properties with small changes in the right-hand sides, and indicated the necessary and sufficient conditions for the system to be rough. The most natural generalization of these conditions identifies the class of dynamic systems (flows and cascades) called *Morse-Smale*. Morse-Smale systems received their name after the publication of the paper [47] in 1960. In this paper S. Smale proved the fulfillment of relations similar to Morse inequalities for flows on n-manifolds whose nonwandering set is hyperbolic and consists of a finite number of fixed points and periodic trajectories. It is well known [38], [39]that Morse-Smale dynamical systems on manifolds of any dimension are indeed rough (structurally stable), but are not typical, with the exception of Morse-Smale flows on surfaces [1], [40] and Morse-Smale cascades on the circle [27].

One of the first questions that arises when studying a dynamic system is the question of the behavior of its trajectories and the possibility of qualitatively, up to topological equivalence (conjugacy), distinguishing this behavior from the behavior of the trajectories of another system. The solution to these problems constitutes a *topological classification* of dynamic systems and consists, firstly, in isolating information about the system, unambiguously defining its class of topological equivalence (conjugacy) and called *a complete topological invariant*, and, secondly, in *realization* - the construction, based on the selected information, of a standard representative in each class of topological equivalence (conjugacy). The feasibility of realization also allows one to model systems with specified properties.

For example, the equivalence class of a Morse-Smale flow on a circle is uniquely determined by the number of its fixed points. For cascades on a circle, a complete topological invariant was obtained by A.G. Mayer [27] in 1939 and consists of the number of periodic orbits and the Poincare rotation number. In 1955 E.A. Leontovich and A.G. Mayer [24] introduced a flow scheme with a finite number of singular trajectories on a two-dimensional sphere as a complete topological invariant. In 1971, M. Peixoto formalized the concept of a Leontovich-Mayer scheme and proved that for a flow on an arbitrary surface, the complete topological invariant is the isomorphism class of an orientable graph, the vertices of which are in one-to-one correspondence with equilibrium states and closed trajectories, and the edges correspond to some connected components of invariant manifolds of equilibrium states and closed trajectories, while the isomorphism of the graphs includes the preservation of specially selected subgraphs[1].

When passing from flows to Morse-Smale surface diffeomorphisms, a new type of trajectories appears, called *heteroclinic* and belonging to the intersection of $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ invariant

[1]In the work [37] A.A. Oshemkova and V.V. Charcot noticed the inaccuracy of the Peixoto invariant due to the fact that the graph isomorphism does not distinguish between an inequivalent bundle on the trajectories of regions bounded by two periodic orbits.

manifolds of different saddle points. Moreover, the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is a countable set and each point of this set is called *heteroclinic point*, and the orbit heteroclinic point is called *heteroclinic orbit*.

For any heteroclinic point $x \in W^s_{\sigma_1} \cap W^u_{\sigma_2}$ we define an ordered pair of vectors $(\vec{v}^u_x, \vec{v}^s_x)$, where:

- $\vec{v}^u_x$ — is the tangent vector to the unstable manifold of the point σ_2 at the point x and directed from x to $f^{m^u}(x)$, where m^u — is a period of connected component of the set $W^u_{\sigma_2} \setminus \{\sigma_2\}$ contains x;

- $\vec{v}^s_x$ — is the tangent vector to the unstable manifold of the point σ_1 at the point x and directed from x to $f^{m^s}(x)$, where m^s — is the tangent vector to the unstable manifold of the point $W^s_{\sigma_1} \setminus \{\sigma_1\}$ contains x.

A heteroclinic intersection of a Morse-Smale diffeomorphism $f : M^2 \to M^2$ is called *orientable* if ordered pairs of vectors $(\vec{v}^u_x, \vec{v}^s_x)$ specify the same orientation of the load-bearing surface M^2. Otherwise, the heteroclinic intersection is called *non-orientable.*

A diffeomorphism f is called *gradient-like* if $W^s_{\sigma_1} \cap W^u_{\sigma_2} = \varnothing$ for any different saddle points $\sigma_1, \sigma_2 \in \Omega_f$.

To obtain a complete classification of gradient-like diffeomorphisms on surfaces, in 1985 A.N. Bezdenezhnykh and V.Z. Grines [4, 5, 6, 15] introduced *equiped graphs* similar to M. Peixoto graphs [41] for gradient-like flows. The equipment included information about the boundaries of diffeomorphism cells. A modified version of the graph is described in the work [18]. Also, similar to the approach of A.A. Oshemkova, V.V. Sharko [37] for gradient-like flows V.Z. Grines, S.H. Kapkaeva and O.V. Repair [16] described a three-color graph as an invariant for gradient-like cascades.

The next logical step in the classification of Morse-Smale diffeomorphisms was the classification of orientation-preserving diffeomorphisms on orientable surfaces with a finite set of heteroclinic orbits. In 1993 V.Z. Grines [15], based on the Peixoto graph, described a topological invariant for Morse-Smale diffeomorphisms with a finite number of heteroclinic orbits, exhausted by the orbits of the intersection points of connected fundamental segments of separatrices. The given invariant was a graph equipped with some heteroclinic substitution describing the pattern of intersection of invariant manifolds.

A little later, R. Langevin [23] proposed considering the space of orbits of the snik basin and the projections of unstable separatrices of saddle points onto the resulting space of orbits. This approach was generalized and successfully applied by Chr. Bonatti, V.Z. Grines, V.V. Medvedev, E. Pecu, F. Laudenbach and O.V. Pochinka in the works [8, 7, 18, 43, 44] for the topological classification of Morse-Smale diffeomorphisms on 3-manifolds. Langevin's approach was also used in the two-dimensional case. So, in 2010 T.M. Mitryakova and O.V. Pochinka [30] applied it to the topological classification of diffeomorphisms of surfaces with a finite number of orbits of heteroclinic tangency. They constructed a topological invariant, called a *scheme*, consisting of a finite number of two-dimensional tori with a set of simple closed curves. They also proved that this invariant is complete for the class of diffeomorphisms under consideration with a non-wandering set consisting of a finite number of fixed points, and in the work [29] they solved the problem of their realization according to an admissible *abstract scheme*.

In 1998, a different approach was used by Chr. Bonatti and R. Langvin [9], considering Smale diffeomorphisms of compact surfaces - structurally stable diffeomorphisms with zero-dimensional basis sets. They proved that each Smale diffeomorphism corresponds to a finite combinatorial object, which is a set of geometric types of Markov partitions. Morse-Smale diffeomorphisms are a special case of Smale diffeomorphisms; however, they were not singled out for separate consideration, and therefore such a classification turned out to be unreasonably laborious for them.

Section 1 provides an overview of the classification results available to date for non-gradient-like Morse-Smale diffeomorphisms on surfaces. This shows that, even in the case of a finite number of heteroclinic orbits for Morse-Smale diffeomorphisms on surfaces, there are no comprehensive classification results. The main problem is that all currently known complete topological invariants of such systems have not been brought to the realization stage. In Section 4 of this paper, we obtain a complete topological classification of orientation-preserving Morse-Smale diffeomorphisms with a finite number of heteroclinic orbits on orientable surfaces, including realization. The invariant is a pair of schemes similar to the schemes introduced in the paper [30].

In Section 5.1 it is established that the orientability condition for a heteroclinic intersection of an orientation-preserving Morse-Smeil diffeomorphism on an orientable surface guarantees that the number of its heteroclinic orbits is finite. In addition, the fact that the intersection is orientable made it possible to establish for such diffeomorphisms in Section 5.2 the existence of combinatorial invariants—effectively distinguishable graphs. A set of admissible graphs is also identified, for each of which a Morse-Smale diffeomorphism is realized on the surface, which has an orientable heteroclinic intersection.

Sections 2 and 3 of this work are devoted to the connection between Morse-Smale surface diffeomorphisms and the Nielsen-Thurston homotopy theory. According to the *Nielsen-Thurston classification* (see, for example, [10], [2] or [18, p. 284]), the set of all homotopy classes of homeomorphisms of an orientable surface is represented as four disjoint sets T_1, T_2, T_3, T_4. A homotopy class from each subset is characterized by the existence in it of a homeomorphism called *Thurston's canonical form*, namely: 1) a periodic homeomorphism, 2) a reducible non-periodic homeomorphism of algebraically finite type, 3) a reducible homeomorphism that is not a homeomorphism of algebraically finite type, 4) pseudo-Anosov homeomorphism.

Thurston's canonical forms are not structurally stable diffeomorphisms, however, S.Kh. Aranson and V.Z. Grines [2] assumed that every homotopy class of a surface homeomorphism also contains a structurally stable diffeomorphism. For class T_1, confirmation of the Aranson-Grines hypothesis was obtained by A.N. Bezdenezhnykh and V.Z. Grines [6]. Namely, in each homotopy class from the set T_1 a gradient-like diffeomorphism is constructed and a complete topological classification of such diffeomorphisms [4, 5] is obtained. The article [48] announced the existence in each homotopy class T_4 of a structurally stable diffeomorphism, the non-wandering set of which consists of a finite number of source orbits and a single one-dimensional attractor. Necessary and sufficient conditions for the topological conjugacy of two such diffeomorphisms are found in [17]. Section 2 of this paper describes the realization of each homotopy class of type T_2 by a Morse-Smale diffeomorphism with an orientable heteroclinic intersection.

According to [2], any homeomorphism of homotopy type T_3 or T_4 has an infinite set of periodic points. Due to this, Morse-Smale surface diffeomorphisms cannot belong to homotopy classes of types T_3 and T_4. Moreover, all gradient-like diffeomorphisms belong to homotopy classes of type T_1. Whereas Morse-Smale diffeomorphisms with heteroclinic intersections can belong to homotopy classes of both types T_1 and T_2. In Section 3 of this work, an algorithm is constructed that allows one to determine the Nielsen-Thurston homotopy type for a non-gradient-like Morse-Smale diffeomorphism on a surface based on the signature of its heteroclinic intersection.

Aims and objectives of the research

The main aims of the thesis are to obtain new classification and implementation results for the class of orientation-preserving Morse-Smale diffeomorphisms.

The problems are divided into two directions: homotopy theory and topological classification of orientation-preserving maps of surface.

Within the framework of homotopy theory, this study solves the following problems:

- Construct a structurally stable representative in each homotopy class of the second Nielsen-Thurston type;

- Develop an algorithm to determine the Nielsen-Thurston homotopy type for an orientation-preserving Morse-Smale diffeomorphism of a surface.

Within the framework of the topological classification of orientation-preserving surface mappings, this study solves the following problems:

- Obtain a complete topological classification for Morse-Smale diffeomorphisms with a finite number of heteroclinic orbits, including realization;

- Prove that a Morse-Smale diffeomorphism with orientable heteroclinic intersections has a finite number of heteroclinic orbits;

- Obtain a combinatorial invariant for Morse-Smale diffeomorphisms with orientable heteroclinic intersections;

- Prove the existence of an effective algorithm for distinguishing the resulting combinatorial invariant.

