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Integrability property in combinatorics of symmetric groups

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Contents

Introduction

The thesis is devoted to the study of various combinatorial objects and their invariants. The objects of interest arise in the framework of problems on the topology of spaces of maps of 1-dimensional objects over field of real and complex numbers, which are mappings of the circle and complex algebraic curves. In the real case we study invariants of knots in the 3-dimensional sphere, in the complex case we deal with meromorphic functions on algebraic curves. The corresponding combinatorial objects may be naturally described in terms of algebraic structures, Hopf algebras, and by considering generating functions in infinitely many variables. These generating functions often happen to be solutions of integrable hierarchies of partial differential equations, which arise in mathematical physics. This property not only sheds light on the nature of the underlying geometric objects, but also provides some efficient ways for explicitly computing the generating functions.

The weight system $\mathfrak{sl}_2$ is a function on chord diagrams, which satisfies the 4-term relation. For every chord diagram we construct its intersection graph, such that its vertices correspond to chords of the diagram and the two vertices are connected by an edge if the corresponding chords intersect. The 4-term relation on chord diagrams corresponds to the 4-term relation on the intersection graphs. The value of the weight system $\mathfrak{sl}_2$ on a chord diagram is determined by its intersection graph [4]. This leads to the natural question (S.K. Lando): is it possible to extend the weight system $\mathfrak{sl}_2$ on graphs, which satisfy the 4-term relation (on graphs)? We developed the algorithms, which imply the positive answer for this question in the case of graphs with $n \leq 8$ vertices.

The $\mathfrak{sl}_2$-weight system is a specialization of the more general $\mathfrak{gl}$-system. There are good reasons to expect, that the result of averaging the universal $\mathfrak{gl}$-system over the permutations is a τ-function for the Kadomtsev-Petviashvili hierarchy, which may further clarify the nature of the $\mathfrak{sl}_2$-weight system.

Recently, S. V. Chmutov, M. E. Kazarian and S. K. Lando [6] introduced a class of graph invariants called shadow invariants. These invariants are graded homomorphisms from the Hopf algebra of graphs to the Hopf algebra of polynomials on the infinite number of variables. They proved that the result of averaging of almost any such invariant over all graphs, after a suitable rescaling of the variables, turns into a linear combination of one-part Schur functions and thus becomes a τ-function of the integrable Kadomtsev-Petviashvili hierarchy. We prove a similar statement for the Hopf algebra of framed graphs. At the same time, we show that the analogous statement is not true for a number of other Hopf algebras of similar nature, including Hopf algebras of weighted graphs, chord diagrams, and binary delta-matroids. Thus, it turns out that Hopf algebras of graphs and framed graphs play a special role among graded Hopf algebras of combinatorial nature.

Going back to A. Hurwitz, the theory of complex Hurwitz numbers, which enumerates branched coverings of the complex projective line with a prescribed branching data became one of the central areas of mathematics in recent decades. One of the natural directions of development of Hurwitz theory is its extension to the case of real branched coverings of a projective line. The simple real Hurwitz numbers enumerate real meromorphic functions on real algebraic curves, such that all finite critical values being simple. M. E. Kazarian, S. K. Lando and S. M. Natanzon [13] constructed algebras of transition types for which these numbers are structural constants, and derived the transposition equations for their derivative functions. We study the structure of transition type algebras and develop approaches to compute the simple real Hurwitz numbers efficiently.

1 Hopf algebras of combinatorial objects

The structure of many invariants of combinatorial objects is closely related to the structures of the corresponding Hopf algebras. In this section we give descriptions of Hopf algebras of objects of combinatorial nature, which are the subject of the present thesis.

1.1 Definition of a Hopf algebra

Our definition of a Hopf algebra follows [3]. All vector spaces considered in what follows are over a field $\mathbb{F}$ of characteristic 0. For simplicity, we may assume that this is a field of $\mathbb{C}$ of complex numbers.

Let A be a vector space. A *multiplication* ν on a vector space A is a linear map $\nu : A \otimes A \to A$. A multiplication ν *is associative* if the diagram

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\nu \otimes id"] \arrow[d, "id \otimes \nu"'] & A \otimes A \arrow[d, "\nu"] \\
A \otimes A \arrow[r, "\nu"] & A
\end{tikzcd}$$

is commutative. Here and in what follows, id denotes the identity self-map of the vector space. A *unit* for ν is a linear map $\iota : \mathbb{F} \to A$ such that the diagram

$$\begin{tikzcd}
\mathbb{F} \otimes A \arrow[r, "\iota \otimes id"] & A \otimes A \arrow[d, "\nu"] \\
A \arrow[u] \arrow[r, equal] & A
\end{tikzcd}$$

is commutative.

A vector space A together with linear maps $\delta : A \to A \otimes A$ (comultiplication) and $\epsilon : A \to \mathbb{F}$ (counit) is called a *coalgebra* if the following diagrams are commutative:

$$\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "\delta \otimes id"'] \\
A \otimes A \arrow[u, "id \otimes \delta"] & A \arrow[l, "\delta"] \arrow[u, "\delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F} \otimes A \arrow[d] & A \otimes A \arrow[l, "\epsilon \otimes id"'] \\
A \arrow[r, equal] & A \arrow[u, "\delta"']
\end{tikzcd}$$

All algebras and coalgebras considered in this paper are commutative (cocommutative), i.e., the following diagrams are commutative:

$$\begin{tikzcd}
A \otimes A \arrow[r, "\nu"] & A \arrow[d, equal] \\
A \otimes A \arrow[u, "\tau"'] \arrow[r, "\nu"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[d, "\tau"'] & A \arrow[l, "\delta"'] \arrow[d, equal] \\
A \otimes A & A \arrow[l, "\delta"']
\end{tikzcd}$$

where $\tau : A \otimes A \to A \otimes A$ is the transposition of factors in the tensor product, i.e., $\tau(a \otimes b) = b \otimes a$.

A *bialgebra* is a vector space A together with an algebra structure (ν, ι) and a coalgebra structure (δ, ϵ) such that

1. $\epsilon(1) = 1$

2. $\delta(1) = 1 \otimes 1$
3. $\epsilon(ab) = \epsilon(a)\epsilon(b)$
4. $\delta(ab) = \delta(a)\delta(b)$

A bialgebra A is said to be *graded* if it decomposes into a direct sum of vector spaces, i.e.,

$$A = \bigoplus_{k \geq 0} A_k,$$

and multiplication and comultiplication in A are compatible with the grading, i.e.,$\nu(A_k \otimes A_l) \subset A_{k+l}$ for all $k, l = 0, 1, 2, \dots$ and $\delta(A_n) \subset A_0 \otimes A_n + A_1 \otimes A_{n-1} + \dots + A_n \otimes A_0$ for all $n = 0, 1, 2, \dots$.

A graded vector space A is said to be a vector space of *finite type* if all A_n are finite-dimensional.

A graded bialgebra A is *connected* if $\iota : \mathbb{F} \to A$ is an isomorphism between the field $\mathbb{F}$ and $A_0 \subset A$.

A *graded Hopf algebra* is a connected graded bialgebra of finite type together with a linear map $S : A \to A$ such that

$$\nu \circ (S \otimes 1) \circ \delta = \nu \circ (1 \otimes S) \circ \delta = \iota \circ \epsilon.$$

The map S is called the *antipode*.

1.2 The Hopf algebra of polynomials

The simplest example of a Hopf algebra we are interested in is the Hopf algebra of polynomials.

Let $\mathcal{R}$ be an algebra of polynomials in possible infinitely many variables, $\mathcal{R} = \mathbb{C}[q_1, q_2, ...]$. The algebra $\mathcal{R}$ is graded, $\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus ...$, where $\mathcal{R}_n$ is a subspace in $\mathcal{R}$ generated by monomials of degree n. The power n of a monomial $\prod\limits_{a=1}^{b} q_a^{d_a}$ is defined as a sum of powers of its variables, $n = \sum\limits_{a=1}^{b} d_a \deg(q_a)$. Natural powers of variable $\deg(q_a)$ are predefined, and the set of variables with powers no greater than n is finite for every natural number n.

