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as a manuscript

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Approximation by doubly periodic functions

Summary of the PhD thesis
for the purpose of obtaining academic degree
Doctor of Philosophy in Mathematics

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Saint Petersburg - 2024

Introduction

Issues related to polynomial approximation in the complex plane have been studied since the 1920s. The fundamental result of the 1950s is the theorem of S. N. Mergelyan [1]. Since the 1960s, many mathematicians have been working on this issue; let us mention, for example, V.K. Dzyadyk, N.K. Lebedev, N.A. Shirokov, I.A. Shevchuk, V.M. Miklyukov, V. . Bely, P. M. Tamrazov, V. V. Andreevsky, V. V. Maymeskul, D. E. Saff, V. Totik, J. Muller, T. Ganelius [3-6], [7-8], [9-10], [14-22], [23-25], [11].
An important topic studied in the works of the mentioned mathematicians was the problem of a constructive description of classes of functions that are analytic in a domain and included in any smoothness class in a closed domain in terms of the rate of their approximation by polynomials. To illustrate the problem, consider a bounded Jordan domain D with boundary Γ.
Let $H^{\alpha}(\overline{D})$ denote the class of functions f that are analytic in D and satisfy the condition $|f(z)-f(\xi)| \leq c_f|z-\xi|^{\alpha}, z, \xi \in \overline{D}, 0<\alpha<1$. Let the function $\xi=\varphi(z)$ map $\mathbb{C} \setminus \overline{D}$ to $\mathbb{C} \setminus \overline{\mathbb{D}}, \mathbb{D}$ is the unit circle such that $\varphi(\infty)=\infty, \varphi'(\infty)>0, z=\Psi(\xi)$ is the inverse mapping. For $h>0$ we set $\Gamma_h=\Psi((1+h)\Pi), \Pi=\partial\mathbb{D}$ to be the unit circle.
First results on a constructive description of the class $H^{\alpha}(\overline{D})$ for the case of a sufficiently smooth Γ [2] or a curve Γ with a finite number of angles [3], [6]] were formulated as follows. For $z \in \Gamma$ we set $\rho_h(z)=dist(z, \Gamma_h)$.
Theorem A. In order for $f \in H^{\alpha}(\overline{D})$, it is necessary and sufficient that for any $n=1,2,...$ there were polynomials $P_n(z), degP_n \leq n$ and some constant c_f, independent of z and n, such that the relation

$$|f(z)-P_n(z)| \leq c_f \rho^{\alpha}_{\frac{1}{n}}(z), z \in \Gamma. \tag{a}$$

Subsequent efforts of mathematicians were aimed at weakening the conditions on the Γ boundary under which criterion (a) would continue to be satisfied. The most powerful result was obtained in [19], in which the Γ curve was assumed to be quasiconformal. In parallel with this, it turned out [13] that the presence of external angles equal to zero or 2π on the boundary can lead to the absence of a constructive description of the class $H^{\alpha}(\overline{D})$ of type (a). Moreover, the presence of external angles for D equal to 2π on the curve Γ in the case of sufficient smoothness of the arcs touching at the corner point, description (a) preserved the constructive description [6].
Along with the constructive description of classes of analytic functions in smooth domains, a constructive description of holomorphic functions in domains $\mathbb{C}^n$, for $n \geq 2$, was considered [29]. The boundaries of the corresponding regions were assumed to be C^2 - smooth.

There is a situation intermediate between $\mathbb{C}$ and $\mathbb{C}^n$, namely, regions on elliptic curves in $\mathbb{C}^2$, one-dimensional regions in the complex sense, immersed in $\mathbb{C}^2$, require for a constructive description of the holomorphic classes in them of functions of polynomials in two variables.

The first results of this kind were obtained in [11], [12]. In these works it was assumed that the boundary of the domain satisfies the condition of commensurability of the arc and the chord, and it was the class $H^{\alpha}(\overline{D})$ with $0 < \alpha < 1$. that was studied

The construction of the required polynomials took place on the $\mathbb{C}$ plane; the polynomials were constructed from doubly periodic Weierstrass functions and their derivatives.

In this thesis, the classes $H^{r+\omega}(\overline{D})$ are constructively described for moduli of continuity ω satisfying the Dini condition (in particular, for the classes $H^{r+\alpha}(\overline{D}), r \geq 0, 0 < \alpha < 1$) and for regions D that can have a finite number of external angles with respect to D, equal to 2π. The so-called direct approximation theorem for classes $C^r(D)$.