Scientific novelty of the results

All results are new. Exactly:

- A realization of Morse-Smale diffeomorphisms with an orientable heteroclinic intersection in each homotopy class of the second Nielsen-Thurston type is obtained.
- An algorithm is obtained for recognizing whether a given non-gradient-like Morse-Smale diffeomorphism belongs to the first T_1 or second T_2 Nielsen-Thurston set by its heteroclinic intersection.
- A complete topological classification of the set of Morse-Smale diffeomorphisms with a finite number of heteroclinic orbits, including the realization, is obtained.
- It is proven that an orientation-preserving Morse-Smale diffeomorphism with orientable heteroclinic intersections has a finite number of heteroclinic orbits.
- A graph that is a complete topological invariant for the class of Morse-Smale diffeomorphisms with orientable heteroclinic intersections is constructed.
- The existence of an effective algorithm for establishing isomorphism of graphs is proven.

Theoretical and practical significance of the conducted research

In the course of the work, fundamental mathematical results in the field of dynamic systems were obtained. Many natural sciences use mathematical models based on Morse-Smale diffeomorphisms on 2-manifolds. Morse-Smale diffeomorphisms are of great practical importance in the field of data processing. For example, work [13] explores topological analysis and processing of large-scale distributed data generated by sensor networks. Algorithms built on the basis of the Morse-Smale decomposition have an advantage in performance when processing data. Also, the Morse-Smale complex provides a topologically significant decomposition of the region. The work of [14] implements a discrete approximation of the Morse-Smale complex, which allows better work with multidimensional data sets. Morse Smale diffeomorphisms can be used to describe the behavior of magnetic fields. For example, in the work [19] the properties of the Morse–Smale diffeomorphism are applied to solve the problem of the existence of separators in the magnetic field of a plasma.

Methodology and research methods

During the study, new invariants were developed to solve the problems. To determine the Nielsen-Thurston type for a Morse-Smale diffeomorphism, the heteroclinic intersection index ξ is introduced. A new idea of graph construction is used for Morse-Smale diffeomorphism with orientable heteroclinic intersections. Another original idea is to use a pair of schemes to classify Morse-Smale diffeomorphisms.

Among the classical methods of the theory of dynamical systems used in the work, the following can be distinguished: factorization, linearizing neighborhoods, knot theory, sequential construction of the conjugate mapping.

Provisions for defense

- The realiztion of a Morse-Smale diffeomorphism with an orientable heteroclinic intersection in each homotopy class $\{h\} \in T_2$ is described (Theorem 1).

- An algorithm is described for recognizing whether a given non-gradient-like diffeomorphism of class $MS(M^2)$ belongs to the Nielsen-Thurston set T_1 or T_2 by its heteroclinic intersection (Theorem 2).

- A complete topological classification of the set $MS_1(M^2)$ of Morse-Smale diffeomorphisms with a finite number of heteroclinic orbits ($beh(f) = 1$), including the realization (Theorems 3,4), is given.

- It is proven that an orientation-preserving Morse-Smale diffeomorphism with orientable heteroclinic intersections has a finite number of heteroclinic orbits (Theorem 5).

- The graph (T_f, P_f) has been developed, which is a complete topological invariant for diffeomorphisms of the class $MS_+(M^2)$ (Theorem 6).

- The existence of an effective algorithm for establishing isomorphism of graphs (T_f, P_f) is proven (Theorem 7).

Structure and scope of work

The dissertation consists of an introduction, five chapters, a conclusion and a list of references. The work contains 91 pages, including 44 drawings. The bibliography contains 72 titles.

Author's personal contribution

All results presented in the dissertation were obtained by the author independently. Scientific supervisor O.V. Pochinka is responsible for setting tasks and general management of the dissertation candidate's research activities in order to prepare for the defense of the dissertation. V.Z. Grines and D.S. Malyshev were consultants on graph theory issues.

1 The results

This work is devoted to the topological and homotopy classification of Morse-Smale diffeomorphisms f on orientable surfaces M^2; let $MS(M^2)$ denote the class of such diffeomorphisms.

Let $\mathcal{O}_i$, $\mathcal{O}_j$ be periodic orbits of the diffeomorphism $f \in MS(M^2)$. In the work [46] S. Smale introduced *partial order relations* $\prec$ for periodic orbits

$$\mathcal{O}_i \prec \mathcal{O}_j \iff W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j} \neq \varnothing.$$

For saddle orbits $\mathcal{O}_i \prec \mathcal{O}_j$, any non-empty intersection point $W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j}$ is called *heteroclinic*. Sequence of different saddle periodic orbits $\mathcal{O}_i = \mathcal{O}_{i_0}, \mathcal{O}_{i_1}, \dots, \mathcal{O}_{i_k} = \mathcal{O}_j$ $(k \geq 1)$, such that $\mathcal{O}_{i_0} \prec \mathcal{O}_{i_1} \prec \dots \prec \mathcal{O}_{i_k}$ is called *a chain of length k connecting periodic orbits $\mathcal{O}_i$ and $\mathcal{O}_j$*. The maximum length of the saddle chain connecting the orbit $\mathcal{O}_i$ and $\mathcal{O}_j$ is denoted as $beh(\mathcal{O}_j|\mathcal{O}_i)$ (*beh* from English *behavior* (relation)). Set $beh(\mathcal{O}_j|\mathcal{O}_i) = 0$ if $W^u_{\mathcal{O}_j} \cap W^s_{\mathcal{O}_i} = \varnothing$. Let

$$beh(f) = max\{beh(\mathcal{O}_j|\mathcal{O}_i)\}.$$

A diffeomorphism $f \in MS(M^2)$ is called *gradient-like* if $beh(f) = 0$.

We denote by $MS_1(M^2)$ the diffeomorphisms $f \in MS(M^2)$ with $beh(f) = 1$.

If $beh(f) > 0$, then for any heteroclinic point $x \in W^s_{\sigma_1} \cap W^u_{\sigma_2}$ of the diffeomorphism $f \in MS(M^2)$ we define an ordered pair of vectors $(\vec{v}^u_x, \vec{v}^s_x)$, where:

- $\vec{v}^u_x$ — is the tangent vector to the unstable manifold of the point σ_2 at the point x and directed from x to $f^{m^u}(x)$, where m^u — is a period of connected component of the set $W^u_{\sigma_2} \setminus \{\sigma_2\}$ contains x;

- $\vec{v}^s_x$ — is the tangent vector to the unstable manifold of the point σ_1 at the point x and directed from x to $f^{m^s}(x)$, where m^s — is the tangent vector to the unstable manifold of the point $W^s_{\sigma_1} \setminus \{\sigma_1\}$ contains x.

A heteroclinic intersection of a Morse-Smale diffeomorphism $f : M^2 \to M^2$ is called *orientable* if ordered pairs of vectors $(\vec{v}^u_x, \vec{v}^s_x)$ specify the same orientation of the load-bearing surface M^2. Otherwise, the heteroclinic intersection is called *non-orientable.*

Let $MS_+(M^2)$ denote the set of diffeomorphisms $f \in MS(M^2)$ with an orientable heteroclinic intersection.

Chapter 1 provides the necessary information from the theory of dynamical systems, as well as provides an overview of the results available on this topic.

Chapter 2 is devoted to the realization of homotopy classes of type T_2 according to Nielsen-Thurston by diffeomorphisms of the class $MS_+(M^2)$.

According to the *Nielsen-Thurston classification* (see, for example, [10], [2] or [18, p. 284]), the set of all homotopy classes of homeomorphisms of an orientable surface is represented as four disjoint sets T_1, T_2, T_3, T_4. A homotopy class from each subset is characterized by the existence in it of a homeomorphism called *Thurston's canonical form*, namely: 1) a periodic homeomorphism, 2) a reducible non-periodic homeomorphism of algebraically finite type, 3) a reducible homeomorphism that is not a homeomorphism of algebraically finite type, 4) pseudo-Anosov homeomorphism.

Thurston's canonical forms are not structurally stable diffeomorphisms, however, S.Kh. Aranson and V.Z. Grines [2] assumed that every homotopy class of a surface homeomorphism also contains a structurally stable diffeomorphism. For class T_1, confirmation of the Aranson-Grines hypothesis was obtained by A.N. Bezdenezhnykh and V.Z. Grines [6].

The main result of Chapter 2 is the following theorem.

Theorem 1 ([22]*, Theorem 1) *Each homotopy class $\{h\} \in T_2$ of homeomorphisms of the surface M^2 contains a diffeomorphism $f \in MS_+(M^2)$.*

According to [2], any homeomorphism of homotopy type T_3 or T_4 has an infinite set of periodic points. Due to this, Morse-Smale surface diffeomorphisms cannot belong to homotopy classes of types T_3 and T_4. Moreover, all gradient-like diffeomorphisms lie in homotopy classes of type T_1. Whereas Morse-Smale diffeomorphisms with heteroclinic intersections can belong to homotopy classes of both types T_1 and T_2.

In **Chapter 3** an algorithm for recognizing whether a given non-gradient-like diffeomorphism $f \in MS(M^2)$ belongs to the Nielsen-Thurston set T_1 or T_2 by its heteroclinic intersection is presented.

Let $f \in MS(M^2)$. Since $\{f\} \in T_i$ if and only if $\{f^k\} \in T_i$, $k \neq 0$ (see, [34, 35, 36]) . Then, without loss of generality, we can assume that the nonwandering set Ω_f of the diffeomorphism f consists of fixed points and f preserves orientation on its invariant manifolds. Then the set Σ^σ_f of saddle points of the diffeomorphism f can be ordered according to the Smale partial order relation [46] as follows:

$$\textit{if } W^u_{\sigma_j} \cap W^s_{\sigma_i} \neq \varnothing, \ \textit{then } i < j.$$

Denote numbering $\sigma_1, \dots, \sigma_{k_1}$ on the set Σ^σ_f accordingly to the Smale partial order relation.

Let us denote by Σ^ω_f, Σ^α_f the set of all sink and source points of the diffeomorphism f, respectively, and by k_ω, k_α is the number of points in the corresponding sets. Let

$$A_{f,0} = \Sigma^\omega_f, \ A_{f,i} = \Sigma^\omega_f \cup \bigcup_{j=1}^{i} W^u_{\sigma_j}, \ i = 1, \dots, k_1,$$

$$R_{f,i} = \Sigma^\alpha_f \cup \bigcup_{j=i+1}^{k_1} W^s_{\sigma_j}, \ i = 0, \dots, k_1 - 1, \ R_{f,k_1} = \Sigma^\alpha_f.$$

From [42], [20] it follows that each of the sets $A_{f,i}$ ($R_{f,i}$) is an *attractor (repeller)*, i.e. it has a *grabbing neighborhood* $M_{f,i}$ ($N_{f,i}$), which is a compact surface with boundary such that

$$f(M_{f,i}) \subset \operatorname{int} M_{f,i}, \quad \bigcap_{n\in\mathbb{N}} f^n(M_{f,i}) = A_{f,i}$$

$$\left(f^{-1}(N_{f,i}) \subset \operatorname{int} N_{f,i}, \bigcap_{n\in\mathbb{N}} f^{-n}(N_{f,i}) = R_{f,i}\right).$$

Moreover, the attractor $A_{f,i}$ and the repeller $R_{f,i}$ are *dual*, .e. the neighborhoods of $N_{f,i}$ and $M_{f,i}$ can be chosen in such a way that $N_{f,i} = M^2 \setminus \operatorname{int} M_{f,i}$ (see [42]). Since $A_{f,0} \subset A_{f,1} \subset \cdots \subset A_{f,k_1}$, then, by [42], the grabbing neighborhoods can be chosen so that:

- $M_{f,i} \subset f(M_{f,i+1})$, $i = 0, \dots, k_1 - 1$;
- $\partial M_{f,i}$ does not contain heteroclinic points;
- each connected component v of the set $K_{f,i} = M_{f,i} \setminus \operatorname{int} f(M_{f,i})$ is diffeomorphic to the two-dimensional anulus $[0,1]\times\mathbb{S}^1$ by means of some diffeomorphism $h_v : [0,1]\times\mathbb{S}^1 \to v$ such that $h_v^{-1} f h_v|_{\{0\}\times\mathbb{S}^1}(0,s) = (1,s)$, $s \in \mathbb{S}^1$ (see Fig. 1).