The comultiplication δ of polynomials is a homomorphism of algebras and defined as follows:

$$\delta(q_i) = q_i \otimes 1 + 1 \otimes q_i.$$

The counit is given by the mapping $\epsilon : \mathcal{R} \to \mathbb{C}$, which puts in correspondence to every polynomial $r \in \mathcal{R}$ its free term $r(0, 0, ...)$. Hence the algebra $\mathcal{R}$ has a structure of bialgebra. This bialgebra is commutative and cocommutative, and the multiplication and comultiplication respect the grading. This turns it into the Hopf Algebra of polynomials.The action of the antipode is given by the following: $S(q_i) = -q_i$.

1.3 The Hopf algebra of graphs

The Hopf algebra of graphs was introduced in [10]. Let $\mathcal{G}$ be the vector space over $\mathbb{F}$ generated by simple graphs (graphs without loops and multiple edges). All graphs in this paper considered up to isomorphism. The space $\mathcal{G}$ is graded:

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus ... = \langle \varnothing \rangle \oplus \langle \bullet \rangle \oplus \langle \bullet \quad \bullet \, , \; \bullet\!\!-\!\!\!-\!\!\bullet \rangle \oplus ...,$$

where each $\mathcal{G}_n$ is the finite-dimensional vector space generated by all simple graphs on n vertices.

A multiplication $\nu : \mathcal{G} \otimes \mathcal{G} \longrightarrow \mathcal{G}$ is defined as the disjoint union of graphs; it is extended to the linear combinations of graph by linearity. A comultiplication $\delta : \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{G}$ for graphs is defined as follows: given a graph G,

$$\delta(G) = \sum_{J_1 \sqcup J_2 = V(G)} G|_{J_1} \otimes G|_{J_2};$$

here the summation is over all ordered partitions of the vertex set $V(G)$ of G into two disjoint subsets and $G|_J$, where $J \subset V(G)$, denotes the subgraph of G induced on the set J of vertices. The comultiplication is extended to the linear combinations of graphs by linearity.

Note that the multiplication and comultiplication of graphs are compatible with the grading:

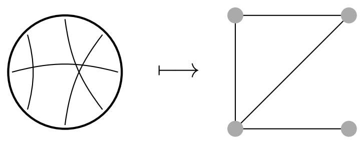
$$\nu : \mathcal{G}_k \otimes \mathcal{G}_l \longrightarrow \mathcal{G}_{k+l},$$

$$\delta : \mathcal{G}_n \longrightarrow \mathcal{G}_0 \otimes \mathcal{G}_n \oplus \mathcal{G}_1 \otimes \mathcal{G}_{n-1} \oplus \cdots \oplus \mathcal{G}_n \otimes \mathcal{G}_0.$$

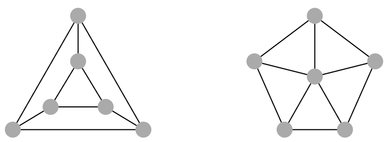
The multiplication and comultiplication operations turn the vector space $\mathcal{G}$ into a graded bialgebra. This bialgebra is commutative and cocommutative; its unit is the empty graph and counit is the map $\epsilon : \mathcal{G} \to \mathbb{F}$ that takes the empty graph to the identity element of the field and each nonempty graph to the zero element. According to the Milnor–Moore theorem [17], any graded cocommutative bialgebra is a Hopf algebra; therefore, we can regard the bialgebra of graphs as a Hopf algebra.

1.4 The Hopf algebra of chord diagrams

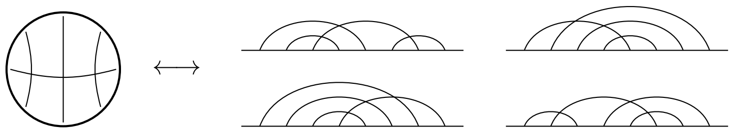
Chord diagram D of order n is oriented circle together with a collection of n chords, considered up to an orientation preserving diffeomorphism of the circles. In the pictures below, we assume that the circle is oriented counterclockwise The *intersection graph* $\Gamma(D)$ of a chord diagram D is a graph whose vertices correspond to the chords of the diagram D, and there is an edge connecting two vertices, provided that the corresponding chords intersect. An example of such correspondence is depicted below.



In contrast, not each graph is the intersection graph of a chord diagram. All graphs with 0 through 5 vertices are intersection graphs. Two graphs with 6 vertices that are not intersection graphs, are depicted below. The percentage of such graphs grows rapidly with the number of vertices.

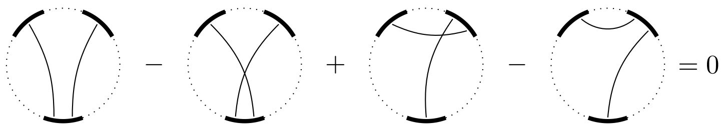


An *arc diagram* is a representation of chord diagram, in which the vertices of the chord diagram are placed along an oriented line with edges drawn as semicircles in one of the two halfplanes bounded by the line. Each arc diagram corresponds to a chord diagram, which is the result of closing a line into a circle. In contrast, a chord diagram with n chords admits up to $2n$ representations as an arc diagram. Any chord diagram can be made into an arc diagram by cutting the circle at some point. For example, to the following chord diagram four arc diagrams are associated.



On the contrary, the closure of a straight line into a circle uniquely associates any arc diagram with a chord diagram.

Denote by $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \ldots$ the graded vector space of chord diagrams; the component of grading n is the vector space $\mathcal{C}_n$ generated by chord diagrams with n chords. Chord diagrams arise naturally in the theory of V.A. Vasiliev of invariants of knots of finite order [22]. In this theory every invariant of knots of order at most n is associated with a function on chord diagrams with n chords satisfying the following 4-term relation:



Here and below the punctured line denotes the parts of the circumference of a chord diagram, where the ends of the fixed set of chords, equal for every diagram, may be placed. The function acting on the chords is omitted from the pictures.

A *product* of chord diagrams is the chord diagram corresponding to the arc diagram obtained by the concatenation of two corresponding arc diagrams. The result of the product of chord diagrams is independent of the choice of the arc representations of the factors modulo four-term relations.

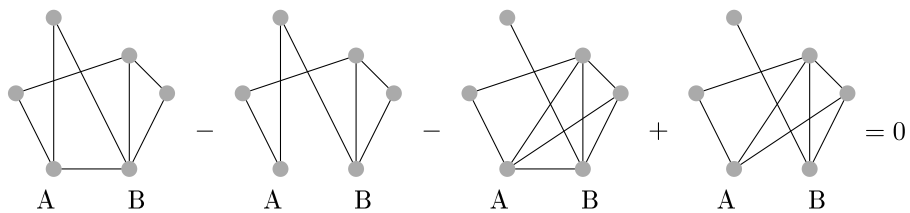
The *coproduct* δ of chord diagram D is defined as follows

$$\delta(D) := \sum_{X \subset V(D)} D|_X \otimes D|_{V(D)\setminus X},$$

where $D|_X$ denotes the chord diagram formed by a subset $X \subset V(D)$ of the set of chords $V(D)$.

Multiplication and comultiplication are extended to linear combinations of chord diagrams by linearity and preserve the grading. These operations turn the vector space $\mathcal{C}$ modulo four-term relations into a graded Hopf algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots$, $\mathcal{A}_i = \mathcal{C}_i / <$4-term relations$>$.

In [14] the *four-term relation for graphs is introduced.*



The relation is constructed as follows. Choose an arbitrary edge, say AB, of the graph — it is the first graph on the left hand-side in the relation above. The second graph on the left hand-side is the same graph with the AB deleted. We now describe how the third and fourth graphs are obtained. Consider the set of edges (excluding AB) sharing a common vertex B. Denote them by $BC_1, BC_2, ..., BC_n$. Now if the vertices A and C_i are connected by an edge in the initial graph, we delete this edge; otherwise – if the vertices A and C_i lack an edge connecting them – we add this edge. In this way the third graph is obtained. The fourth graph differs from the third one in that it lacks the edge AB. The linear combination in the left hand-side in the relation is called *4-term element*.

The factor space of the space of graphs over the subspace generated by the 4-term elements is denoted $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus ...$, где $\mathcal{F}_n = \mathcal{G}_n / <\text{4-term elements}>$. It admits the structure of the Hopf Algebra, induced from the structure on $\mathcal{G}$. The mapping which puts in correspondence to a chord diagram its intersection graph may be naturally extended to a graded homomorphism of Hopf algebras $\mathcal{A} \to \mathcal{F}$.