Goals and objectives of the study

The purpose of the study is to construct an approximation by polynomials of a function of two variables defined on the continuum of an elliptic curve in $\mathbb{C}^2$ and holomorphic in its interior, to prove direct and inverse approximation theorems by doubly periodic Weierstrass functions, as well as to prove a direct approximation theorem for classes $C^r(D)$. To solve this problem, the following steps are planned:

- Derivation and proof of the direct theorem for approximating a function by doubly periodic Weierstrass functions;
- Definition and study of the considered domains and classes of functions, analytic in given domains and continuous on the closure of the specified domains, formulation of the main theorems - the main result of the first half of the work;
- Construction of an approximating polynomial $\mathbf{P}_n(\mathfrak{P}'(z), \mathfrak{P}(z))$;
- Checking the estimate of the values $\rho_n(1/n)(z)$ in the vicinity of the sharpening points;
- Completion of the proof of the direct theorem;
- Derivation and proof of the converse theorem for approximating a function by doubly periodic Weierstrass functions on subsets of domains with peaks;
- Construction of an approximating polynomial;
- Specifying a Bernstein type inequality;
- Completion of the proof of the converse theorem.
- Consideration of the class of functions $f \in C^r(D)$ (the class of functions whose derivative of some order is bounded).

• Proof of a direct theorem for the approximation by polynomials of doubly periodic Weierstrass functions of functions that are analytic in the domains under consideration and for which the derivative of a given order is bounded.

Scientific novelty of the results

All results submitted for defense are new.

As part of the thesis, results were obtained for approximating a function defined on an elliptic curve in $\mathbb{C}^2$ using polynomials of Weierstrass functions, as well as their derivatives. Direct and inverse theorems on the approximation of the indicated type of functions are stated and proven, and a direct approximation theorem is proved for classes of functions $C^r(D)$. Of fundamental importance is the fact that the approximation problem was transferred to flat areas, and the approximation itself was represented using Weierstrass functions.

Theoretical and practical significance of the results

The work is of a theoretical nature. The results of the thesis can be used in further studies of polynomial approximation on subsets of elliptic curves.

Methodology and research methods

To obtain the stated results, the theory of elliptic functions was applied, specifically the theory of Weierstrass functions, the theory of approximation, as well as the properties of conformal mappings of areas onto the exterior of a circle.

Provisions for defense

1. A direct theorem is obtained for the approximation of functions from the class $H^{r+\omega}(\overline{D})$ by polynomials in doubly periodic Weierstrass functions for domains D that can have a finite number of external ones with respect to to D boundary angles equal to 2π.

2. An inverse approximation theorem is obtained for functions from the class $H^{r+\omega}(\overline{D})$ for the domains mentioned above.

3. A direct approximation theorem is obtained for the classes $C^r(D)$.

Reliability of results

The reliability of the results obtained is ensured by strict mathematical proofs.

Structure and scope of work

The dissertation consists of an introduction, 3 chapters and a conclusion.
The full volume of the dissertation is 61 pages. The bibliography contains 35 titles.

Author's personal contribution

All results of the dissertation work were obtained by the author independently.

1 The content of the work

In **the first chapter** a direct theorem for the approximation of a function defined on an elliptic curve using Weierstrass polynomials is formulated and proven.

Let Q be a parallelogram on the complex plane $\mathbb{C}$ with vertices 0, $2\omega_1, 2\omega_2,\ 2\omega_3 \stackrel{\text{def}}{=} 2(\omega_1{+}\omega_2), Im\frac{\omega_2}{\omega_1} > 0, \mathfrak{P}$ - classical Weierstrass function with periods $2\omega_1, 2\omega_2$ [2, Chapter 1]. $\mathring{Q}$ is the interior of Q, D is a Jordan domain, on the boundary of which some conditions will be imposed below, $\overline{D} \subset \mathring{Q}$.

Let $\omega : \{x \in R : x \geq 0\} \to \{x \in R : x \geq 0\}$ be a strictly increasing function, $\omega(0) = 0$, which we will consider as the modulus of continuity. Let $H^\omega(\overline{D})$ denote the set of functions that are analytic in D, continuous in $\overline{D}$, for which the relation $|f(z_1) - f(z_2)| \leq c_f\omega(|z_1 - z_2|)$, for any $z_1, z_2 \in \overline{D}$.

