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Quaternion-Solvable Hypercomplex Nilmanifolds

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Introduction

The present dissertation is dedicated to the study of certain aspects of the geometry of (hyper-)complex nilmanifolds. Nilmanifolds are compact quotients of nilpotent Lie group G by cocompact lattice Γ. We denonote them $N = \Gamma \backslash G$.

One of the goals of this dissertation is to study submanifolds in complex nilmanifolds, in particular, the presence or absence of complex curves. This question pertains to the field of classical algebraic and complex geometry; however, solving it using standard methods presents significant difficulties.

Nevertheless, nilmanifolds have a unique property: questions about their geometry can be translated into the language of the theory of nilpotent Lie algebras. To understand the geometric structure of nilmanifolds, we study the corresponding nilpotent Lie algebras, which are finite-dimensional vector spaces. The central tenet is that the geometry of the compact, complex nilmanifold $\Gamma \backslash G$ can be described by the linear algebra of $\mathfrak{g} = \mathrm{Lie}(G)$, complex structure operator $I \in \mathrm{End}(\mathfrak{g})$ and the rational subalgebra of $\mathfrak{g}$ generated by $\log \Gamma \subset \mathfrak{g}$.

The (hyper-)complex structure on a nilmanifold $\Gamma \backslash G$ is induced by the corresponding left-invariant (hyper-)complex structure on the Lie group, which is identified with a complex structure operator $I \in \mathrm{End}(\mathfrak{g})$ on the corresponding Lie algebra $\mathfrak{g}$. In the work [AV], A. Abasheva and M. Verbitsky considered hypercomplex nilmanifolds with an abelian hypercomplex structure (one for which the $\sqrt{-1}$-eigenspace of the complex structure I, induced by the quaternions, form an abelian Lie subalgebra). Abelian complex structures were first described in the work of M. L. Barberis [Ba]. The study [AV] provided a characterization of the geometry of submanifolds in nilmanifolds with an abelian hypercomplex structure.

In the first part of this dissertation, we develop the approach outlined in [AV]. We consider complex nilmanifolds, with a complex structure induced by quaternions and the corresponding Lie algebra is $\mathbb{H}$-solvable (Definition 0.0.2).

The condition of $\mathbb{H}$-solvability of the corresponding Lie algebra is of independent interest. For example, any Lie algebra with an abelian hypercomplex structure is $\mathbb{H}$-solvable. Less obvious is the existence of $\mathbb{H}$-solvable Lie algebras whose hypercomplex structure is not abelian. We provide such an example using the quaternionic double construction described in [SV].

The question of the $\mathbb{H}$-solvability of a hypercomplex Lie algebra is the subject of study in the second part of the dissertation. We consider hypercomplex nilmanifolds with flat Obata connection. The Obata connection is the unique torsion-free connection in the tangent bundle that preserves the hypercomplex structure [Ob], [K].

Curves in Hypercomplex Nilmanifolds

Before delving into the research topic, we briefly consider the reasons for our interest in nilmanifolds, as well as highlighting the reasoning in the study of the existence of curves.

The question of the existence of low-dimensional objects very often holds key importance in the field of algebraic geometry. For example, when studying moduli spaces of vector bundles or sheaves, we look for curves in algebraic varieties. However, there are manifolds which do not contain curves. Even in the Enriques-Kodaira classification, some surfaces lack complex curves, leaving a gap in the classification that persists to this day. Let's briefly recall some results in this area.

Among compact complex surfaces, an important class is that of surfaces of **class VII**. These are non-Kähler surfaces with Kodaira dimension $-\infty$ and first Betti number $b_1 = 1$ [In]. A minimal surface S of class VII is called **class $\mathbf{VII}_0$**. In the works [Bog1], [Bog2], it was shown and then clarified in [Te], [LYZ], that S with second Betti number $b_2(S) = 0$ is biholomorphically equivalent to a *Hopf surface* or an *Inoue surface*.

A Hopf surface H is a class VII_0 surface with the universal cover $\mathbb{C}^2\backslash\{0\}$. Hopf surface can be obtained as the quotient $H \cong \mathbb{C}^2\backslash\{0\}/\langle\gamma\rangle$ by a cyclic group generated by a holomorphic contraction γ. A Hopf surface always contains at least one complex curve, for example, consider an elliptic curve $C = \mathbb{C}^* \times \{0\}/\langle\gamma\rangle$.