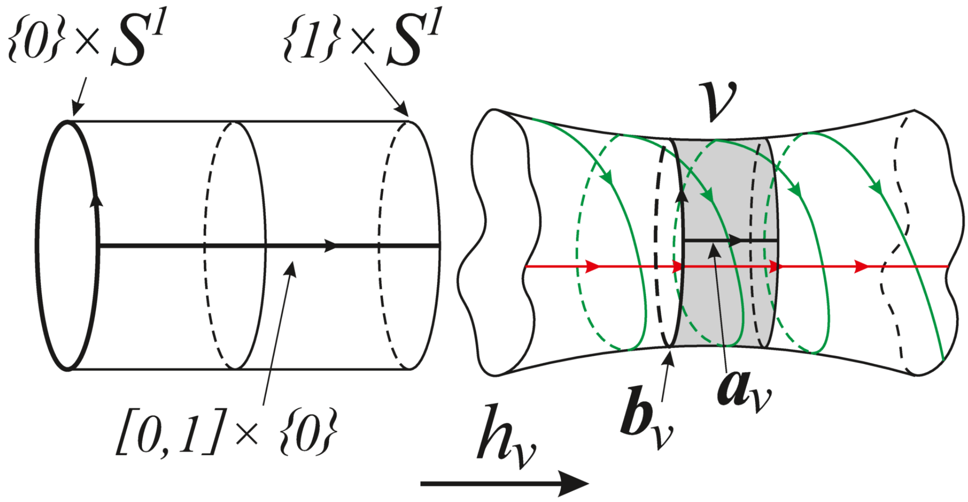

Figure 1: Action of diffeomorphism $h_v : [0,1] \times \mathbb{S}^1 \to v$.

Let $a_v = h_v([0,1] \times \{0\})$, $b_v = h_v(\{0\} \times \mathbb{S}^1)$. We orient the curve a_v along the direction on the segment $[0,1]$ from 0 to 1. Then the orbit space $\hat{v} = v/f$ is diffeomorphic to the two-dimensional torus and the natural projection $p_v : v \to \hat{v}$ induces on the torus $\hat{v}$ generators

$$\hat{a}_v = p_v(a_v),\ \hat{b}_v = p_v(b_v).$$

Let v be the connected component of the set $K_{f,i}$ and $C_v = \bigcup_{n\in\mathbb{Z}} f^n(v)$. Let $L^s_{f,v}$ ($L^u_{f,v}$) denote the union of stable (unstable) separatrices of saddle points σ_j, $j \leqslant i\,(j > i)$, completely contained in the set C_v (see Fig. 2).

Let $\hat{L}^s_{f,v} = p_v(L^s_{f,v})$, $\hat{L}^u_{f,v} = p_v(L^u_{f,v})$. Then each connected component of these sets is a knot on the torus $\hat{v}$ of homotopy type $\langle 1, d^s_{f,v}\rangle$, $\langle 1, d^u_{f,v}\rangle$ in the selected generators $\hat{a}_v$, $\hat{b}_v$ (see

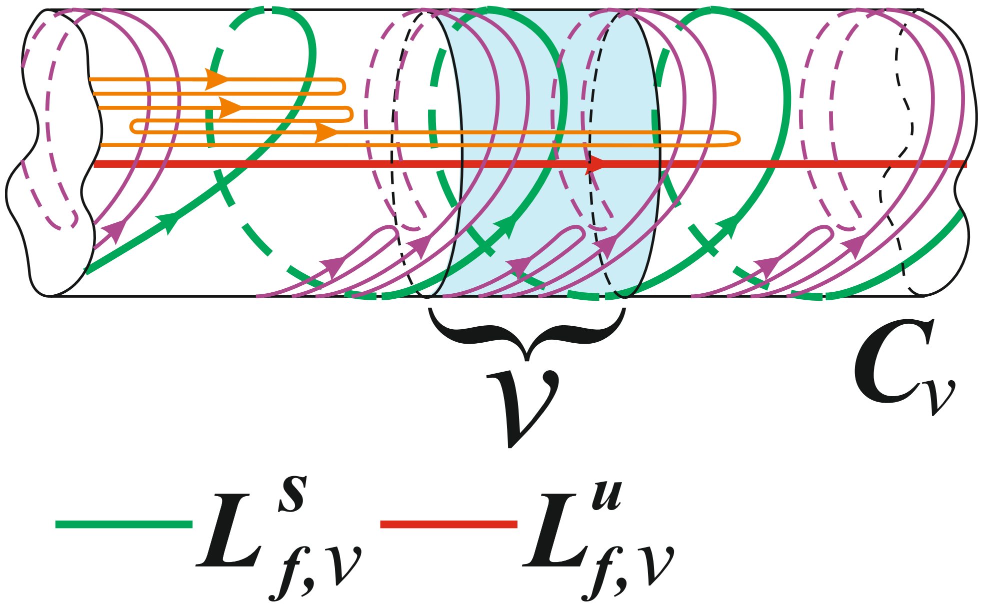

Figure 2: An example of the set C_v, where the purple and orange separatrices do not belong to the sets $L^s_{f,v}$ and $L^u_{f,v}$, respectively.

Fig. 3). According to [45, Chapter 2.D Exercise 7.] the number $d^s_{f,v}(d^u_{f,v})$ does not depend on the choice of a knot from the set $\hat{L}^s_{f,v}(\hat{L}^u_{f,v})$ Let

$$\xi_{f,v} = d^s_{f,v} - d^u_{f,v}.$$

According to [45, Chapter 2.C] the number $\xi_{f,v}$ does not depend on the choice of basis

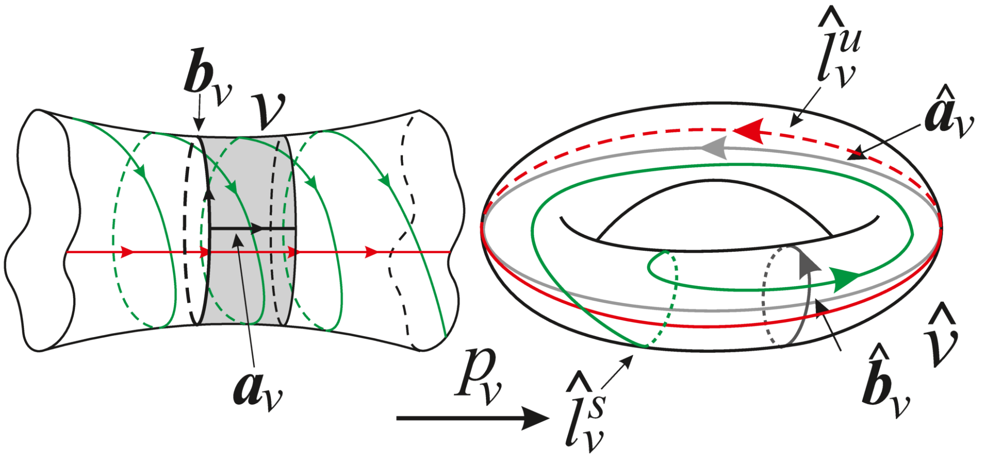

Figure 3: The figure shows the location of the knots $\hat{l}^s_v \in \hat{L}^s_{f,v}, \hat{l}^u_v \in \hat{L}^u_{f,v}$ on the torus $\hat{v}$.

$\hat{a}_v, \hat{b}_v$. We will call a component v *a heteroclinic anulus* if the set $L^u_{f,v}$ contains at least one unstable saddle separatrix σ_{i+1} intersecting the set $L^s_{f,v}$. Note that each set $K_{f,i}$ contains at most two heteroclinic annulos. We will call the heteroclinic anulus v (see Fig. 4):

- *contractible* if the curve b_v is homotopic to zero on the surface M^2;
- *trivial* if $\xi_{f,v} = 0$;

- *essential* if v is neither contractible nor trivial.

Let $\mathcal{V}_f$ denote the set of essential heteroclinic annulos v. Let us define the *index of heteroclinic intersections* ξ_f of the diffeomorphism f. If the set $\mathcal{V}_f$ is empty, then we set $\xi_f = 0$. Otherwise, on the set $\mathcal{V}_f$ we introduce the following equivalence relation: components $v \subset K_{f,i}$, $v' \subset K_{f,i'}$ will be called *equivalent* if the curves b_v, $b_{v'}$ are homotopic (see an example of a diffeomorphism containing annulos of different equality classes in Fig. 4). We denote by $[v]$ the equivalence class of the anulus v, and by $[\mathcal{V}_f]$ the set of equivalence classes. Assuming that the curves b_v, $b_{v'}$ are consistently oriented for equivalent annulos v, v', we set

$$\xi_{f,[v]} = \sum_{v \in [v]} \xi_{f,v}, \ \ \xi_f = \sum_{[v] \in [\mathcal{V}_f]} |\xi_{f,[v]}|.$$

The main result of Chapter 3 is the following theorem.

Theorem 2 ([21]*, Theorem 1) *Let $f \in MS(M^2)$. Then $\{f\} \in T_1$ $(\{f\} \in T_2)$ $\Leftrightarrow$ $\xi_f = 0$ $(\xi_f \neq 0)$* [2].

In Chapter 4 a complete topological classification (including realization) of diffeomorphisms of the class $MS_1(M^2)$ is obtained.

Let $f \in MS_1(M^2)$. Then the set Σ_f of periodic orbits of the diffeomorphism f can be partitioned into subsets Σ^i_f, $i \in \{\omega, s, u, \alpha\}$ as follows:

- Σ^ω_f — set of all sink points;
- Σ^s_f — the set of saddle points whose unstable manifolds do not contain heteroclinic points;
- Σ^u_f — the set of all other saddle points;
- Σ^α_f — the set of all source points.

From the properties of the introduced order $\prec$ it follows that each orbit $\mathcal{O}_i \in \Sigma^u_f$ is in the relation $\mathcal{O}_j \prec \mathcal{O}_i$ with some periodic orbit $\mathcal{O}_j \in \Sigma^s_f$. Let

$$\mathcal{A}_f = \Sigma^\omega_f \cup W^u_{\Sigma^s_f}, \ \mathcal{R}_f = \Sigma^\alpha_f \cup W^s_{\Sigma^u_f}, \ V_f = M^2 \setminus (\mathcal{A}_f \cup \mathcal{R}_f).$$

The article [20] shows that the sets $\mathcal{A}_f$, $\mathcal{R}_f$ are an attractor and the system repeller, respectively. Let

$$\hat{V}_f = V_f / f.$$

According to the article [42], each connected component of the orbit space $\hat{V}_f$ is homeomorphic to the two-dimensional torus. Let $p_f : V_f \to \hat{V}_f$ denote the natural projection, which is also a covering for the space $\hat{V}_f$.