Let $H^{\omega+r}(\overline{D})$ be the set of functions that are analytic in D and such that $f^{(r)} \in H^\omega(\overline{D})$; $H^{\omega+0}(\overline{D}) \stackrel{\text{def}}{=} H^\omega(\overline{D})$.

We assume that the modulus of continuity $\omega(t)$ satisfies the inequality:

$$\int_0^x \frac{\omega(t)}{t}dt + x\int_x^\infty \frac{\omega(t)}{t^2}dt \leq c\omega(x), x > 0.$$

We assume that the Jordan domain D has the following properties:

1) there is a finite number of points $z_1, ..., z_m \in \Gamma \stackrel{\text{def}}{=} \partial D, m \geq 1$, and their neighborhoods $\Omega_1, ..., \Omega_m,\ z_j \in \Omega_j$, such that $\overline{\Omega}_k \cap \overline{\Omega}_l = \emptyset,\ k \neq l$;

2) arcs $\Gamma_1, ..., \Gamma_m$ lie on $\Gamma, \Gamma_j \subset \Gamma \backslash \bigcup_{r=1}^m \Omega_r,\ \Gamma_k \bigcap \Gamma_l = \emptyset, k \neq l, \bigcup_{j=1}^m \Gamma_j = \Gamma \backslash \bigcup_{r=1}^m \Omega_r$ and the ends of Γ_j are the points z^0_{2j-1} and z^0_{2j}, where $z_j, z^0_{2j-1}, z^0_{2j}, z_{j+1}$ follow in the order of positive traversal of the curve Γ, $z_{m+1} \stackrel{\text{def}}{=} z_1$. We assume that there exists $b > 0$, where $\Gamma(\xi_1, \xi_2)$ is an arc with endpoints ξ_1, ξ_2, contained in the arc of the curve Γ, with endpoints z^0_{2j} and z^0_{2j-1}, not containing the point z_j, the relation

$$|\Gamma(\xi_1, \xi_2)| \leq b|\xi_2 - \xi_1|. \tag{b}$$

Further, we will call condition b) the condition of commensurability of the length of the arc and the chord.

An arc Γ with ends at the points z^0_{2j-1} and z^0_{2j-2} containing the point z_j has a tangent at each point.

Arcs $\Gamma(z^0_{2j-1}, z_j) \subset \Omega_j,\ \Gamma(z_j, z^0_{2j-2}) \subset \Omega_j$ have the following property: if $\Theta(\xi)$ is the angle of inclination of the oriented tangent to Γ in the positive direction of the real axis, then with some $b_1 > 0, \sigma > 0$ there is the relation

$$|\Theta(\xi_2) - \Theta(\xi_1)| \leq b_1|\xi_2 - \xi_1|^\sigma, \tag{c}$$

The external angle η^* with respect to D between the tangents to the arcs $\Gamma(z_{2j-1}^0, z_j)$ and $\Gamma(z_j, z_{2j-2}^0)$ at the point z_j is equal to

$$\eta^* = 2\pi, j = 1...m.$$

Let $\varphi(z)$ denote the function that conformally maps $\mathbb{C}\backslash\overline{D}$ onto the exterior of the unit circle $\mathbb{D}$ so that $\varphi(\infty) = \infty$, $\varphi'(\infty) > 0, z = \Psi(\xi)$ - inverse mapping. For $t > 0$ we set $\Gamma_{1+t} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : z = \Psi(\xi), |\xi| = 1 + t\}$, for $z \in \Gamma$ we set $\rho_t(z) \stackrel{\text{def}}{=} dist(z, \Gamma_{1+t})$.

Main result:

Theorem 1.1. *Let D satisfy the above conditions, the modulus of continuity ω satisfies relation (1), $f \in H^{\omega+r}(\overline{D})$, $r \in \mathbb{Z}, r \geq 0$. Then there is a constant $c = c(f)$ such that for n = 1,2... there is a polynomial $\mathbf{P}_n(u, v)$ in two variables, $deg\mathbf{P}_n \leq n$, such that the following relation holds:*

$$|f(z) - \mathbf{P}_n(\mathfrak{P}(z), \mathfrak{P}'(z))| \leq c\rho^r_{\frac{1}{n}}(z)\omega(\rho_{\frac{1}{n}}(z)), \quad z \in \Gamma.$$

The first chapter of the work is devoted to the proof of Theorem 1.1.