1.5 The Hopf algebra of framed graphs

The Hopf algebra of framed graphs was introduced in [16] as a tool for constructing invariants of finite order of plane curves. A *framed graph* is a simple graph G together with a *framing*, that is, a map $V(G) \to \{0, 1\}$ from the vertex set $V(G)$ of G to the two-element set $\{0, 1\}$.

Let $\mathcal{G}^f$ be the vector space over $\mathbb{F}$ generated by the isomorphism classes of framed graphs. This space is a Hopf algebra as well; its structure is similar to that of the Hopf algebra $\mathcal{G}$ of graphs.

The space $\mathcal{G}^f$ is graded as

$$\mathcal{G}^f = \mathcal{G}^f_0 \oplus \mathcal{G}^f_1 \oplus \mathcal{G}^f_2 \oplus ... = \langle \varnothing \rangle \oplus \langle \textcircled{0}, \textcircled{1} \rangle \oplus \langle \textcircled{0}\ \ \textcircled{0}, \textcircled{1}\ \ \textcircled{0}, \textcircled{1}\ \ \textcircled{1}, \textcircled{0}\!-\!\textcircled{0}, \textcircled{1}\!-\!\textcircled{0}, \textcircled{1}\!-\!\textcircled{1} \rangle \oplus ...,$$

where each $\mathcal{G}^f_n$ is the finite-dimensional vector space generated by all framed graphs on n vertices.

The multiplication $\nu : \mathcal{G}^f \otimes \mathcal{G}^f \longrightarrow \mathcal{G}^f$ of framed graphs is defined as disjoint union: given graphs G_1 and G_2,

$$\nu(G_1, G_2) = G_1 \sqcup G_2.$$

The comultiplication $\delta : \mathcal{G}^f \longrightarrow \mathcal{G}^f \otimes \mathcal{G}^f$ of framed graphs is defined by setting, given a framed graph G,

$$\delta(G) = \sum_{J_1 \sqcup J_2 = V(G)} G|_{J_1} \otimes G|_{J_2},$$

where the summation is over all ordered partitions of the vertex set $V(G)$ of G into two disjoint subsets and $G|_J$ denotes the framed subgraph of G induced on the set J of vertices. The framings of vertices in induced subgraphs are preserved. The comultiplication is extended to the linear combinations of framed graphs by linearity.

Like in the case of simple graphs, the multiplication and comultiplication of framed graphs are compatible with the grading:

$$\nu : \mathcal{G}^f_{l_1} \otimes \mathcal{G}^f_{l_2} \longrightarrow \mathcal{G}^f_{l_1+l_2},$$

$$\delta : \mathcal{G}^f_n \longrightarrow (\mathcal{G}^f_0 \otimes \mathcal{G}^f_n) \oplus (\mathcal{G}^f_1 \otimes \mathcal{G}^f_{n-1}) \oplus \cdots \oplus (\mathcal{G}^f_n \otimes \mathcal{G}^f_0).$$

The unit, counit, and antipode are introduced by analogy with the corresponding elements of the Hopf algebra of simple graphs.

A *framed chord diagram* is a chord diagram of order n with each chord marked by an element of $\{0, 1\}$. The 4-term relations for framed graphs and framed chord diagrams are introduced in [16], where the structure of a Hopf algebra on the graded vector spaces is also described, obtained as a factor by the 4-term relations.

2 The $\mathfrak{sl}_2$ weight system on graphs

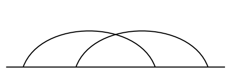
One of the major sources of weight systems is Lie algebras. The simplest non-trivial Lie algebra provides a fundamental example of the weight system $\mathfrak{sl}_2$. Even this seemingly basic case exhibits complex and interesting behavior. We are interested in its possible extensions to the invariant on graphs, which satisfy the 4-term relation on them.

2.1 The $\mathfrak{sl}_2$ weight system on chord diagrams

Let $\mathfrak{G}$ be a Lie algebra, $\dim \mathfrak{G} = m$, endowed with a nondegenerate invariant bilinear form $(\cdot,\cdot)$. Invariance means that $(x,[y,z]) = ([x,y],z)$ for all $x,y,z \in \mathfrak{G}$. Let $U(\mathfrak{G})$ be an universal enveloping algebra of the algebra Lie $\mathfrak{G}$.

Pick an orthonormal basis $\{e_1,...,e_m\}$ in $\mathfrak{G}$ with respect to the scalar product $(\cdot,\cdot)$. Consider the mapping $w_{\mathfrak{G}} : \mathcal{A} \to U(\mathfrak{G})$ of the algebra of chord diagrams modulo 4-term relations to the universal enveloping algebra of algebra Lie $\mathfrak{G}$. Fix an arc diagram a, assosiated to the chord diagram D. We construct an element $w_{\mathfrak{G}}(D)$ of the universal enveloping algebra $U(\mathfrak{G})$ as follows. For a given mapping ϕ of the set of arcs of the diagram a to the set $\{1,...,m\}$, at the ends of each arc we place an element $e_i \in \mathfrak{G}$ if this arc goes to i. The summation over all such mappings gives us the image of the chord diagram D in the universal enveloping algebra $U(\mathfrak{G})$. We extend the mapping $w_{\mathfrak{G}}$ to the whole $\mathcal{A}$ by linearity.

For example, for $m = 3$:



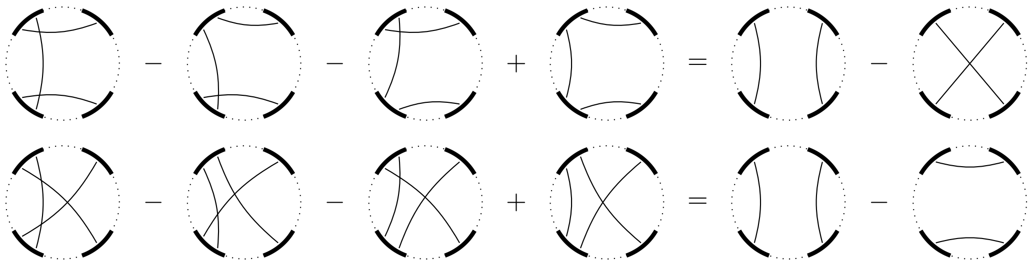
$$\longmapsto \quad \begin{array}{c} e_1^4 + e_1e_2e_1e_2 + e_1e_3e_1e_3 + e_2e_1e_2e_1 + e_2^4 + \\ + e_2e_3e_2e_3 + e_3e_1e_3e_1 + e_3e_2e_3e_2 + e_3^4 \end{array}$$

Theorem 1. [2, 11] *Let $\mathfrak{G}$ be a Lie algebra together with the nondegenerate invariant scalar product $(\cdot,\cdot)$. Then the mapping $w_{\mathfrak{G}} : \mathcal{A} \to U(\mathfrak{G})$ possesses the following properties:*
(1) the value $w_{\mathfrak{G}}(D)$ of the mapping $w_{\mathfrak{G}}$ does not depend on the choice of the orthonormal basis $e_1,...,e_m$;
(2) the value $w_{\mathfrak{G}}(D)$ of the mapping $w_{\mathfrak{G}}$ does not depend on choosing an arc representation of chord diagram D;
(3) the image of $w_{\mathfrak{G}}$ lies in the center of the universal enveloping algebra $U(\mathfrak{G})$;
(4) the mapping $w_{\mathfrak{G}}$ satisfies the 4-term relation for chord diagrams.

Note that if a chord diagram is the product of two nonempty diagrams, then the value of the map $w_{\mathfrak{G}}$ on it is the product of its values on the factors. Thus, $w_{\mathfrak{G}}$ is an algebra homomorphism, $w_{\mathfrak{G}} : \mathcal{A} \to ZU(\mathfrak{G})$.

In the simplest nontrivial case, namely, in the case of the Lie algebra $\mathfrak{sl}_2$ and the Killing form, the center of the universal enveloping algebra $U(\mathfrak{sl}_2)$ is generated by a single element, the *Casimir element* $c = e_1^2 + e_2^2 + e_3^2$, $ZU(\mathfrak{sl}_2) = \mathbb{C}[c]$. In this particular case, the function $w_{\mathfrak{G}}$ is determined by the Chmutov–Varchenko recurrence relations, which can be considered as an alternative definition of the weight system $\mathfrak{sl}_2$.