[2]For Morse-Smale diffeomorphisms with a finite set of heteroclinic orbits defined on a two-dimensional torus, the theorem was proven in [31].

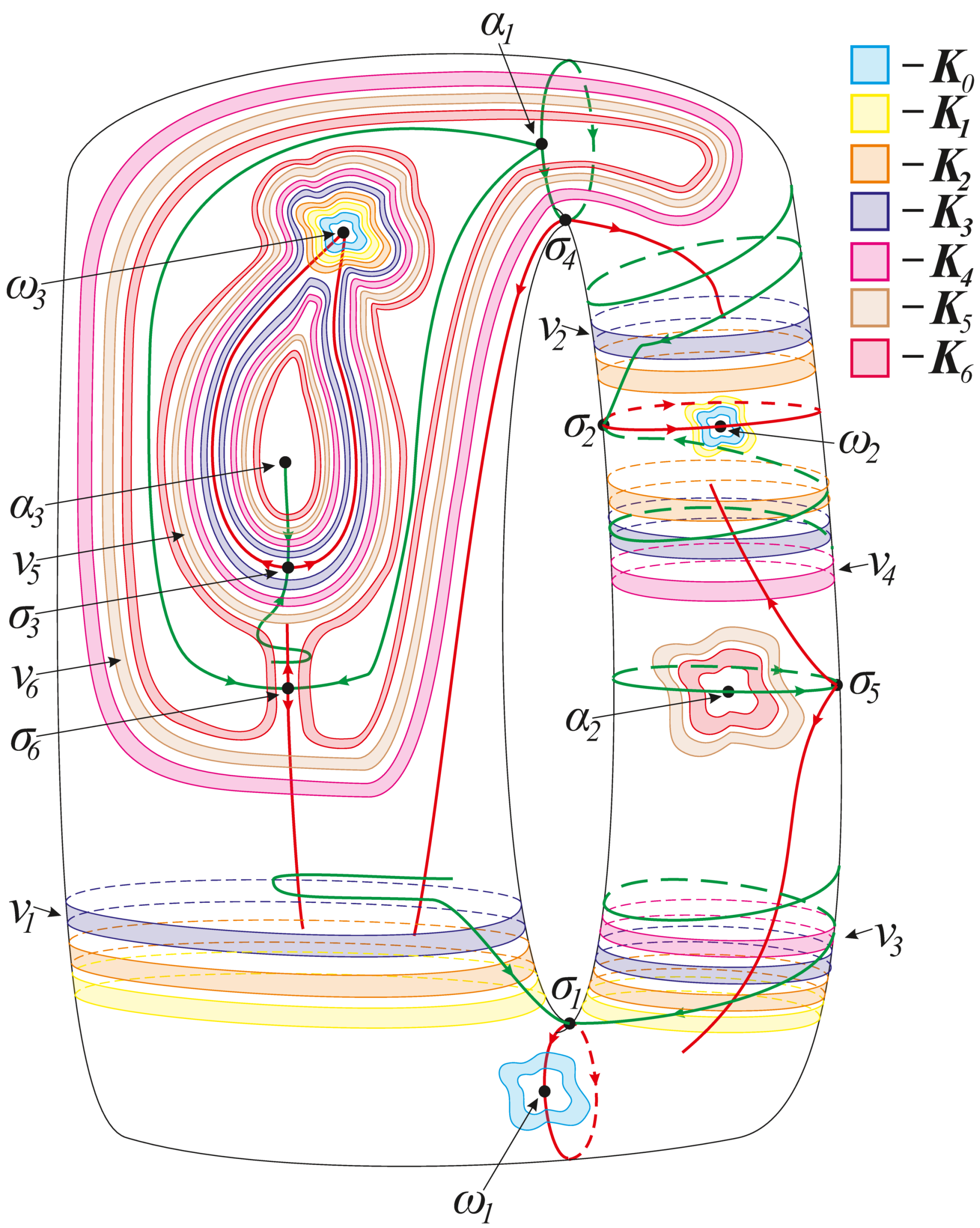

Figure 4: Heteroclinic annulos of Morse-Smale diffeomorphism on a torus.

We denote by $\hat{V}_i$, $i \in \{1, 2, ..., n\}$ the connected components of the orbit space $\hat{V}_f$. Let us set $V_i = p_f^{-1}(\hat{V}_i)$ and denote the natural projection by $p_i : V_i \to \hat{V}_i$. The covering p_i induces a nontrivial homomorphism $\eta_i : \pi_1(\hat{V}_i) \to m_i\mathbb{Z}$, which assigns to the homotopy class of the curve $[\hat{c}] \in \pi_1(\hat{V}_i)$ the number μm_i such that any lift of the curve $\hat{c}$ connects the point $x \in V_i$ with the point $f^{\mu m_i}(x)$. Let

$$\hat{V}_f = \hat{V}_1 \sqcup \cdots \sqcup \hat{V}_n$$

and denote by η_f the mapping composed of homomorphisms $\eta_1, \ldots, \eta_n$.

Denote by $m_f \in \mathbb{N}$ the smallest number such that all points of the non-wandering set of the diffeomorphism f^{m_f} are fixed and the diffeomorphism f^{m_f} preserves the orientation on W^u_σ for all $\sigma \in \Sigma_f$. Let us put $\tilde{f} = f^{m_f}$ and note that the sets $\mathcal{A}_f$ and $\mathcal{R}_f$ are also an attractor and repeller for the diffeomorphism $\tilde{f}$. Let

$$\tilde{V}_f = V_f/\tilde{f}.$$

Denote by $\tilde{p}_f : V_f \to \tilde{V}_f$ is a natural projection, which is also a covering.

Let us introduce the following notation:

- $\mathcal{L}^s_f$, $\mathcal{L}^u_f$ – sets of all stable, unstable, respectively, saddle separatrices of the diffeomorphism f and $\mathcal{L}_f = \mathcal{L}^s_f \cup \mathcal{L}^u_f$;
- $L^s_f = \bigcup\limits_{l \in \mathcal{L}^s_f} l$, $L^u_f = \bigcup\limits_{l \in \mathcal{L}^u_f} l$ and $L_f = L^s_f \cup L^u_f$;
- $\hat{\mathcal{L}}^s_f = \{\hat{l} = p_f(l) | \, l \in \mathcal{L}^s_f\}$, $\hat{\mathcal{L}}^u_f = \{\hat{l} = p_f(l) | \, l \in \mathcal{L}^u_f\}$ and $\hat{\mathcal{L}}_f = \hat{\mathcal{L}}^s_f \cup \hat{\mathcal{L}}^u_f$;
- $\hat{L}^s_f = \bigcup\limits_{\hat{l} \in \hat{\mathcal{L}}^s_f} \hat{l}$, $\hat{L}^u_f = \bigcup\limits_{\hat{l} \in \hat{\mathcal{L}}^u_f} \hat{l}$ and $\hat{L}_f = \hat{L}^s_f \cup \hat{L}^u_f$;
- $\hat{\mathcal{P}}_f$ – an involution on the set $\hat{\mathcal{L}}_f$ such that for any element $\hat{l} \in \hat{\mathcal{L}}_f$ is executed $\hat{l} \cup \hat{\mathcal{P}}_f(\hat{l}) = p_f(W^\delta_\sigma \setminus \sigma)$ for some $\delta \in \{s, u\}$ and $\sigma \in (\Sigma^s_f \cup \Sigma^u_f)$;
- $\tilde{\mathcal{L}}^s_f = \{\tilde{l} = \tilde{p}_f(l) | \, l \in \mathcal{L}^s_f\}$, $\tilde{\mathcal{L}}^u_f = \{\tilde{l} = \tilde{p}_f(l) | \, l \in \mathcal{L}^u_f\}$ and $\tilde{\mathcal{L}}_f = \tilde{\mathcal{L}}^s_f \cup \tilde{\mathcal{L}}^u_f$;
- $\tilde{L}^s_f = \bigcup\limits_{\tilde{l} \in \tilde{\mathcal{L}}^s_f} \tilde{l}$, $\tilde{L}^u_f = \bigcup\limits_{\tilde{l} \in \tilde{\mathcal{L}}^u_f} \tilde{l}$ and $\tilde{L}_f = \tilde{L}^s_f \cup \tilde{L}^u_f$;
- $\tilde{\mathcal{P}}_f$ – an involution on the set $\tilde{\mathcal{L}}_f$ such that for any element $\tilde{l} \in \tilde{\mathcal{L}}_f$ is executed $\tilde{l} \cup \tilde{\mathcal{P}}_f(\tilde{l}) = \tilde{p}_f(W^\delta_\sigma \setminus \sigma)$ for some $\delta \in \{s, u\}$ and $\sigma \in (\Sigma^s_f \cup \Sigma^u_f)$.

For any diffeomorphism $f \in MS_1(M^2)$ we set

$$\hat{S}_f = (\hat{V}_f, \hat{\mathcal{L}}_f, \hat{\mathcal{P}}_f), \; \tilde{S}_f = (\tilde{V}_f, \tilde{\mathcal{L}}_f, \tilde{\mathcal{P}}_f).$$

Definition 1 (Diffeomorphism scheme)**.** *We will call the pair*

$$\mathcal{S}_f = (\hat{S}_f, \tilde{S}_f)$$

scheme *of the diffeomorphism $f \in MS_1(M^2)$.*

The sets $\hat{S}_f$, $\hat{S}_{f'}$ for diffeomorphisms $f, \, f' \in MS_1(M^2)$ will be called *equivalent* (see Fig. 5), if there is a homeomorphism $\hat{\varphi} : \hat{V}_f \to \hat{V}_{f'}$ such that $\eta_{f} = \eta_{f'} \hat{\varphi}_*$ and

(1) $\hat{L}^s_{f'} = \hat{\varphi}(\hat{L}^s_f)$, $\hat{L}^u_{f'} = \hat{\varphi}(\hat{L}^u_f)$;

(2) homeomorphism $\hat{\varphi}$ by the formula $\hat{\varphi}_\star(\hat{l}) = \hat{\varphi}(\hat{l})$ induces a one-to-one correspondence $\hat{\varphi}_\star : \hat{\mathcal{L}}_f \to \hat{\mathcal{L}}_{f'}$ such that $\hat{\varphi}_\star \hat{\mathcal{P}}_f = \hat{\mathcal{P}}_{f'} \hat{\varphi}_\star$.