Inoue surfaces S are class VII_0 surfaces with the universal cover $\mathbb{C} \times \mathbb{H}$, where $\mathbb{H}$ is the upper half-plane. Inoue surfaces S are biholomorphic to $\mathbb{C} \times \mathbb{H}/\Gamma$, where Γ is a discrete subgroup acting holomorphically on $\mathbb{H} \times \mathbb{C}$. Inoue showed that these surfaces have zero algebraic dimension and contain no complex curves. If we assume the *Global Spherical Shell conjecture*, then they are the only non-Kähler surfaces without curves.

A **Global Spherical Shell (GSS)** on a complex surface is an open subset biholomorphic to a neighborhood of S^3 in $\mathbb{C}^2\backslash 0$ such that its complement is connected. The **Global Spherical Shell conjecture** claims that all surfaces of **class $\mathbf{VII}_0$** with positive second Betti number contain a GSS. The hypothesis has been proven for $b_2 = 0$ and $b_2 = 1$ by A. Teleman in [Te2]. Furthermore, Dloussky, Oeljeklaus, and Toma showed that the presence of a GSS in a surface S implies that all class $VII_0^{b_2>0}$ surfaces contain exactly $b_2(S)$ rational curves [DOT].

There is a higher-dimensional generalization of Inoue surfaces. They are related to certain number fields and are called Oeljeklaus-Toma manifolds. For many number fields, these manifolds do not contain subvarieties at all [OV1],[Ve].

We study compact complex non-Kähler manifolds without curves. We have already seen an example: an Inoue surface. In particular, both Inoue surfaces and Oeljeklaus-Toma manifolds fall under the category of *solvmanifolds*. These are smooth manifolds obtained from a solvable Lie group. All solvmanifolds are fibered over a torus with a fiber diffeomorphic to a *nilmanifold* [Mos]. Usually, it is hard to study solvmanifolds straightforward, so we deal with nilmanifolds instead. It should be noted that *complex nilmanifolds*, except compact tori, do not admit Kähler structure [BG].

To obtain a complex nilmanifold without curves, we recall a standard technique called the *twistor deformation*.

A smooth manifold X is called **hypercomplex** if there exist three integrable almost complex structures I, J, and K in $\mathrm{End}(TX)$ satisfying $I^2 = J^2 = K^2 = -Id$ and $IJ = -JI = K$. For any $(a, b, c) \in S^2$, the linear combination $L := aI + bJ + cK$ defines another complex structure on X. This results in a $\mathbb{C}P^1$-family of complex structures. It is called a **twistor deformation**.

Using the twistor deformation and foliation theory, we prove the following theorem:

Theorem 0.0.1: Let (N, I, J, K) be a hypercomplex nilmanifold, and suppose the corresponding Lie algebra is $\mathbb{H}$-*solvable* (Definition 0.0.2). Then, for *a generic complex structure* L induced by quaternions, the complex manifold (N, L) has no complex curves[1].

First, we introduce the definition of $\mathbb{H}$-solvable Lie algebra.

A **hypercomplex structure** on a Lie algebra $\mathfrak{g}$ is a triple of complex structure operators I, J, and K on $\mathfrak{g}$, satisfying the quaternionic relations.

[1]Here, generic means outside a countable set

Let $\mathfrak{g}$ be a nilpotent hypercomplex Lie algebra. We define inductively the $\mathbb{H}$-invariant subalgebras of the Lie algebra as follows:

$$\mathfrak{g}_i^{\mathbb{H}} := \mathbb{H}[\mathfrak{g}_{i-1}^{\mathbb{H}}, \mathfrak{g}_{i-1}^{\mathbb{H}}], \tag{0.0.1}$$

where $\mathfrak{g}_0^{\mathbb{H}} = \mathbb{H}\mathfrak{g} = \mathfrak{g}$ and $\mathfrak{g}_1^{\mathbb{H}} := \mathbb{H}[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}] + I[\mathfrak{g}, \mathfrak{g}] + J[\mathfrak{g}, \mathfrak{g}] + K[\mathfrak{g}, \mathfrak{g}]$.

Definition 0.0.2: A hypercomplex nilpotent Lie algebra $\mathfrak{g}$ is called **$\mathbb{H}$-solvable** if the sequence (0.0.1) converges to zero.

We will consider left-invariant foliations on the Lie group G, generated by subalgebras of the Lie algebra $\mathfrak{g}$.

For the reader's convenience, we recall the definition of a foliation.