Let v denote the *weight system* $\mathfrak{sl}_2$. It associates to a chord diagram with n chords a polynomial of degree n in the variable c. The value of v on a chord diagram with one chord equals c. If a chord diagram contains a chord that intersects precisely one other chord, in which case we call the former chord a *leaf*, then the value of v on the initial chord diagram is equal to that on the chord diagram obtained from the initial one by deleting the leaf times $(c-1)$. If a chord diagram contains no leaves, then the *Chmutov–Varchenko recurrence relations* for the values of v on it hold:



By means of these relations, the value of the weight system $\mathfrak{sl}_2$ on any chord diagram can be computed recursively. However, complexity of such a computation is exponential: at each step the diagram is replaced with 5 simpler diagrams.

If a chord diagram is a product of two non-empty diagrams, then the value of the $\mathfrak{sl}_2$ weight system on it is a product of values on its factors.

Theorem 2. [5]
(1) *The function v defined by the recurrence relations above is well-defined.*
(2) *The function v coincides with the weight system constructed from the Lie algebra $\mathfrak{sl}_2$.*

2.2 The problem of the extension of the $\mathfrak{sl}_2$ weight system on graphs

A function on graphs, which satisfies the 4-term relations, is referred to as a *4-invariant of graphs.* One of the first examples of a 4-invariant is the chromatic polynomial for graphs. Every 4-invariant of graphs defines a weight system: the value of the weight system on a chord diagram is given by the value of the 4-invariant on its intersection diagram.

The following assertion allows one to define the value of the weight system $\mathfrak{sl}_2$ on intersection graphs.

Theorem 3. [4] *The value of the weight system $\mathfrak{sl}_2$ on a chord diagram is determined by the intersection graph of the diagram.*

The statement above leads to the natural question (S.K. Lando): is there a 4-invariant of graphs, such that its values on the intersection graphs equals to the values of the $\mathfrak{sl}_2$-weight system on them?

In spite of a large number of works on this question, it remains open. Our major result gives positive answer for this question in case of graphs with ≤ 8 vertices.

Theorem EK21-1. *There is a polynomial graph invariant of graphs with up to 8 vertices satisfying the 4-term relations whose values on intersection graphs coincide with that of the weight system $\mathfrak{sl}_2$; such a graph invariant is unique.*

The question of existence and uniqueness of an extension for graphs with more vertices remains open and is to be studied further. One of possible ways to approach this case is to define the $\mathfrak{sl}_2$-weight system on graphs based on its known values on some series of graphs, see e.g.[23].

The proof of the theorem EK21-1 is based on machine computations.

3 Generating functions of combinatorial objects as solutions of KP hierarchy

The Kadomtsev–Petviashvili hierarchy (below, KP) is an integrable system of partial differential equations for functions depending on infinitely many variables. Combinatorial solutions of this hierarchy are well described, for example, in [12]. The lowest KP hierarchy equation has the form

$$\frac{\partial^2 F}{\partial p_2^2} = \frac{\partial^2 F}{\partial p_1 \partial p_3} - \frac{1}{2}\left(\frac{\partial^2 F}{\partial p_1^2}\right)^2 - \frac{1}{12}\frac{\partial^4 F}{\partial p_1^4}.$$

In this section we provide the description of the space of solutions of KP hierarchy and study some families of functions from this the space, which are related to the considered combinatorial objects .

3.1 Semi-infinite wedge power

Let V be the infinite-dimensional space of Laurent series in one variable z. By definition, the *half-infinite wedge power* $\Lambda^{\infty/2}V$ is the vector space spanned by the vectors

$$v_\mu = z^{m_1} \wedge z^{m_2} \wedge z^{m_3} \wedge \dots, \quad m_1 > m_2 > m_3 > \dots, \quad m_i = \mu_i - i,$$

where $\mu = (\mu_1, \mu_2, \mu_3, \dots)$, $\mu_1 \geq \mu_2 \geq \mu_3 \geq \dots \geq 0$, is a partition in which all but finitely many parts equal 0. In particular, the empty partition $\mu = (0, 0, 0, \dots) = \emptyset$ corresponds to the vacuum vector $v_\varnothing = z^{-1} \wedge z^{-2} \wedge z^{-3} \wedge \dots$.

3.2 Schur polynomials

Let $\mu \vdash n$ be a partition. The *Schur polynomial* $\mathcal{S}_\mu$ is defined as follows.

- For a single-part partition $n^1 \vdash n$, the Schur polynomial $\mathcal{S}_n$ is determined from the decomposition

$$\begin{aligned} \mathcal{S}_0 + \mathcal{S}_1 z + \mathcal{S}_2 z^2 + \mathcal{S}_3 z^3 + \dots &= \exp(p_1 z + p_2 \frac{z^2}{2} + p_3 \frac{z^3}{3} + \dots) \\ &= 1 + p_1 z + \frac{1}{2}(p_1^2 + p_2) z^2 + \dots. \end{aligned}$$

  Thus,

$$\begin{aligned} \mathcal{S}_0 &= 1 \\ \mathcal{S}_1 &= p_1 \\ \mathcal{S}_2 &= \frac{1}{2}(p_1^2 + p_2) \\ \dots &= \dots \\ \mathcal{S}_n &= \frac{1}{n!}\sum_{\alpha \vdash n} \prod_{\alpha_i \in \alpha} (\alpha_i - 1)! p_{\alpha_i} \\ \dots &= \dots \end{aligned}$$

- For any partition $\mu = (\mu_1, \mu_2, \mu_3, \dots), \mu_1 \geq \mu_2 \geq \mu_3 \geq \dots$, the Schur polynomial $\mathcal{S}_\mu$ is a determinant:

$$\mathcal{S}_\mu = \det ||\mathcal{S}_{\mu_j - j + i}||.$$

3.3 The solution space of a KP hierarchy

We say that a function is a *solution* of a KP hierarchy if it belongs to the space of solutions of this hierarchy. The space of (formal) solutions of a KP hierarchy can be described by using the boson–fermion correspondence ϕ; see [12] for details.

Consider an isomorphism $\phi : \Lambda^{\infty/2}V \to \mathbb{F}[p_1, p_2, \dots]$ between the half-infinite wedge power and the space of power series in infinitely many variables. For any partition μ, this isomorphism takes the basis vector v_μ corresponding to μ to the Schur polynomial $\mathcal{S}_\mu$. To the half-infinite plane spanned by vectors $\beta_1(z), \beta_2(z), \dots$ we assign the vector $\beta_1(z) \wedge \beta_2(z) \wedge \dots \in \Lambda^{\infty/2}V$ (this is Plucker's embedding). We represent every such vector as a linear combination of the basis vectors of the space $\Lambda^{\infty/2}V$, divide this linear combination by the coefficient of the vacuum vector, and replace each basis vector in it by the corresponding Schur polynomial. The formal power series in $p_1, p_2, \dots$ thus obtained forms the space of τ-functions of the KP hierarchy, and their logarithms form the space of its solutions.

It is well known that any linear combination $\sum_{i=0}^{\infty} a_i \mathcal{S}_i$ of single-part Schur polynomials in which $\mathcal{S}_0$ has coefficient 1 is a τ-function of the KP hierarchy.

3.4 Family of KP hierarchy solutions

In [6] the Hopf algebra of graphs was associated with a solution of an integrable Kadomtsev–Petviashvili hierarchy of partial differential equations.

Let $\mathbb{C}[q_1, q_2, q_3, ...]$ be a graded Hopf algebra of polynomials, where the weight of the variable q_i equals i, $i = 1, 2, 3, ...$

Theorem 4. [6] *Let I be a graph invariant with values in the ring of polynomials in infinitely many variables $q_1, q_2, \dots$, $I : G \mapsto I_G(q_1, q_2, \dots)$, extending to a graded homomorphism of Hopf algebras. Suppose also that all numbers i_n in defined by*

$$i_n = n! \sum_{G, |V(G)|=n} \frac{[q_n] I_G(q_1, q_2, \dots)}{|\operatorname{Aut}(G)|}$$

(here $|V(G)|$ denotes the number of vertices in the graph G, $[q_n]P$ is the coefficient of the monomial q_n in a polynomial $P = P(q_1, q_2, \dots)$, and $|\operatorname{Aut}(G)|$ is the order of the automorphism group of G) are nonzero.