Supporting conclusions and statements:

Lemma 1. *Let $Q_{\mathbf{t}} \stackrel{\text{def}}{=} \omega_3 + \mathbf{t}(Q - \omega_3)$ – parallelogram with center at point ω_3 and sides of length $2\mathbf{t}|\omega_1|, 2\mathbf{t}|\omega_2|, 0 < \mathbf{t} < 1$. Let the condition overlineD $\subset Q_{\mathbf{t}}$. Let the function F be analytic in $Q_{\mathbf{t}}$, belong to the class $H^{1+\alpha}$ in $\overline{Q}_{\mathbf{t}}$, and, moreover, $F(z) = F(2\omega_3 - z)$ for arbitrary $z \in \overline{Q}_{\mathbf{t}}$. Then for an arbitrary n we can specify a polynomial $Q_n, degQ_n \leq n$, such that*

$$|F(z) - Q_n(\mathfrak{P}(z))| \leq c \cdot n^{-(\alpha+1)},$$

$$|F'(z) - (Q_n(\mathfrak{P}(z)))'| \leq c \cdot n^{-\alpha}, z \in \overline{Q}_{\mathbf{t}}$$

The following theorem is in the spirit of the results of E.M. Dynkin [7], but he does not find it for the areas considered in this work.

Theorem 1.3. *There is a continuation f_0 of the function f to $\mathbb{C} \setminus D$ such that*

1. $f_0 \in C(\mathbb{C}), f_0 \in C^1(\mathbb{C} \setminus \overline{D}), f_0(z) = 0$, for $|z| \geq R_0$.
2. $|f'_{0\overline{z}}(z)| \leq c \cdot dist^{r-1}(z, \Gamma)\omega(dist(z, \Gamma)), c$ does not depend on $z, x \in \mathbb{C} \setminus D$.

As shown in the dissertation (see Chapter 1, p. 20), one can define a polynomial $Q_{m_0}(u)$ such that the function $s(z) = \mathfrak{P}'(z)Q_{m_0}(\mathfrak{P}(z))$ is univalent in the parallelogram $Q_{\mathbf{t}}$.

Lemma 2. *For $z \in D$ the relation holds*

$$f(z) = -\frac{1}{\pi} \int_{\overline{Q}_t \backslash \overline{D}} \frac{f'_{0\overline{\xi}}(\xi)s'(\xi)}{s(\xi) - s(z)} dA(\xi),$$

where dA is the two-dimensional Lebesgue measure.

The proof of Lemma 2 is based on the use of the properties of the function f_0, the properties of the modulus of continuity, as well as the application of the Cauchy formula. Let $G = s(D)$, the function Φ conformally maps $\mathbb{C} \setminus \overline{G}$ onto $\mathbb{C} \setminus \overline{\mathbb{D}}$ so that $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$, ψ is the inverse of Φ display.
For $t \in \overline{\mathbb{C}} \setminus \overline{G}, \Theta \in [0; 2\pi], R > 1$ we set

$$t_{R,\Theta} \stackrel{\text{def}}{=} \psi(Re^{-i\Theta}\Phi(t)), t_R \stackrel{\text{def}}{=} t_{R,0} = t_R(t)$$

For $w \in \overline{G}, t \in \mathbb{C} \setminus \overline{G}$ we set $R = 1 + \frac{1}{n}$ and $K_n(w, t, R, \Theta) \stackrel{\text{def}}{=} \sum_{\nu=1}^{k} \frac{(t_{R,\Theta} - t)^{\nu-1}}{(t_{R,\Theta} - w)^{\nu}}$,

$$\Pi_n(w, t) = C_{n,q} \int_{-\pi}^{\pi} \left(\frac{\sin n\Theta}{\sin \Theta} \right)^q K(w, t, R, \Theta)\, d\Theta,$$

$$\mathbf{P}_n(\mathfrak{P}(z), \mathfrak{P}'(z)) = -\frac{1}{\pi} \int_{\overline{Q}_t \setminus \overline{D}} f'_{0\bar{\xi}}(\xi) s'(\xi) \Pi_n(s(z), s(\xi))\, dA(\xi).$$

It is then essential to obtain estimates for the expressions $(t_{R,\Theta} - t)$ and $(t_{R,\Theta} - \omega)$. The following results apply.