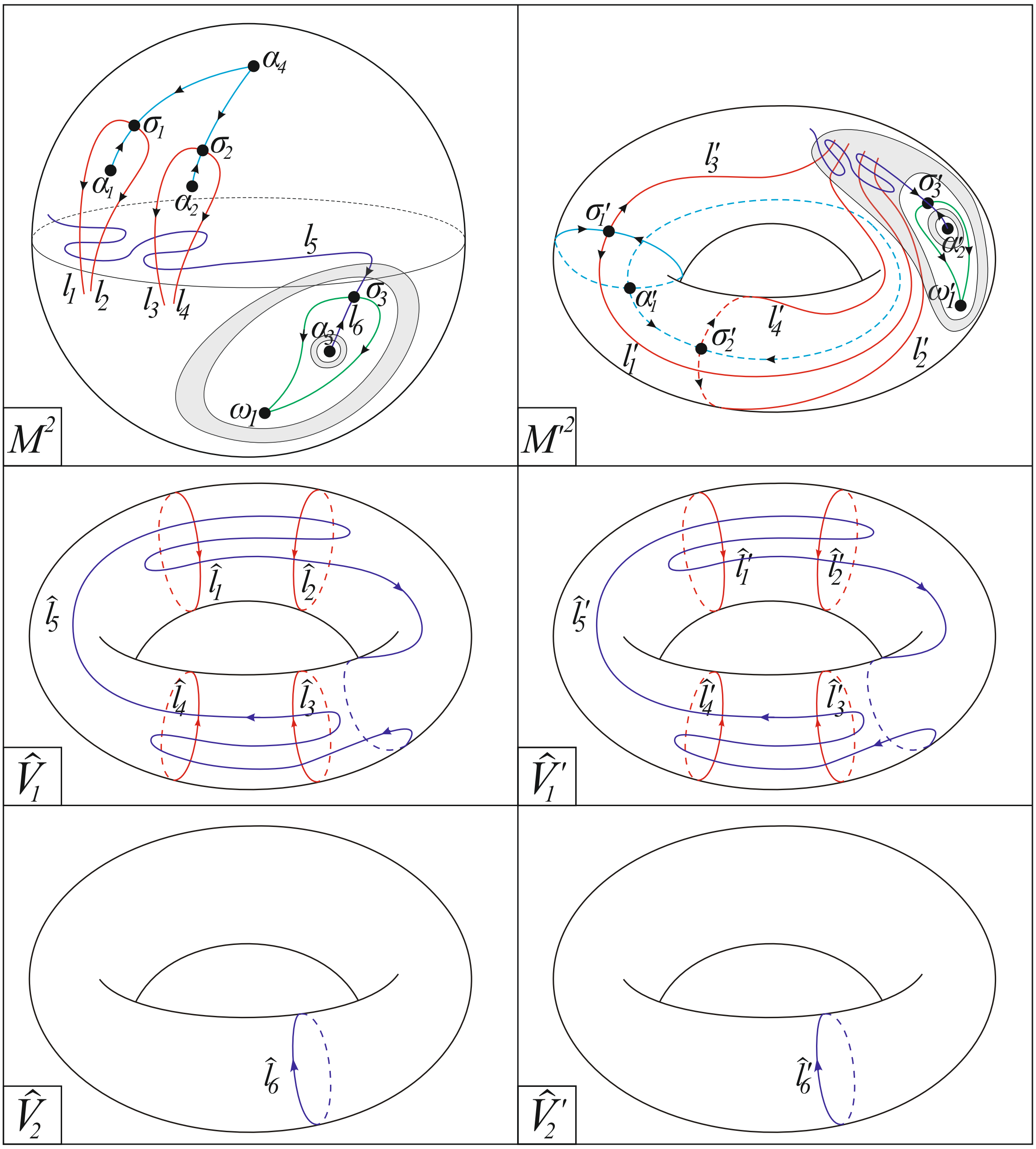

Figure 5: Phase portraits of topologically non-conjugate diffeomorphisms $f,\ f' \in MS_1(M^2)$ with nonequivalent sets $\hat{S}_f,\ \hat{S}_{f'}$

Figure 5 shows diffeomorphisms $f,\ f' \in MS_1(M^2)$. On the left is a diffeomorphism f of a two-dimensional sphere, the non-wandering set of which consists of fixed source points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, fixed saddle points $\sigma_1, \sigma_2, \sigma_3$ and a fixed sink point ω_1. The separatrices $l_1, \dots, l_6 \in \mathcal{L}_f$ are also marked. On the set $\hat{V}_f = \hat{V}_1 \sqcup \hat{V}_2$ the projections of the separatrices $\hat{\mathcal{L}}_f = \{\hat{l}_1, \dots, \hat{l}_6\}$. The involution $\hat{\mathcal{P}}_f$ acts as follows:

$$\hat{\mathcal{P}}_f(\hat{l}_1) = \hat{l}_2, \hat{\mathcal{P}}_f(\hat{l}_3) = \hat{l}_4, \hat{\mathcal{P}}_f(\hat{l}_5) = \hat{l}_6.$$

The figure on the right shows a diffeomorphism f' of a two-dimensional torus, the non-wandering set of which consists of fixed source points α'_1, α'_2, fixed saddle points $\sigma'_1, \sigma'_2, \sigma'_3$ and the fixed sink point ω'_1. The separatrices $l'_1, \dots, l'_6 \in \mathcal{L}_{f'}$ are also marked. On the set $\hat{V}_{f'} = \hat{V}'_1 \sqcup \hat{V}'_2$ the projections of the separatrices $\hat{\mathcal{L}}_{f'} = \{\hat{l}'_1, \dots, \hat{l}'_6\}$. The involution $\hat{\mathcal{P}}_{f'}$ acts as follows:

$$\hat{\mathcal{P}}_{f'}(\hat{l}'_1) = \hat{l}'_3, \hat{\mathcal{P}}_f(\hat{l}'_2) = \hat{l}'_4, \hat{\mathcal{P}}_f(\hat{l}'_5) = \hat{l}'_6.$$

For the tuples $\hat{S}_f, \hat{S}_{f'}$ there is a homeomorphism $\hat{\varphi} : \hat{V}_f \to \hat{V}_{f'}$ such that $\eta_{_f} = \eta_{_{f'}}\hat{\varphi}$ and item (1) of equivalence of sets is satisfied. But for any such homeomorphism, condition (2) $\hat{\varphi}_\star\hat{\mathcal{P}}_f = \hat{\mathcal{P}}_{f'}\hat{\varphi}_\star$ is not satisfied. The topological nonconjugacy of diffeomorphisms f, f' obviously follows from the fact, that the surfaces on which they are defined are not homeomorphic.

The sets $\tilde{S}_f$, $\tilde{S}_{f'}$ for diffeomorphisms $f,\ f' \in MS_1(M^2)$ will be called *equivalent* (see Fig. 6), if there is a homeomorphism $\tilde{\varphi} : \tilde{V}_f \to \tilde{V}_{f'}$ such that:

(1) $\tilde{L}_{f'} = \tilde{\varphi}(\tilde{L}_f)$;

(2) homeomorphism $\tilde{\varphi}$ by the formula $\tilde{\varphi}_\star(\tilde{l}) = \tilde{\varphi}(\tilde{l})$ induces a one-to-one correspondence $\tilde{\varphi}_\star : \tilde{\mathcal{L}}_f \to \tilde{\mathcal{L}}_{f'}$ such that $\tilde{\varphi}_\star\tilde{\mathcal{P}}_f = \tilde{\mathcal{P}}_{f'}\tilde{\varphi}_\star$.

Figure 6 shows diffeomorphisms $f,\ f' \in MS_1(M^2)$. On the left is a diffeomorphism f of a two-dimensional sphere, the non-wandering set of which consists of fixed source points $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$, fixed saddle points $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and a fixed sink point ω_1. The separatrices $l_1, \dots, l_8 \in \mathcal{L}_f$ are also marked. On the set $\hat{V}_f = \hat{V}_1 \sqcup \hat{V}_2$ the projections of the separatrices $\hat{\mathcal{L}}_f = \{\hat{l}_1, \dots, \hat{l}_4\}$. The involution $\hat{\mathcal{P}}_f$ acts as follows:

$$\hat{\mathcal{P}}_f(\hat{l}_1) = \hat{l}_2, \hat{\mathcal{P}}_f(\hat{l}_3) = \hat{l}_4.$$

On the set $\tilde{V}_f = \tilde{V}_1 \sqcup \tilde{V}_2$ the projections of separatrices are marked $\tilde{\mathcal{L}}_f = \{\tilde{l}_1, \dots, \tilde{l}_8\}$. The involution $\tilde{\mathcal{P}}_f$ works as follows:

$$\tilde{\mathcal{P}}_f(\tilde{l}_1) = \tilde{l}_2, \tilde{\mathcal{P}}_f(\tilde{l}_3) = \tilde{l}_4, \tilde{\mathcal{P}}_f(\tilde{l}_5) = \tilde{l}_6, \tilde{\mathcal{P}}_f(\tilde{l}_7) = \tilde{l}_8.$$

The figure on the right shows a diffeomorphism f' of a two-dimensional torus, the non-wandering set of which consists of fixed source points $\alpha'_1, \alpha'_2, \alpha'_3$, fixed saddle points $\sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4$ and the fixed sink point ω'_1. The separatrices $l'_1, \dots, l'_8 \in \mathcal{L}_{f'}$ are also marked. On the set $\hat{V}_{f'} = \hat{V}'_1 \sqcup \hat{V}'_2$ the projections of the separatrices $\hat{\mathcal{L}}_{f'} = \{\hat{l}'_1, \dots, \hat{l}'_4\}$. The involution $\hat{\mathcal{P}}_{f'}$ acts as follows:

$$\hat{\mathcal{P}}_{f'}(\hat{l}'_1) = \hat{l}'_2, \hat{\mathcal{P}}_f(\hat{l}'_3) = \hat{l}'_4.$$

On the set $\tilde{V}_{f'} = \tilde{V}'_1 \sqcup \tilde{V}'_2$ the projections of the separatrices $\tilde{\mathcal{L}}'_f = \{\tilde{l}'_1, \dots, \tilde{l}'_8\}$. The involution $\tilde{\mathcal{P}}_{f'}$ works as follows:

$$\tilde{\mathcal{P}}_{f'}(\tilde{l}'_1) = \tilde{l}_4, \tilde{\mathcal{P}}_{f'}(\tilde{l}'_2) = \tilde{l}'_5, \tilde{\mathcal{P}}_{f'}(\tilde{l}'_3) = \tilde{l}'_6, \tilde{\mathcal{P}}_{f'}(\tilde{l}'_7) = \tilde{l}'_8.$$

For the tuples $\hat{S}_f, \hat{S}_{f'}$ there is a homeomorphism $\hat{\varphi} : \hat{V}_f \to \hat{V}_{f'}$ such that $\eta_f = \eta_{f'}\hat{\varphi}$ and items (1) and (2) of equivalence of sets are satisfied. Also, for the sets $\tilde{S}_f, \tilde{S}_{f'}$ there is a homeomorphism $\tilde{\varphi} : \tilde{V}_f \to \tilde{V}_{f'}$ such that item (1) of equivalence of sets is satisfied. But for any such homeomorphism, condition (2) $\tilde{\varphi}_\star \tilde{\mathcal{P}}_f = \tilde{\mathcal{P}}_{f'} \tilde{\varphi}_\star$ is not satisfied. The topological nonconjugacy of diffeomorphisms f, f' obviously follows from the fact, that the surfaces on which they are defined are not homeomorphic.

Definition 2 (Schemes equivalence)**.** *Schemes $\mathcal{S}_f, \mathcal{S}_{f'}$ of diffeomorphisms $f, f' \in MS_1(M^2)$ will be called* equivalent *if there is a homeomorphism $\hat{\varphi} : \hat{V}_f \to \hat{V}_{f'}$, which implements the equivalence of the sets $\hat{S}_f$, $\hat{S}_{f'}$ and rises to a homeomorphism $\tilde{\varphi} : \tilde{V}_f \to \tilde{V}_{f'}$, which implements the equivalence of the sets $\tilde{S}_f$, $\tilde{S}_{f'}$.*

The main result of Chapter 4 is the following theorem.