Consider the generating functions

$$\begin{aligned} \mathcal{I}^\circ(q_1, q_2, \dots) &= \sum_G \frac{I_G(q_1, q_2, \dots)}{|\operatorname{Aut}(G)|}, \\ \mathcal{I}(q_1, q_2, \dots) &= \sum_{G \text{ -- connected}} \frac{I_G(q_1, q_2, \dots)}{|\operatorname{Aut}(G)|}, \end{aligned}$$

where the first sum is over all graphs and the second, over all connected graphs. Then, after rescaling the variables as $q_n = \frac{2^{n(n-1)/2}(n-1)!}{i_n} p_n$, the generating function $\mathcal{I}$ becomes a solution of the KP hierarchy with respect to the new variables p_i, and $\mathcal{I}^\circ$ becomes a τ-function of the KP hierarchy. This τ-function does not depend on the chosen invariant I.

Among the graph invariants satisfying the assumptions of the theorem are many important ones, such as the Stanley symmetrized chromatic polynomial [21], the Abel polynomial introduced in [6], and many other invariants. The technique of combinatorial Hopf algebras developed in [1] makes it possible to construct such invariants from any multiplicative graph invariant.

Note that the generating functions $\mathcal{I}^\circ$ and $\mathcal{I}$ are connected by the standard relation

$$\mathcal{I} = \log \mathcal{I}^\circ.$$

between a τ-function and the corresponding solution of the hierarchy.

The theorem stated above gives rise to the natural question: What Hopf algebras of combinatorial nature other that the Hopf algebra of graphs have the same property? We show that a similar assertion holds for the Hopf algebra of framed graphs, which was introduced by Lando in [16], and does not hold for a whole series of other Hopf algebras of a similar nature, including the Hopf algebras of weighted graphs, of chord diagrams, and of binary delta-matroids.

Let $I^f : \mathcal{G}^f \to \mathbb{F}[q_1, q_2, \dots]$ be a graded homomorphism of the Hopf algebra of framed graphs to the Hopf algebra of polynomials in infinitely many variables, and let $I^f_G(q_1, q_2, \dots)$ be the invariant of the framed graph G being the value of this homomorphism at G.

Consider the generating functions

$$\begin{aligned} \mathcal{I}^{f\circ}(q_1, q_2, \dots) &= \sum_G \frac{I^f_G(q_1, q_2, \dots)}{|\operatorname{Aut}(G)|}, \\ \mathcal{I}^f(q_1, q_2, \dots) &= \sum_{G \text{ — connected}} \frac{I^f_G(q_1, q_2, \dots)}{|\operatorname{Aut}(G)|}, \end{aligned}$$

here the first sum is over all framed graphs and the second is over all connected framed graphs, and $|\operatorname{Aut}(G)|$ denotes the order of the automorphism group of a framed graph G (that is, the group of the framing-preserving automorphisms of G). As in the case of usual graphs, we have

$$\mathcal{I}^f = \log \mathcal{I}^{f\circ}.$$

Let us define constants i^f_n, $n = 0, 1, 2, \dots$ by

$$i^f_n = n! \sum_{\substack{G \text{ — connected} \\ |V(G)|=n}} \frac{[q_n] I^f_G(q_1, q_2, \dots)}{|\operatorname{Aut}(G)|},$$

where each $[q_n]P$ denotes the coefficient of the monomial q_n in the polynomial $P = P(q_1, q_2, \dots)$. The main result of this paper is the following theorem.

Theorem EK19-1. *If $i^f_n \neq 0$ for all $n = 0, 1, 2, \dots$, then, after the rescaling $q_n = \frac{2^{n(n-1)/2}(n-1)!}{i^f_n} p_n$ of the variables, the generating function $\mathcal{I}^f$ becomes a solution of the KP hierarchy in the variables p_n and $\mathcal{I}^{f\circ}$ becomes a τ-function of the KP hierarchy. This τ-function is a linear combination of single-part Schur polynomials.*

Remark. This result, as well as its proof, does not change under the replacement of the Hopf algebra of framed graphs by the Hopf algebra of graphs with vertices marked by the elements of an arbitrary finite set rather than by the elements of the set $\{0, 1\}$ (the multiplication and comultiplication in this Hopf algebra are defined in a similar way). We, however, consider only the case of framed graphs, because this Hopf algebra is related to invariants of knots and plane curves. In particular, it is this Hopf algebra which is used in [18] and [15] to construct extensions of graph invariants to invariants of embedded graphs and binary delta-matroids and, as a consequence, of knot invariants to link invariants.

Similar to the Theorem EK19-1 statements do not hold for the Hopf algebra of weighted graphs. A *weighted graph* is a graph in which every vertex is assigned a positive integer.

Theorem EK19-2. *Let $I^w : \mathcal{G}^w \to \mathbb{F}[q_1, q_2, \dots]$ be a graded homomorphism of the Hopf algebra of weighted graphs to the Hopf algebra of polynomials in infinitely many variables, and let $I^w_G(q_1, q_2, \dots)$ be the corresponding invariant of a weighted graph G.*

Consider the generating functions

$$\begin{aligned} \mathcal{I}^{w\circ}(q_1, q_2, \dots) &= \sum_G \frac{I^w_G(q_1, q_2, \dots)}{|\operatorname{Aut}(G)|}, \\ \mathcal{I}^{w}(q_1, q_2, \dots) &= \sum_{G\,-\,connected} \frac{I^w_G(q_1, q_2, \dots)}{|\operatorname{Aut}(G)|}, \end{aligned}$$

where the first sum is over all weighted graphs, the second sum is over all connected weighted graphs, and $|\operatorname{Aut}(G)|$ denotes the order of the automorphism group of the weighted graph G.

Whatever rescaling of variables $q_n = a_n p_n$, $a_n \in \mathbb{F}$, $a_n \neq 0$, $n = 0, 1, 2, \dots$, the generating function $\mathcal{I}^{w\circ}$ is not a linear combination of single-part Schur polynomials. Moreover, whatever rescaling of variables $q_n = a_n p_n$, $a_n \in \mathbb{F}$, the generating function $\mathcal{I}^{w\circ}$ is not a τ-function of the KP hierarchy in the variables p_n and, therefore, $\mathcal{I}^w$ is not a solution of the KP hierarchy.

4 Hurwitz numbers

Another set of results of this thesis refers to Hurwitz numbers. Complex Hurwitz numbers and their various generalizations play key roles in the study of the geometry of moduli spaces of algebraic curves [8], in the topological recursion theory [7] and in various enumerative problems. Their real analogues are much less studies, in spite of a broad range of possible applications. Below we provide some key notions and major theorems from the theory of complex Hurwitz numbers, then we state the results on the real analogues.

4.1 Complex Hurwitz numbers

Complex Hurwitz numbers enumerate meromorphic functions with a given set of critical values, ramification over each is prescribed. *General Hurwitz numbers* for a tuple $(\mu_1, ..., \mu_m)$ of partitions of d is the sum

$$\sum_{f:M\to S^2} \frac{1}{|\operatorname{Aut}(f)|},$$

where the summation is over all ramified coverings $f : M \to S^2$ of the sphere S^2 by the surface M of degree d with prescribed ramification types $(\mu_1, ..., \mu_m)$ over the marked points $t_1, ..., t_m \in S^2$. The *connected Hurwitz number* is defined in a similar way, but the covering surface is connected.

The *simple Hurwitz numbers* $h^\circ_{m,\mu}$ enumerate ramified coverings of the sphere S^2 with a given set of critical values, ramification over one of which is a prescribed partition μ, while all the other m critical values are simple. The *simple Hurwitz numbers* are equal

$$h^\circ_{m,\mu} = \frac{1}{n!}|\{(\tau_1, ...\tau_m), \tau_i \in C_2(\mathbb{S}_n)|\tau_m \circ ... \circ \tau_1 \in C_\mu(\mathbb{S}_n)\}|,$$

here $C_2(\mathbb{S}_n)$ denotes the set of all transpositions in $\mathbb{S}_n$, and $C_\mu(\mathbb{S}_n)$ is the set of all permutations of cyclic type $\mu \vdash n$ in $\mathbb{S}_n$. *The connected simple Hurwitz numbers* are equal

$$h_{m,\mu} = \frac{1}{n!}|\{(\tau_1, ...\tau_m), \tau_i \in C_2(\mathbb{S}_n)|\tau_m \circ ... \circ \tau_1 \in C_\mu(\mathbb{S}_n), \langle\tau_1, ...\tau_m\rangle \text{ acts transitively}\}|.$$

Consider the exponential generating functions for simple Hurwitz numbers:

$$H^\circ(u;p_1,p_2...)=\sum_{m=0}^{\infty}\sum_{\mu}h^\circ_{m,\mu}p_{\mu_1}p_{\mu_2}...\frac{u^m}{m!};$$

$$H(u;p_1,p_2...)=\sum_{m=0}^{\infty}\sum_{\mu}h_{m,\mu}p_{\mu_1}p_{\mu_2}...\frac{u^m}{m!},$$

where in each case μ runs over the set of all partitions of all numbers.