A subbundle on a smooth manifold X is called a distribution $\Sigma \subset TX$. A distribution is called **involutive** if it is closed under the Lie bracket. A **leaf** of the distribution Σ is the maximal connected immersed submanifold $L \subset N$ such that L is tangent to Σ at every point. If Σ is involutive, then the set of all its leaves is called **a (smooth) foliation**.

For each $i \in \mathbb{Z}_{>0}$, consider the left-invariant foliation Σ_i on the Lie group G generated by the subalgebra $\mathfrak{g}_i^{\mathbb{H}}$.

Theorem 0.0.3: Let C_L be a complex curve in the complex nilmanifold (N, L), where $L \in \mathbb{C}\mathrm{P}^1$ is a generic complex structure. Suppose that C_L is tangent to the foliation Σ_{i-1}. Then it is also tangent to Σ_i.

Corollary 0.0.4: Let (N, I, J, K) be a hypercomplex nilmanifold, and suppose the corresponding Lie algebra is $\mathbb{H}$-solvable. Then, for a generic complex structure L induced by the quaternions, the complex manifold (N, L) has no complex curves.

$\mathbb{H}$-solvable Lie Algebras and Algebraic Monodromy

The second part of the dissertation is devoted to the question of the $\mathbb{H}$-solvability of the Lie algebra of a hypercomplex nilmanifold. We work with hypercomplex nilmanifolds which admit flat Obata connection. The Obata connection is the unique torsion-free connection on the tangent bundle that preserves the hypercomplex structure.

Recall that manifolds with a flat connection in a tangent bundle are called **(flat) affine manifolds**.

Let X be a compact affine manifold whose linear holonomy representation is unipotent. Then X admits a parallel volume form. The partial inverse also holds and was proven by Goldman, Fried, and Hirsch [FGH, Theorem A]:

Theorem 0.0.5: Let X be a compact affine manifold with a parallel volume form. Assume the affine holonomy group is nilpotent. Then the linear holonomy representation is unipotent.

We use the following reformulation of Theorem 0.0.5:

Theorem 0.0.6: Let X be a compact affine manifold with a parallel volume form, and let its fundamental group be nilpotent. Then its monodromy representation is unipotent.

Nilmanifolds are $K(\Gamma, 1)$-spaces and $\pi_1(N) = \Gamma$, which is, obviously, nilpotent. To be able to apply Theorem 0.0.5, we also need a parallel volume form on a nilmanifold. Its existence guaranteed by the following theorem:

Theorem 0.0.7: [BDV, Theorem 3.2] Let $N = \Gamma\backslash G$ be a hypercomplex nilmanifold, $n = \dim_{\mathbb{C}} G$. Then G admits a left-invariant non-zero holomorphic section Ω of the canonical bundle $\Lambda^{n,0}G$. Moreover, $\nabla\Omega = 0$, where ∇ is the **Obata connection**.

Note that Theorem 0.0.6 is only valid for nilmanifolds, that is, when the corresponding Lie algebra has a rational structure, i.e. rational subalgebra $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g}$, such that $\mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R} = \mathfrak{g}$.

In the case when a Lie group G admits a left-invariant hypercomplex structure with a flat Obata connection, we might attempt to prove that $\mathfrak{g}_1^{\mathbb{H}}$ is a proper subalgebra of $\mathfrak{g}$, and then apply induction as $\mathfrak{g}$ is finite-dimensional. However, we must be concerned with the existence of a rational structure in $\mathfrak{g}_i^{\mathbb{H}}$, which is not guaranteed to exist, hence it is not possible to apply Theorem 0.0.5. Consequently, there is no way to resolve this issue directly via induction.

Instead, we use a different approach.

We introduce the notion of *algebraic monodromy*:

Definition 0.0.8: Let $\mathfrak{g}$ be a Lie algebra, and $\nabla : \mathfrak{g} \longrightarrow \operatorname{End}(\mathfrak{g})$ an $\mathbb{R}$-linear map. **An algebraic monodromy group** $\mathscr{H}\!ol^a_\nabla$ is a subgroup of $\mathrm{GL}(B)$ generated by the matrix exponents:

$$\mathscr{H}\!ol^a_\nabla := \langle e^{t\nabla_X} \mid t \in \mathbb{R}, \text{ for all } X \in \mathfrak{g} \rangle.$$

Informally, it allows us to measure how much the monodromy action of the flat connection differs from the action of the differential of the left shift on the Lie group. Using this notion, we prove the following theorem, which is a key ingredient of the proof of main result:

Theorem 0.0.9: Let $\Gamma\backslash G$ be a hypercomplex nilmanifold with the flat Obata conection ∇ on TG. Then the action of the algebraic holonomy on $\mathfrak{g} = \mathrm{Lie}(G)$ is unipotent.