Theorem 3 ([32]*, Theorem 1.) *Two diffeomorphisms $f \in MS_1(M^2), f' \in MS_1(M'^2)$ are topologically conjugate if and only if their schemes $\mathcal{S}_f, \mathcal{S}_{f'}$ are equivalent.*

Based on the properties of the diffeomorphism scheme $f \in MS_1(M^2)$, the concept of an abstract scheme $\mathcal{S}$ is introduced and the following theorem is proved.

Theorem 4. ([32]*, Theorem 2) *For any abstract scheme $\mathcal{S}$ there is a diffeomorphism $f \in MS_1(M^2)$ whose scheme $\mathcal{S}_f$ is equivalent to the scheme $\mathcal{S}$.*

Using any abstract scheme $\mathcal{S} = (\hat{S}, \tilde{S})$ one can determine the type of surface on which the diffeomorphism $f \in MS_1(M^2)$ is realized according to this scheme. To do this, we construct two sets of circles S_ω, S_α from the set $\tilde{S}$ as follows.

By [18, Lemma 3.2.1] for each connected component $\tilde{V}_i$ of a set $\tilde{V}_f$ such that $\tilde{\mathcal{L}}^s \cap \tilde{V}_i \neq \varnothing$, there is a knot that intersects every knot $\tilde{l} \in \tilde{\mathcal{L}}^s \cap \tilde{V}_i$ at a single point (see Fig. 7), let's call such a knot *equator*. Let us choose the equator $\tilde{\beta}^s_i$ for each component of the connectivity $\tilde{V}_i$, such that $\tilde{\mathcal{L}}^s \cap \tilde{V}_i \neq \varnothing$. Denote by B^s the union of all equators $\tilde{\beta}^s_i$. The points $\tilde{l} \cap B^s$, $\tilde{\mathcal{P}}(\tilde{l}) \cap B^s$ will be called *paired*. Let us denote by S_ω the set of knots obtained from B^s by taking a connected sum along pairwise disjoint neighborhoods of paired points, and also by adding knots $\tilde{p}(p^{-1}(\hat{b}_i))$ for $i : \mathcal{L}^s \cap \tilde{V}_i = \varnothing$ (see Fig. 7).

In a similar way, a set of knots S_α is constructed from equators B^u to knots $\tilde{\mathcal{L}}^u$. Let us denote by k^s, k^u, k^ω, k^α the number of knots in the sets $\tilde{\mathcal{L}}^s$, $\tilde{\mathcal{L}}^u$, S_ω, S_α, respectively.

The following lemma allows us to determine the genus of the supporting surface of the diffeomorphism $f \in MS_1(M^2)$ according to the scheme $\mathcal{S}$.

Lemma 4.1 ([32]*, Lemma 1) *Genus g of the supporting surface of the diffeomorphism $f \in MS_1(M^2)$ with scheme $\mathcal{S}$ is calculated using the formula*

$$2 - 2g = k^\alpha + k^\omega - \frac{1}{2}\left(k^s + k^u\right).$$

Chapter 5 covers the class $MS_+(M^2)$. First, it is established that any diffeomorphism of the class under consideration has a finite number of orbits.

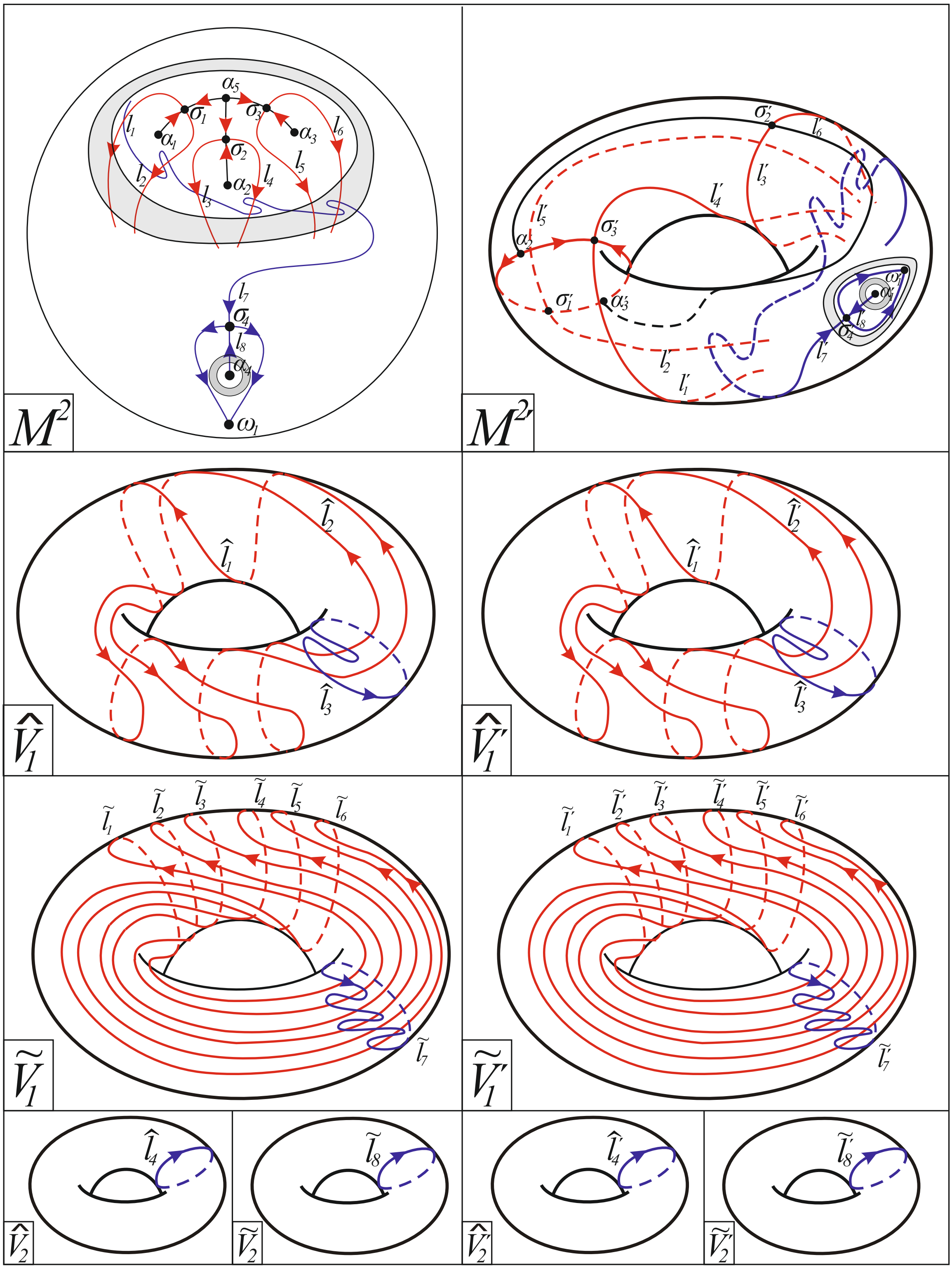

Figure 6: Phase portraits of topologically non-conjugate diffeomorphisms $f,\ f' \in MS_1(M^2)$ with nonequivalent sets $\tilde{S}_f,\ \tilde{S}_{f'}$

Theorem 5 ([33]*, Theorem 1) *Если диффеоморфизм* $f \in MS_+(M^2)$, *то* $beh(f) = 1$.[3]

[3]Independent proof of this fact was given in the work of A.N. Bezdenezhnykh [3] using deep results of A.G. Mayer on the connection between the number of pairwise disjoint self-limiting trajectories and the

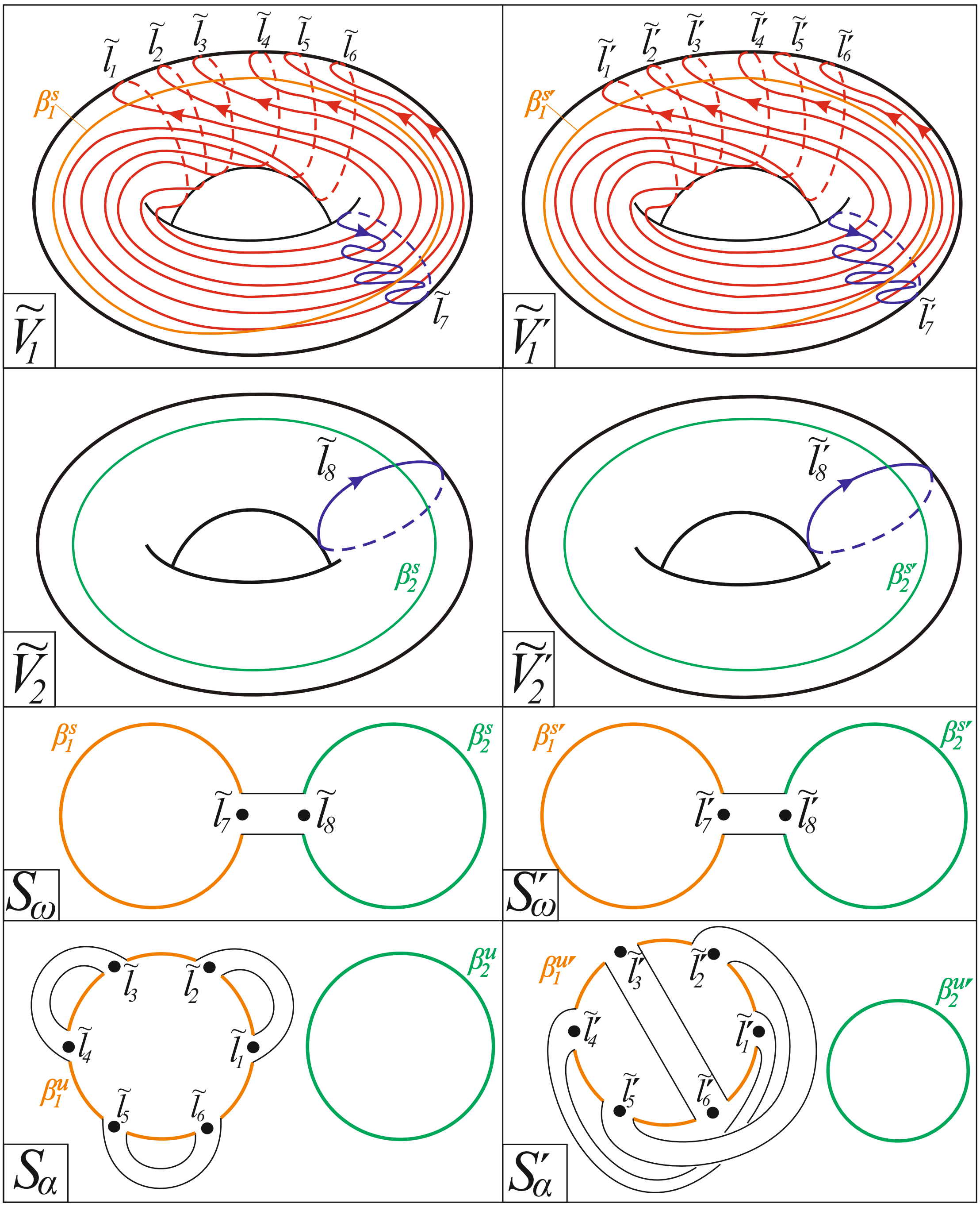

Figure 7: Spaces $\tilde{V}_f, \tilde{V}_{f'}$ for diffeomorphisms f, f' shown in figure 6, with marked knots $\tilde{\beta}^s_1, \tilde{\beta}^s_2, \tilde{\beta}'^s_1, \tilde{\beta}'^s_2$, as well as sets of knots S_ω, S_α of the diffeomorphism f and S'_ω, S'_α of the diffeomorphism f' .