A very general combinatorial construction relating connected and disconnected objects justifies the following relationship between these two generating functions:

Theorem 5. *The following equation holds:* $H^\circ=\exp(H)$.

We have

Theorem 6. [20] *The generating function H° is a 1-parameter family of τ-functions to the Kadomtsev–Petviashvili hierarchy and the generating function H is a 1-parameter family of solutions to the KP hierarchy.*

Denote by $H^\circ_m(p_1,p_2...)$ the coefficient of $\frac{u^m}{m!}$ in generating function $H^\circ(u;p_1,p_2...)$.

$$H^\circ(u;p_1,p_2...)=\sum_{m=0}^{\infty}H^\circ_m(p_1,p_2...)\frac{u^m}{m!}.$$

Note, that $H^\circ_0=H^\circ(0;p_1,p_2...)=e^{p_1}$. The following theorem states the way to compute the Hurwitz numbers recursively, given this base case.

Theorem 7. [9] *(Goulden and Jackson) The generating function H° for simple Hurwitz numbers satisfies the following partial differential equation:*

$$\frac{\partial H^\circ}{\partial u}=WH^\circ,$$

where $W=\frac{1}{2}\sum_{n=1}^{\infty}\sum_{i+j=n}\left((i+j)p_ip_j\frac{\partial}{\partial p_{i+j}}+ijp_{i+j}\frac{\partial^2}{\partial p_i\partial p_j}\right)$ *is a transposition operator (cut-and-join operator).*

Следствие 8. *The cut-and-join equation can be rewritten as the recurrence* $H^\circ_{m+1}=WH^\circ_m$.

Okounkov's theorem 6 is derived from the theorem 7: Schur functions form an eigenbasis of the transposition operator.

4.2 Real meromorphic functions

Real Hurwitz numbers enumerate real meromorphic functions on real algebraic curves. We introduce different kinds of real Hurwitz numbers,for different classes of real meromorphic functions.

For an antiholomorphic involution $\tau:C\to C$, the pair (C,τ) is called a *real algebraic curve.* Denote by C^τ the set of fixed points of the involution τ. A real curve (C,τ) is said to be *separating* provided $C\backslash C^\tau$ is disconnected, and *nonseparating* otherwise. For a separating real curve, the complement $C\backslash C^\tau$ consists of two connected components. By a *framing* of a separating real curve we mean a choice of one of the two components of $C\backslash C^\tau$.

A *real holomorphic mapping* of a real curve (C_1, τ_1) to a real curve (C_2, τ_2) is a holomorphic mapping $f : C_1 \to C_2$, such that $f \circ \tau_1 = \tau_2 \circ f$. In particular, a *real meromorphic function* on (C, τ) is a real meromorphic mapping from (C, τ) to $(\mathbb{C}P^1, \sigma)$, where $\sigma : \mathbb{C}P^1 \to \mathbb{C}P^1$ is the standard complex conjugation, $\sigma : z \mapsto \overline{z}$.

A real meromorphic function $f : C \to \mathbb{C}P^1$ is said to be *simple* if all its finite critical values are simple. A real meromorphic function $f : C \to \mathbb{C}P^1$ is said to be *purely real* if all its finite critical values are real.

4.3 Case of separating real curves

4.3.1 Purely real simple Hurwitz numbers

Framed real meromorphic function is a real meromorphic function $f : (C, \tau) \to (\mathbb{C}P^1, \sigma)$ defined on a framed separating real curve (C, τ). Denote by C^f the connected component, chosen by the framing.

We are going to define the ramification type of a framed real meromorphic function $f : (C, \tau) \to (\mathbb{C}P^1, \sigma)$ over a point $\infty \in \mathbb{R}P^1$. The poles of f are split into real ones and pairs of τ-conjugate nonreal poles. In each pair exactly one of τ-conjugate poles belongs to the domain C^f. The orders of the τ-conjugate poles form a partition $\lambda = (\ell_1, \ell_2, ...)$. A real pole of the function f is said to be *positive* if the function f increases to the left of the pole, and *negative*, if the function f decreases to the left. The orders of positive and negative real poles form partitions $\kappa^+ = (k_1^+, k_2^+, ...)$ and $\kappa^- = (k_1^-, k_2^-, ...)$ respectively. The *ramification type* of f over infinity is the triple of partitions $\mu = (\kappa^+, \kappa^-, \lambda)$.

Framed simple purely real connected Hurwitz numbers $h^{\mathbb{R}\circ}_{m;\mu}$ enumerating real meromorphic functions having ramification type $\mu = (\kappa^+, \kappa^-, \lambda)$ over infinity and m given nondegenerate real critical values. Formally,

$$h^{\mathbb{R}\circ}_{m;\mu} = \sum_f \frac{1}{\#\mathrm{Aut}(f)},$$

where $\#\mathrm{Aut}(f)$ denotes the order of the automorphism group of the function f.

We denote by $h^{\mathbb{R}}_{m;\mu}$ the number of all simple framed purely real meromorphic functions with connected domain.

4.3.2 Transposition operator

Let us associate with the ramification type

$$\mu = (\kappa^+, \kappa^-, \lambda) = ((k_1^+, k_2^+, ...), (k_1^-, k_2^-, ...), (\ell_1, \ell_2, ...))$$

the monomial

$$p_\mu = p^+_{k_1^+} p^+_{k_1^+} ... p^-_{k_1^-} p^-_{k_2^-} ... q_{\ell_1} q_{\ell_2} ...$$

in the variables $p_i^+, p_i^-, q_i, i = 1, 2, ...$ Introduce the generating functions

$$H^{\mathbb{R}}(u; p_1^+, ..., p_1^-, ..., q_1 ...) = \sum_{m=0}^{\infty} \sum_{\mu} h^{\mathbb{R}}_{m;\mu} p_\mu \frac{u^m}{m!},$$

$$H^{\mathbb{R}\circ}(u; p_1^+, ..., p_1^-, ..., q_1 ...) = \sum_{m=0}^{\infty} \sum_{\mu} h^{\mathbb{R}\circ}_{m;\mu} p_\mu \frac{u^m}{m!};$$

where the summation on the right runs over all triples of partitions $\mu = (\kappa^+, \kappa^-, \lambda)$ and all nonnegative values of m.

Theorem 9. *We have $H^{\mathbb{R}\circ} = \exp(H^{\mathbb{R}})$.*

The following theorem introduces the transposition equation for the real case — the analogue of Goulden-Jackson equation for complex Hurwitz numbers.

Theorem 10. [13] *The generating function $H^{\mathbb{R}\circ}$ satisfies the differential equation*

$$\frac{\partial H^{\mathbb{R}\circ}}{\partial u} = W^+(H^{\mathbb{R}\circ}),$$

where $W^+ = \sum_{i,j=1}^{\infty} \left(p_i^{\bar{i}} p_j^+ \frac{\partial}{\partial p_{i+j}^{\bar{i}}} + p_{i+j}^{\bar{i}} \frac{\partial^2}{\partial p_i^{\bar{i}} \partial p_j^+} \right) + \sum_{i=1}^{\infty} \left(i p_{2i}^+ \frac{\partial}{\partial q_i} + q_i \frac{\partial}{\partial p_{2i}^+} \right)$, where, for a positive integer i, notation $\bar{i}$ stands for the sign $+$ provided i is even, and for the sign $-$ otherwise.