As a consequence of Theorem 0.0.9, we have the following:

Theorem 0.0.10: The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ of a hypercomplex nilmanifold $\Gamma\backslash G$ with a flat Obata connection is $\mathbb{H}$-solvable.

Lie pencils

In this dissertation, the comprehensive resolution of the question of $\mathbb{H}$-solvability remains an open challenge. This section addresses an issue that eludes complete resolution. We announce a novel perspective on the problem concerning $\mathbb{H}$-solvability.

We introduce the following definitions:

Definition 0.0.11: Let V be a vector space, and $S \subset \mathrm{Hom}(\Lambda^2 V, V)$ a subspace, such that for any $w \in S$, the map $w(x, y)$, denoted in the sequel as $[x, y]_w$, satisfies the Jacobi condition $[[x, y]_w, z]_w + [[y, z]_w, x]_w + [[z, x]_w, y]_w = 0$. Then S is called **a Lie pencil**. When $\dim S = k$, we call it **a k-pencil**.

Definition 0.0.12: A Lie pencil $S \subset \mathrm{Hom}(\Lambda^2 V, V)$ is **S-solvable** if V admits a filtration $V = V_0 \supset V_1 \supset ... \supset V_n = 0$ such that $[V_i, V_i]_w \subset V_{i-1}$ for all $w \in S$.

Question 0.0.13: ("the main conjecture")
Let $S \subset \mathrm{Hom}(\Lambda^2 V, V)$ be a Lie pencil. Assume that the Lie algebra $(V, [\cdot, \cdot]_w)$ is nilpotent for all $w \in S$. Will it follow that that (V, S) is S-solvable?

Remark 0.0.14: When $\dim S = 2$, the answer is affirmative.

We are interested in this conjecture only when $S = \mathbb{H}$ and the Lie pencil comes from a hypercomplex structure on a Lie algebra, but it might be true in all generality.

Theoretical significance of the results

The results of this study have interesting implications and can be built upon and expanded. One immediate effect is the creation of many new examples of compact complex manifolds without complex curves. Additionally, another outcome partially characterizes hypercomplex nilmanifolds that have a flat Obata connection.

Practical significance of the results

The dissertation is entirely theoretical in nature.

Personal contribution

All of the main results were obtained by the author.

Approbation of the results of the dissertation research

1. Seminário de Geometria Diferencial, talk “Complex curves in hypercomplex manifolds”, IMPA, Rio de Janeiro, Brazil;
2. Geometric Structures and Moduli Spaces, poster ”Complex curves in hypercomplex nilmanifolds with $\mathbb{H}$-solvable Lie algebras”, UNC, Cordoba, Argentina;
3. Brazil-China Joint Mathematical Meeting, poster “Flat hypercomplex nilmanifolds are quaternionic-solvable”, Foz do Iguacu, Brazil, July, 2023;
4. Estruturas geométricas em variedades, talk “Flat hypercomplex nilmanifolds are $\mathbb{H}$-solvable ”, IMPA, Rio de Janeiro, Brazil, August, 2023;
5. Geometry Seminar, talk “Quaternionic-solvable hypercomplex nilmanifolds”, UFRJ, Rio de Janeiro, Brazil, November, 2023;
6. Algebraic Geometry, Lipschitz Geometry and Singularities, talk “Complex curves in nilmanifolds”, Pipa, Brazil, December 2023
7. Conference on Singularity and Birational Geometry, talk “Complex curves in nilmanifolds”, Yonsei University in Seoul, Korea, January, 2024.
8. Special Holonomy and Geometric Structures on Complex Manifolds, poster “Complex curves in hypercomplex nilmanifolds”, IMPA Rio de Janeiro, Brazil, March 2024
9. Algebraic geometry seminar, talk “Complex curves in nilmanifolds”, HSE, Moscow, Russia, April 2024

10. Algebra seminar, talk “Complex curves in hypercomplex nilmanifolds”, KU Leuven, Leuven, Belgium, May 2024

Publications

The results of the thesis are published in two articles:

a. Yulia Gorginyan, Complex curves in hypercomplex nilmanifolds with $\mathbb{H}$-solvable Lie algebras, Journal of Geometry and Physics, Volume 192, October 2023, 104900

b. Yulia Gorginyan, Flat hypercomplex nilmanifolds are $\mathbb{H}$-solvable, Functional Analysis and Its Applications, Volume 58, 3 issue, 2024

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