Thus, for diffeomorphisms of the class $MS_+(M^2)$ all classification results presented in Section 4 for diffeomorphisms of the class $f \in MS_1(M^2)$ are valid. However, the orientability of the heteroclinic intersection allows us to bring the complete invariant (diffeomorphism scheme) to a combinatorial description.

Namely, the procedure for uniquely constructing the graph T_f with the substitution P_f

type of bearing surface [26]. The proof of Theorem 5 given in this paper is based only on the property of linearizing neighborhoods and quotient spaces.

according to the scheme $\mathcal{S}_f$ of the diffeomorphism $f \in MS_+(M^2)$ is described. In more detail.

Consider the set $\tilde{S}_f$ of the scheme $\mathcal{S}_f$ described in Chapter 4. Let $\tilde{V}_i = \tilde{p}_f p_f^{-1}(\hat{V}_i),\ i \in \{1, \dots, n\}$ and denote by $\tilde{V}_{i,\kappa}$ connected components of the set $\tilde{V}_i,\ \kappa \in \{0, \dots, m_i - 1\}$. Let

$$\tilde{\mathcal{L}}^s_{i,\kappa} = \{\tilde{l} \in \tilde{\mathcal{L}}^s_f | \tilde{l} \cap \tilde{V}_{i,\kappa} \neq \emptyset\},\ \ \tilde{\mathcal{L}}^u_{i,\kappa} = \{\tilde{l} \in \tilde{\mathcal{L}}^u_f | \tilde{l} \cap \tilde{V}_{i,\kappa} \neq \emptyset\}.$$

For separatrices from the set $\tilde{\mathcal{L}}^s_{i,\kappa}, (\tilde{\mathcal{L}}^u_{i,\kappa})$ is determined by their homotopy type $\langle 1, \tilde{\nu}^s_i \rangle\, (\langle 1, \tilde{\nu}^u_i \rangle)$, which is the same for all $\kappa \in \{0, \dots, m_i - 1\}$. Let us denote by $\tilde{r}^s_i, \tilde{r}^u_i$ the number of connected components of the sets $\tilde{\mathcal{L}}^s_{i,\kappa}, \tilde{\mathcal{L}}^u_{i,\kappa}$, respectively. Since the curves of the family $\tilde{\mathcal{L}}^s_{i,\kappa}(\tilde{\mathcal{L}}^u_{i,\kappa})$ do not intersect , then we will assume that the numbering was chosen in a natural way, according to the orientation of the generator $\tilde{b}_{i,\kappa}$.

Similarly, consider the set $\hat{S}_f$ of the scheme $\mathcal{S}_f$, define the corresponding sets of projections of separatrices $\hat{\mathcal{L}}^s_i, \hat{\mathcal{L}}^s_i$, denote by $\langle \mu^s_i, \nu^s_i \rangle$, $\langle \mu^u_i, \nu^u_i \rangle$ their homotopy types are denoted by r^s_i, r^u_i is the number of elements in the corresponding set.

Let's construct a graph T_f with vertices B_f and edges E_f as follows.

1. Vertices B_f. All vertices B_f of the graph T_f are located on the plane $\mathbb{R}^2$ and are in the following correspondence ζ with the diffeomorphism f.

1.1. For $i \in \{1, \dots, n\}$, $\kappa \in \{0, 1, \dots, m_i - 1\}$ we place the vertex $\delta_{i,\kappa}$ at a point with coordinates $(5(i-1), 5\kappa)$. Let us set $\zeta(\tilde{V}_{i,\kappa}) = \delta_{i,\kappa}$.

1.2. If $\tilde{r}^s_i \neq 0$, then for $j_s \in \{0, ..., \tilde{r}^s_i - 1\}, \kappa \in \{0, 1, \dots, m_i - 1\}$ place the vertices δ^s_{i,κ,j_s} on the unit circle $C^s_{i,\kappa}$ with center $\delta_{i,\kappa}$ so that point δ^s_{i,κ,j_s} has coordinates

$$\left(cos\left(\frac{2\pi j_s}{\tilde{r}^s_i}\right) + 5(i-1),\ sin\left(\frac{2\pi j_s}{\tilde{r}^s_i}\right) + 5\kappa \right).$$

Let us set $\zeta(\tilde{l}^s_{i,\kappa,j_s}) = \delta^s_{i,\kappa,j_s}$.

1.3. If $\tilde{r}^u_i \neq 0$, then for $j_u \in \{0, ..., \tilde{r}^u_i - 1\}, \kappa \in \{0, 1, \dots m_i - 1\}$ place the vertices δ^u_{i,κ,j_u} on a circle $C^u_{i,\kappa}$ of radius 2 with center $\delta_{i,\kappa}$ so that point δ^u_{i,κ,j_u} has coordinates

$$\left(2cos\left(\frac{2\pi j_u}{\tilde{r}^u_i}\right) + 5(i-1),\ 2sin\left(\frac{2\pi j_u}{\tilde{r}^u_i}\right) + 5\kappa \right).$$

Let us set $\zeta(\tilde{l}^u_{i,\kappa,j_u}) = \delta^u_{i,\kappa,j_u}$.

2. Edges E_f.

2.1. Each vertex $\delta_{i,\kappa}$ is connected by edges $(\delta_{i,\kappa}, \delta^s_{i,\kappa,j_s})$ and $(\delta_{i,\kappa}, \delta^u_{i,\kappa,j_u})$ with vertices δ^s_{i,κ,j_s} and δ^u_{i,κ,j_u}, respectively.

2.2. Each pair of vertices $\delta^s_{i,\kappa,j_s}, \delta^s_{i,\kappa,j_s+1}\, (\delta^s_{i,\kappa,\tilde{r}^s_i} = \delta^s_{i,\kappa,0})$ will be connected by an edge (circular arc $C^s_{i,\kappa}$) $(\delta^s_{i,\kappa,j_s}, \delta^s_{i,\kappa,j_s+1})$ of color s. Let us set

$$c^s_{i,\kappa} = \{\delta^s_{i,\kappa,0}, (\delta^s_{i,\kappa,0}, \delta^s_{i,\kappa,1}), \delta^s_{i,\kappa,1}, \dots, (\delta^s_{i,\kappa,\tilde{r}^s_i - 1}, \delta^s_{i,\kappa,0}), \delta^s_{i,\kappa,0}\}.$$

Let's call the cycles $c^s_{i,\kappa}$ *s-cycles*.

2.3. Each pair of vertices $\delta^u_{i,\kappa,j_u}, \delta^u_{i,\kappa,j_u+1}$ $(\delta^u_{i,\kappa,\tilde{r}^u_i} = \delta^u_{i,\kappa,0})$ will be connected by an edge (circular arc $C^u_{i,\kappa}$) $(\delta^u_{i,\kappa,j_u}, \delta^u_{i,\kappa,j_u+1})$ of color u. Let

$$c^u_{i,\kappa} = \{\delta^u_{i,\kappa,0}, (\delta^u_{i,\kappa,0}, \delta^u_{i,\kappa,1}), \delta^u_{i,\kappa,1}, \dots, (\delta^u_{i,\kappa,\tilde{r}^u_i-1}, \delta^u_{i,\kappa,0}), \delta^u_{i,\kappa,0}\}.$$

Let's call the cycles $c^u_{i,\kappa}$ *u-cycles*

2.4. Vertices $\delta^s_{i,\kappa,j_s} = \zeta(\tilde{l}^s_{i,\kappa,j_s}), \delta^s_{\tilde{i},\tilde{\kappa},\tilde{j}_s} = \zeta(\tilde{l}^s_{\tilde{i},\tilde{\kappa},\tilde{j}_s})$, connect by edge $(\delta^s_{i,\kappa,j_s}, \delta^s_{\tilde{i},\tilde{\kappa},\tilde{j}_s})$ if $\tilde{l}^s_{i,\kappa,j_s} = \tilde{\mathcal{P}}_f(\tilde{l}^s_{\tilde{i},\tilde{\kappa},\tilde{j}_s})$. Let's call the vertices $\delta^s_{i,\kappa,j_s}, \delta^s_{\tilde{i},\tilde{\kappa},\tilde{j}_s}$ and the edge incident to them $(\delta^s_{i,\kappa,j_s}, \delta^s_{\tilde{i},\tilde{\kappa},\tilde{j}_s})$ *paired*.

2.5. Vertices $\delta^u_{i,\kappa,j_u} = \zeta(\tilde{l}^u_{i,\kappa,j_u}), \delta^u_{\tilde{i},\tilde{\kappa},\tilde{j}_u} = \zeta(\tilde{l}^u_{\tilde{i},\tilde{\kappa},\tilde{j}_u}))$, we connect with the edge $(\delta^u_{i,\kappa,j_u}, \delta^u_{\tilde{i},\tilde{\kappa},\tilde{j}_u})$, if $\tilde{l}^u_{i,\kappa,j_u} = \tilde{\mathcal{P}}_f(\tilde{l}^u_{\tilde{i},\tilde{\kappa},\tilde{j}_u})$. Let's call the vertices $\delta^u_{i,\kappa,j_u}, \delta^u_{\tilde{i},\tilde{\kappa},\tilde{j}_u}$ and the edge incident to them $(\delta^u_{i,\kappa,j_u}, \delta^u_{\tilde{i},\tilde{\kappa},\tilde{j}_u})$ *paired*. Note that the image of paired vertices, with respect to the automorphism P, is paired vertices.

3. Automorphism P_f.

3.1. The action of the automorphism P_f is induced by the diffeomorphism f, using the correspondence ζ by the formula

$$P_f = \zeta\tilde{p}_f f \tilde{p}_f^{-1}\zeta^{-1} : B_f \to B_f.$$

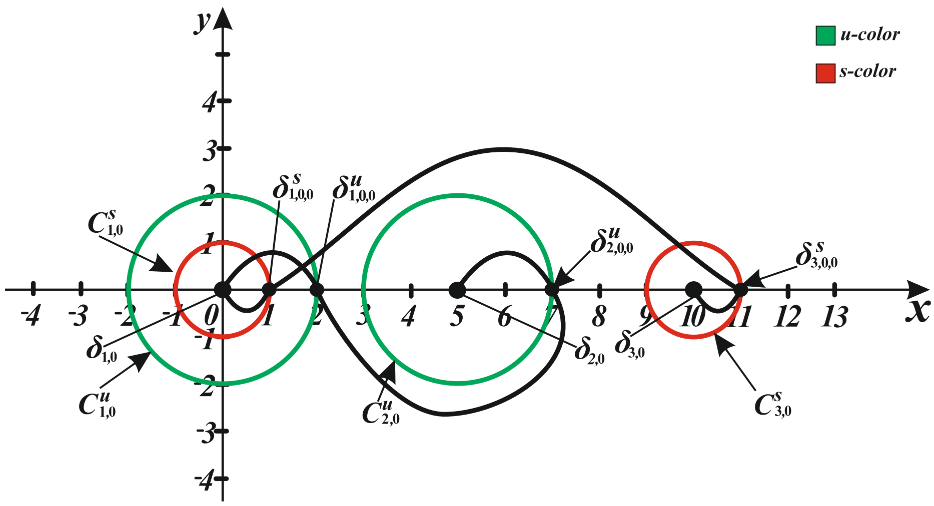

Figure 8: An example of a graph T_f.