The action of the operator W^+ on the initial condition $H^{\mathbb{R}\circ}(0, p_1^{\pm}, p_2^{\pm}, \ldots) = e^{p_1^+ + p_1^- + q_1}$ allows one to compute as many terms of the power series $H^{\mathbb{R}\circ}$ as we like,

$$H^{\mathbb{R}\circ}(u, p_1^{\pm}, p_2^{\pm}, \ldots) = e^{uW^+} e^{p_1^+ + p_1^- + q_1}.$$

4.3.3 Action of the transposition operator

Pick a finite set N of n elements and a representation of N as a disjoint union $N = N^+ \sqcup N^-$ of two subsets N^+ and N^- consisting of n^+ and n^- elements, respectively, $n^+ + n^- = n$. A *state* is a partition of $N = \{1, 2, 3, \ldots, n\}$ into a disjoint union of one and two-element subsets such that each two-element subset contains one element from N^+ and one from N^-. A *transition* is an ordered pair of states. The *type of a transition* is its orbit under the action of the group $\mathbb{S}_{n^+} \times \mathbb{S}_{n^-}$ acting on the set of transitions by separately permuting the elements in N^+ and N^-.

The algebra types of transitions A_{n^+,n^-} is introduced in [13], and the differential operator on the right side of the transposition equation is interpreted as an operator of multiplication by the transposition class in this algebra. It is shown, the operator W^+ on the vector space A_{n^+,n^-} is self-adjoint with respect to a non-degenerate scalar product on A_{n^+,n^-}, hence it has an eigenbasis. Explicit computation of an eigenbasis of the operator W^+ is tedious, since it requires computations of the roots of polynomials of increasing powers, with ration coefficients; instead of computing the eigenvalues and finding the eigenbasis of the operator W^+ we represent it in the block form, which allows to compute the real Hurwitz numbers efficiently.

A_{n^+,n^-} algebra can be presented as a direct sum of subspaces, generated by transitions, such that their left states contain t two-element subsets, $A_{n^+,n^-} = \oplus_t A_{n^+,n^-,t}$. The subspaces $A_{n^+,n^-,t}$ are invariant with respect to the action of the operator W^+.

In sections 3.1.3, 3.1.4 of the major part of the thesis we study properties of representations of a product of two symmetric groups in the algebra A_{n^+,n^-}, which is isomorphic the algebra of polynomials in variables p_k^+, p_k^-, q_l. We describe the decomposition of the transposition operator in the direct sum of operators, corresponding to isotypical decompositions of representations. We show that such operators written in an appropriate basis are matrices with integer entries and then provide the explicit ways to compute them.

Theorem EK23-3. *Action of the operator W^+ on the space of polynomials of degree $n = n^+ + n^-$ can be decomposed as a direct sum of its action on the subspaces of isotypical representations of the group $\mathbb{S}_{n^+} \times \mathbb{S}_{n^-}$.*

In sections 3.1.1, 3.1.2 we describe the methods to decompose the action of the operator W^+ as a direct sum of its actions on the isotypical subspaces, in section 3.1.5 we provide an example of such decomposition for the case of $A_{2,4}$ algebra.

4.4 The case of not necessarily separating real curves

4.4.1 Purely real simple Hurwitz numbers

We are going to define the ramification type of the real meromorphic function f. The poles of f are split into real ones and pairs of complex-conjugate nonreal poles. The sign of a real pole is well defined for a pole of even order only: it is *positive* if the corresponding critical point is a local minimum, and *negative* in case of local maximum. Thus, the *ramification type* of function f at infinity is a quadruple

$$\mu = (\kappa^+, \kappa^-, \kappa, \lambda) = ((k_2^+, k_4^+, ...), (k_2^-, k_4^-, ...), (k_1, k_3, ...), (\ell_1, \ell_2, \ell_3, ...)),$$

where κ^+ and κ^- are the partitions formed by the even parts corresponding to the orders of positive and negative real poles, respectively, κ is the partition formed by odd parts corresponding to the orders of poles of odd orders, and λ is a partition formed by the orders of pairs of conjugate non-real poles.

Simple purely real Hurwitz numbers $\tilde{h}^{\mathbb{R}\circ}_{m;\mu}$ enumerating real meromorphic functions having ramification type $\mu = (\kappa^+, \kappa^-, \kappa, \lambda)$ over infinity and m given nondegenerate real critical values. We denote by $\tilde{h}^{\mathbb{R}}_{m;\mu}$ the simple purely real connected Hurwitz numbers with connected domain.

4.4.2 Transposition operator

Let us associate with the ramification type $\mu = (\kappa^+, \kappa^-, \kappa, \lambda)$ the monomial

$$p_\mu = p^+_{k_2^+} p^+_{k_4^+} ... p^-_{k_2^-} p^-_{k_4^-} ... p_{k_1} p_{k_3} ... q_{\ell_1} q_{\ell_2} ...$$

in the variables $p^+_{2i}, p^-_{2i}, p_{2i-1}, q_i, i = 1, 2, ...$ The generating functions in the case of not necessarily separating real curves have the form

$$\tilde{H}^{\mathbb{R}}(u; p_2^+, ..., p_2^-, ..., p_1 ..., q_1 ...) = \sum_{m=0}^{\infty} \sum_{\mu} \tilde{h}^{\mathbb{R}}_{m;\mu} p_\mu \frac{u^m}{m!},$$

$$\tilde{H}^{\mathbb{R}\circ}(u; p_2^+, ..., p_2^-, ..., p_1 ..., q_1 ...) = \sum_{m=0}^{\infty} \sum_{\mu} \tilde{h}^{\mathbb{R}\circ}_{m;\mu} p_\mu \frac{u^m}{m!};$$

where the summation runs over all quadruples of partitions $\mu = (\kappa^+, \kappa^-, \kappa, \lambda)$ and all nonnegative values of m.

Theorem 11. [19] [13] *The generating function $\tilde{H}^{\mathbb{R}\circ}$ satisfies the differential equation*

$$\frac{\partial \tilde{H}^{\mathbb{R}\circ}}{\partial u} = \tilde{W}^{\mathbb{R}}(\tilde{H}^{\mathbb{R}\circ}),$$

where

$$\tilde{W}^{\mathbb{R}} = \sum_{i,j} \left(p_{2i-1} p_{2j-1} \frac{\partial}{\partial p^-_{2i+2j-2}} + p_{2i-1} p^+_{2j} \frac{\partial}{\partial p_{2i+2j-1}} + p^+_{2i} p^+_{2j} \frac{\partial}{\partial p^+_{2i+2j}} \right) +$$

$$+ \sum_{i,j} \left(2 p_{2i+2j-1} \frac{\partial^2}{\partial p_{2i-1} \partial p^+_{2j}} + \frac{1}{2} p^-_{2i+2j-2} \frac{\partial^2}{\partial p_{2i-1} \partial p_{2j-1}} + 2 p^+_{2i+2j} \frac{\partial^2}{\partial p^+_{2i} \partial p^+_{2j}} \right) + \sum_{i=1}^{\infty} \left(i p^+_{2i} \frac{\partial}{\partial q_i} + q_i \frac{\partial}{\partial p^+_{2i}} \right),$$

with the initial conditions:

$$\tilde{H}^{\mathbb{R}\circ}(0; p_1, p_2, \ldots, q_1, \ldots) = e^{p_1 + q_1/2}.$$

4.4.3 Action of the transposition operator

Denote by N_n a finite set $N_n = \{1, 2, 3, ..., n\}$. A *state* is the involution of the set N_n, that is, the partition of N_n into a disjoint union of one and two-element subsets. A *transition* is an ordered pair of states. The *type of a transition* is its orbit under the action of the group $\mathbb{S}_n$ on the set N_n by permuting the elements.

The algebra types of transitions A_n is introduced in [19], and the differential operator $\tilde{W}^{\mathbb{R}}$ is interpreted as an operator of multiplication by the transposition class in this algebra.

Theorem EK23-6. *The operator $\tilde{W}^{\mathbb{R}}$ on the vector space A_n is self-adjoint with respect to the scalar product defined on A_n.*

Следствие 12. *The operator $\tilde{W}^{\mathbb{R}}$ of degree n is diagonalizable.*

Explicit computation of the eigenbasis for the operator $\tilde{W}^{\mathbb{R}}$ is tedious and requires finding the roots of polynomials of increasing powers with rational coefficients; instead of computing the eigenvalues and the eigenbasis of the operator $\tilde{W}^{\mathbb{R}}$ we present is in the block form, which allows to compute the Hurwitz numbers efficiently.

The algebra A_n can be decomposed in the direct sum of subspaces, generated by transitions, such that their left states contain m one-element subsets, $A_n = \oplus_m A_{n,m}$. The subspaces $A_{n,m}$ are invariant with respect ot the action of the operator $\tilde{W}^{\mathbb{R}}$.

In sections 3.2.3, 3.2.4 of the major part of the thesis we study properties of representations of the symmetric group in the algebra A_n, isomorphic to the algebra of polynomials in the variables p_k, p_k^+, p_k^-, q_l. We describe the decomposition of the transposition operator as a direct sum of operators, corresponding to the isotypical decompositions of representations of the symmetric group. We show that such operators may be written in an appropriate basis as matrices with integer entries and provide explicit ways to compute them.

Theorem EK23-7. *The action of the operator $\tilde{W}^{\mathbb{R}}$ on the space of polynomials of the given degree n can be decomposed as a direct sum of its actions on the subspaces of isotypical representations of the group $\mathbb{S}_n$.*

In sections 3.2.1, 3.2.2 we describe methods to construct the decomposition of the operator $\tilde{W}^{\mathbb{R}}$ as a direct sums of its actions on the isotypical subspaces, and in section 3.2.5 we provide an example of such decomposition in the case of A_6 algebra.

5 Main results of the thesis

Main results of the thesis are stated in the following theorems.

Theorem EK19-1. *Let I^f be an invariant of framed graphs taking values in the ring of polynomials in infinite set of variables $q_1, q_2, \dots$, $I^f : G^f \mapsto \mathbb{F}(q_1, q_2, \dots)$, such that it can be extended to a graded homomorphism of Hopf algebras.*

Let us define constants i_n^f, $n = 0, 1, 2, \dots$ by

$$i_n^f = n! \sum_{\substack{G \text{ — связный} \\ |V(G)|=n}} \frac{[q_n] I_G^f(q_1, q_2, \dots)}{|\operatorname{Aut}(G)|},$$

where each $[q_n]P$ denotes the coefficient of the monomial q_n in the polynomial $P = P(q_1, q_2, \dots)$. The main result of this paper is the following theorem.

Consider the generating functions

$$\begin{aligned} \mathcal{I}^{f\circ}(q_1, q_2, \dots) &= \sum_G \frac{I_G^f(q_1, q_2, \dots)}{|\operatorname{Aut}(G)|}, \\ \mathcal{I}^f(q_1, q_2, \dots) &= \sum_{G \text{ — связный}} \frac{I_G^f(q_1, q_2, \dots)}{|\operatorname{Aut}(G)|}, \end{aligned}$$

here the first sum is over all framed graphs and the second is over all connected framed graphs.

If $i_n^f \neq 0$ for all $n = 0, 1, 2, \dots$, then, after the rescaling $q_n = \frac{2^{n(n-1)/2}(n-1)!}{i_n^f} p_n$ of the variables, the generating function $\mathcal{I}^f$ becomes a solution of the KP hierarchy in the variables p_n and $\mathcal{I}^{f\circ}$ becomes a τ-function of the KP hierarchy.

Theorem EK19-2. *Let $I^w : \mathcal{G}^w \to \mathbb{F}[q_1, q_2, \dots]$ be a graded homomorphism of the Hopf algebra of weighted graphs to the Hopf algebra of polynomials in infinitely many variables, and let $I_G^w(q_1, q_2, \dots)$ be the corresponding invariant of a weighted graph G.*

Consider the generating functions

$$\begin{aligned} \mathcal{I}^{w\circ}(q_1, q_2, \dots) &= \sum_G \frac{I_G^w(q_1, q_2, \dots)}{|\operatorname{Aut}(G)|}, \\ \mathcal{I}^w(q_1, q_2, \dots) &= \sum_{G \text{ — connected}} \frac{I_G^w(q_1, q_2, \dots)}{|\operatorname{Aut}(G)|}, \end{aligned}$$

where the first sum is over all weighted graphs, the second sum is over all connected weighted graphs, and $|\operatorname{Aut}(G)|$ denotes the order of the automorphism group of the weighted graph G.

Whatever rescaling of variables $q_n = a_n p_n$, $a_n \in \mathbb{F}$, $a_n \neq 0$, $n = 0, 1, 2, \dots$, the generating function $\mathcal{I}^{w\circ}$ is not a linear combination of single-part Schur polynomials. Moreover, whatever rescaling of variables $q_n = a_n p_n$, $a_n \in \mathbb{F}$, the generating function $\mathcal{I}^{w\circ}$ is not a τ-function of the KP hierarchy in the variables p_n and, therefore, $\mathcal{I}^w$ is not a solution of the KP hierarchy.

Theorem EK21-1. *There is a polynomial graph invariant of graphs with up to 8 vertices satisfying the 4-term relations whose values on intersection graphs coincide with that of the weight system $\mathfrak{sl}_2$; such a graph invariant is unique.*

Theorem EK23-6. *The operator $\tilde{W}^{\mathbb{R}}$ acting on the vector space A_n is self-adjoint with respect to the scalar product defined on A_n.*

Theorem EK23-3. *The action of the transposition operator W^+ in the case of separating real curves on the space of polynomials of degree $n = n^+ + n^-$ can be decomposed as a direct sum of its action on the subspaces of isotypical representation of the group $\mathbb{S}_{n^+} \times \mathbb{S}_{n^-}$.*

It follows that for the fixed n^+ and n^- the computation of the homogeneous component of the generating function $e^{p_1^+ + p_1^- + q_1}$ can be carried our using the following algorithm:

1. decompose the action of the group $\mathbb{S}_{n^+} \times \mathbb{S}_{n^-}$ on the space of states V_{n^+,n^-} into the irreducible;
2. for each of the subalgebras $A_{n^+,n^-,t} \subset A_{n^+,n^-}$ of the transition algebra deduce its decomposition as direct sum of algebras of endomorphisms of isotypical subspaces of irreducible representations of the group $\mathbb{S}_{n^+} \times \mathbb{S}_{n^-}$;
3. decompose the action of the transposition operator W^+ as a direct sum of actions by multiplication in each of the algebras of endomorphism of isotypical subspaces;
4. decompose the initial condition over the isotypical subspaces;
5. using the known characteristic polynomial of the restriction of the operator on the isotypical subspace, construct the rational generating function for real Hurwitz numbers, determined by the corresponding isotypical subspace.

Theorem EK23-7. *The action of the transposition operator $\tilde{W}^{\mathbb{R}}$ in the case of not necessarily separating real curves on the space of polynomials of a given degree n can be decomposed as a direct sum of its actions on the subspaces of isotypical representations of the group $\mathbb{S}_n$.*

From the above the computation of the homogeneous component of degree n for the generating function $\tilde{H}^{\mathbb{R}\circ} = e^{u\tilde{W}^{\mathbb{R}}} e^{p_1 + q_1/2}$ can be carried out using the following algorithm:

1. decompose the action of the group $\mathbb{S}_n$ on the space of spates V_n into the irreducible;
2. for each of the subalgebras $A_{n,m} \subset A_n$ of the transition algebra deduce its decomposition as a direct sum of algebras of endomorphisms of isotypical subspaces of irreducible representations of the group $\mathbb{S}_n$;
3. decompose the action of the transposition operator $\tilde{W}^{\mathbb{R}}$ as a direct sum of actions by multiplication in each of the algebras of the endomorphisms of isotypical subspaces;
4. decompose the initial conditions over the isotypical subspaces;
5. using the known characteristic polynomial of the restriction of the transposition operator on the isotypical subspace, construct a rational generating function for real Hurwitz numbers, defined by the corresponding isotypical subspace.

6 The main results of the thesis are presented in these papers

EK19 Krasilnikov,E.S., *Invariants of framed graphs and the Kadomtsev–Petviashvili hierarchy*, Functional Analysis and Its Applications, 53, 14–26, 2019, Russian Academy of Sciences, Steklov Mathematical Institute

EK21 Krasilnikov,E.S., *An Extension of the* $\mathfrak{sl}_2$ *Weight System to Graphs with* $n \leq 8$ *Vertices*, Arnold Mathematical Journal, 7, 609–618, 2021, Springer

EK23 Krasilnikov,E.S., *On the structure of the algebra of transition types and the cut-and-join operator*, Algebra i Analiz, 35, 133–170, 2023

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