Definition 6. Graphs $(T_f, P_f)\,(T_{f'}, P_{f'})$ of diffeomorphisms $f, f' \in MS^+_1(M^2)$ will be called *isomorphic* if there is an isomorphism ξ that takes vertices and edges of the graph T_f to vertices and edges of the graph $T_{f'}$ with color preservation and conjugating automorphisms, that is, $P_{f'} = \xi P_f \xi^{-1}$.

The main result of Section 5 is the following fact.

Lemma 5.1 *Graphs $(T_f, P_f), (T_{f'}, P'_f)$ of diffeomorphisms $f, f' \in MS_+(M^2)$ are isomorphic if and only if their schemes $\mathcal{S}_f, \mathcal{S}_{f'}$ are equivalent.*

In the course of the proof of Lemma 5.1, the following lemmas are used and proven.

Lemma 5.2([28] Lemma 1.) Let $c \subset \mathbb{T}^2$ be a non-zero homotopic simple closed curve on the torus such that $[c] =< \mu, \nu >$. If $\hat{\phi} : \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism such that $\hat{\phi}_*([b]) = [b]$, then the curve $\hat{\phi}(c)$ has homotopy type $< \mu, \bar{\nu} >$, where $\bar{\nu} \equiv \nu \pmod{\mu}$. Conversely, for any $\bar{\nu} \equiv \nu \pmod{\mu}$ there is a unique (up to isotopy) homeomorphism $\hat{\phi} : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\hat{\phi}_*([b]) = [b]$ and the curve $\hat{\phi}(c)$ has homotopy type $< \mu, \bar{\nu} >$.

Lemma 5.3 ([28] Lemma 2.) There is a diffeomorphism $\hat{\phi}_i : \hat{V}_i \to \mathbb{T}^2$ such that $\hat{\phi}_{i*}([b_i]) = [b]$, $\hat{\phi}_i(\hat{L}^s_i) = \hat{\Gamma}^s_i$, $\hat{\phi}_i(\hat{L}^u_i) = \hat{\Gamma}^u_i$ and diffeomorphism $\hat{\phi}_i$ rises to a diffeomorphism $\tilde{\phi}_i : \tilde{V}_i \to \mathbb{T}^2$ such that $\tilde{\phi}_{i,\kappa}([b_i]) = [b_i]$, $\tilde{\phi}_{i,\kappa}(\tilde{L}^s_{i,\kappa}) = \tilde{\Gamma}^s_{i,\kappa}$, $\tilde{\phi}_{i,\kappa}(\tilde{L}^u_{i,\kappa}) = \tilde{\Gamma}^u_{i,\kappa}$, where $\tilde{\phi}_{i,\kappa} = \tilde{\phi}_i|_{\tilde{V}_{i,\kappa}}$.

An immediate consequence of Lemma 5.1 is the following classification result.

Theorem 6 ([28]*, Theorem 1) *Diffeomorphisms $f, f' \in MS_+(M^2)$ are topologically conjugate if and only if their graphs (T_f, P_f), $(T_{f'}, P_{f'})$ are isomorphic.*

It is generally accepted that an algorithm is *efficient* if its operating time is limited by a certain polynomial of the length of the input information specified. This understanding of effective solvability comes from A. Cobham [11]. The standard for intractability is the NP-completeness (or NP-hardness) of the problem [12]. At present, neither polynomial solvability nor NP-completeness has been proven for the isomorphism problem in the class of all graphs. At the same time, diffeomorphism graphs from the class $MS_+(M^2)$ are not graphs of general form, which makes it possible to develop an algorithm for solving the isomorphism problem for these graphs that is polynomial in the number of vertices. The main resources for constructing this algorithm are that the maximum degree of the vertices of diffeomorphism graphs from $MS_+(M^2)$, except for the vertices δ_i, does not exceed four and that for any d the isomorphism problem ordinary graphs of maximum degree at most d are polynomially solvable [25].

Theorem 7 ([28]*, Theorem 2) *There is an effective algorithm for establishing isomorphism of graphs (T_f, P_f), $(T_{f'}, P_{f'})$.*

The proof of Theorem 7 consists in sequentially simplifying the graph T_f to the quotient graph T'_f, and then to the simple graph Γ_f, and the weights of the vertices of the graph T_f are preserved into the labels of the graph Γ_f in such a way that the graph T_f is uniquely reconstructed from the graph Γ_f. For two diffeomorphisms $f, f' \in MS_+(M^2)$ with graphs $(T_f, P_f), (T_{f'}, P_{f'})$ accordingly, graph reconstructions are carried out in polynomial time, and the isomorphism of the graphs $\Gamma_f, \Gamma_{f'}$ is checked in polynomial time [25].

Conclusion

In this thesis, we obtain a topological classification of orientation-preserving Morse-Smale diffeomorphisms with a finite number of heteroclinic orbits defined on orientable surfaces, a combinatorial invariant for orientation-preserving Morse-Smale diffeomorphisms with an orientable heteroclinic defined on orientable surfaces, as well as a homotopy classification of a more general class of orientation-preserving Morse-Smale diffeomorphisms defined on orientable surfaces.

A realization of a structurally stable representative (orientation-preserving Morse-Smale diffeomorphism with orientable heteroclinic intersections) in each homotopy class of the second Nielsen-Thurston type is obtained.

Let us list the main results proven in this dissertation:

- The realisation of a Morse-Smale diffeomorphism with an orientable heteroclinic intersection in each homotopy class $\{h\} \in T_2$ is described (Theorem 1).

- An algorithm for recognizing whether a given non-gradient-like diffeomorphism of class $MS(M^2)$ belongs to the Nielsen-Thurston set T_1 or T_2 by its heteroclinic intersection is described (Theorem 2).

- A complete topological classification of the set $MS_1(M^2)$ of Morse-Smale diffeomorphisms with a finite number of heteroclinic orbits ($beh(f) = 1$), including the realization (Theorems 3,4), is given.

- It is proven that an orientation-preserving Morse-Smale diffeomorphism with orientable heteroclinic intersections has a finite number of heteroclinic orbits (Theorem 5).

- The graph (T_f, P_f) is described, which is a complete topological invariant for diffeomorphisms of the class $MS_+(M^2)$ (Theorem 6).

- The existence of an effective algorithm for establishing the isomorphism of graphs (T_f, P_f) is proven (Theorem 7).

Approbation of research results

Participation in conferences

1. *Determination of the homotopy type of a Morse-Smale diffeomorphism on a surface by heteroclinic intersection*, International conference "Dynamics in Siberia", Novosibirsk, 2023.

2. *Determination of the homotopy type of a Morse-Smale diffeomorphism on a surface by heteroclinic intersection*, Complex analysis, mathematical physics and nonlinear equations, Southern Urals, Yakty-Kul (Lake Bannoe), 2023.

3. *Determination of the homotopy type of a Morse-Smale diffeomorphism on a surface by heteroclinic intersection*, International Conference on Nonlinear Dynamics and Integrity and Scientific School “Nonlinear Days”, Yaroslavl, 2022.

4. *Determination of the homotopy type of a Morse-Smale diffeomorphism on a surface by heteroclinic intersection*, International Conference on Differential Equations and Dynamical Systems, Suzdal, 2022.

5. *Determination of the homotopy type of a Morse-Smale diffeomorphism on a surface by heteroclinic intersection*, Second Conference of Mathematical Centers, Moscow, 2022.

6. *Realization of surface homeomorphisms of algebraically finite order by Morse-Smale diffeomorphisms with an orientable heteroclinic*, Conference of World-Class International Mathematical Centers, Sochi, 2021.

7. *Realization of homeomorphism of surfaces of algebraically finite type by Morse-Smale diffeomorphisms with orientable heteroclinic*, International conference “Dynamics in Siberia”, Novosibirsk, 2021.

8. *Realization of homeomorphisms of surfaces of algebraically finite type by Morse-Smale diffeomorphisms with an orientable heteroclinic*, “Nonlinear days in Saratov for young people”, Saratov, 2021.

9. *Conjugacy of orientation preserving Morse-Smale diffeomorphisms graphs*, International conference “Dynamics in Siberia”, Novosibirsk, 2020.

10. *Combinatorial invariant for surface Morse-Smale diffeomorphisms with an orientable heteroclinic*, International Conference “KROMSH 2020”, Batiliman, 2020.

11. *Morse-Smale surfaced diffeomorfisms with orientable heteroclinic*, International conference “Mathematical Spring”, Nizhny Novgorod, 2019.

12. *Morse-Smale surfaced diffeomorfisms with orientable heteroclinic*, International conference “Topological methods in dynamics and related topics”, Нижний Новгород, 2018.

The main results of the dissertation were published in eight articles:

1. Morozov A., Pochinka O. Classification of Morse–Smale diffeomorphisms with a finite set of heteroclinic orbits on surfaces // Moscow Mathematical Journal. 2023. Vol. 23. No. 4. P. 571-590.

2. Grines V., Morozov A., Pochinka O. Determination of the Homotopy Type of a Morse-Smale Diffeomorphism on an Orientable Surface by a Heteroclinic Intersection //Qualitative Theory of Dynamical Systems. – 2023. – Vol. 22. – №. 3. – P. 120.

3. Malyshev D., Morozov A., Pochinka O. Combinatorial invariant for Morse–Smale diffeomorphisms on surfaces with orientable heteroclinic //Chaos: An Interdisciplinary Journal of Nonlinear Science. – 2021. – Vol. 31. – №. 2. – P. 023119.

4. Morozov A., Pochinka O. Morse-Smale surfaced diffeomorphisms with orientable heteroclinic //Journal of Dynamical and Control Systems. – 2020. – Vol. 26. – P. 629-639.

5. Grines V., Morozov A., Pochinka O. Realization of homeomorphisms of surfaces of algebraically finite order by Morse–Smale diffeomorphisms with orientable heteroclinic intersection //Proceedings of the Steklov Mathematical Institute. – 2021. – T. 315. – No. 0. – pp. 95-107.

6. Morozov A. Realization of homotopy classes of torus homeomorphisms by the simplest structurally stable diffeomorphisms // Journal of the Middle Volga Mathematical Society. – 2021. – T. 23. – No. 2. – pp. 171-184.

7. Morozov A., Pochinka O. Combinatorial invariant for surface Morse-Smale diffeomorphisms with an orientable heteroclinic // Middle Volga Mathematical Society Journal. – 2020. – T. 22. – No. 1. – pp. 71-80.

8. Morozov A. Determination of the homotopy type of a Morse-Smale diffeomorphism on a 2-torus by heteroclinic intersection // Russian Journal of Nonlinear Dynamics. 2021. Vol. 17. No. 4. P. 465-473.

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