Lemma 3. *There is an absolute constant $c > 1$ such that for $R = 1 + \frac{1}{n}$, $n \geq 1, |\Theta| \leq \pi$ for $t \in \mathbb{C} \setminus \overline{G}$, $\tau \in G$ the following estimates hold:*

$$c^{-1}(n|\Theta| + 1)^{-4}|t_R - \tau| \leq |t_{R,\Theta} - \tau| \leq c(n|\Theta| + 1)^4 |t_R - \tau|.$$

Lemma 4. [5, Ch. 9], [9, Ch. 2]. *Let $\Omega \subset \mathbb{C}$ be a Jordan domain $z_0 \in \partial\Omega$, f is analytic in Ω, continuous in $\overline{\Omega}$ and satisfies the condition*

$$|f(z)| \leq M \left(1 + \frac{|z - z_0|}{\rho} \right)^A,$$

for $A > 0$ and $z \in \partial\Omega$. Then for $z \in \overline{\Omega}, |z - z_0| \leq \rho$ the relation is true

$$|f(z)| \leq C_0^A M,$$

where C_0 does not depend on the domain Ω, M and the function f.

To complete the proof of Theorem 1.1, we need the function $M(\tau)$, which is equal to $M(\tau) = |f'_{0\bar{\xi}}(\lambda(\tau)| \cdot |\lambda'(\tau)|, \tau \in \overline{Q}_t$, where $\lambda(\tau)$ is the inverse mapping to $s(\xi)$.

$$\begin{aligned} 0 \leq M(\tau) &\leq C_f |\lambda(\tau)| dist^{r-1}(\lambda(\tau), \partial D)) \omega(dist(\lambda(\tau), \partial D)) \\ &\leq b_{20} dist^{r-1}(\tau, \partial G) \omega(dist(\tau, \partial G)), \tau \in s(\overline{Q}_t \setminus \overline{D}). \end{aligned} \tag{d}$$

Due to the previously discovered properties of this function, namely formula d), a result known in the literature, contained in the following lemma (Lemma 5), can be applied to it.

The application of Lemma 5 completes the procedure for estimating the approximation of a function by constant polynomials.

Lemma 5. *For any function M defined on the set $\mathbb{C}\setminus\overline{G}$ and satisfying conditions (d), for t_0^* in∂G and $\rho_0>0$ the estimate is valid*

$$\int_{B_{\rho_0}(t_0^*)}\frac{M(\tau)}{|\tau-t_0^*|}\,dA(\tau)\leq C_M\rho_0^r\omega(\ rho_0).$$

In **the second chapter** of the work the formulation and proof of the inverse approximation theorem is given.

Let's define an elliptic curve, let $E=\{(\zeta,\eta)\in\mathbb{C}^2:\eta^2=4\zeta^3-g_2\zeta-g_3\}\subset\ mathbbC^2_{\zeta,\eta}$. The given curve (with an added point at infinity) is biholomorphically parameterized by a suitable complex torus $\mathbb{C}\setminus(2\omega_1\mathbb{Z}+2\omega_2\mathbb{Z})$ using the vector function $T(z)=(\mathfrak{P}(z),\mathfrak{P}'(z))$.

Let us define a parallelogram of periods of the Weierstrass function, let $Q=\{z\in\mathbb{C}:z=2\boldsymbol{\omega}_1\alpha_1+2\boldsymbol{\omega}_2\alpha_2,\ alpha_1,\alpha_2\in[0,1)\}$ is a parallelogram of periods of the function $\mathfrak{P}(z)$.

Let D be the Jordan region defined in Chapter 1.

Let us set the weights of the approximations of the theorems being proved. Let Φ be a conformal mapping of the set $\mathbb{C}\setminus\overline{D}$ onto $\mathbb{C}\setminus\{|z|\leq1\}$, where $\Phi(\infty)=\infty,\Phi'(\infty)>0$, and let $\Psi=\Phi^{-1}$ - reverse mapping. Let us denote $L_{1+t}=\Psi(|z|=1+t)$, where $t>0$, and let

$$\delta_n(z)=dist(z,L_{1+1/n}),z\in\Gamma.$$

Then we will define a one-to-one mapping of the parallelogram Q onto E as follows

$$T(z)=(\mathfrak{P}(z),\mathfrak{P}'(z))\tag{e}$$

Let's denote

$$\delta_n(\zeta,\eta)=\delta_n(T^{-1}(\zeta,\eta)).$$

Let $G=T(D)$.

We will say that a function Y belongs to the class $H^{\boldsymbol{\omega}+r}(\overline{G})$ if $Y(T^{-1}(\zeta,\eta))\in H^{\boldsymbol{\omega}+r}(\overline{D})$.

Main result:

Theorem 2.1. *Let D be a simply connected domain (defined in (b) and (c) on p. 5), $\overline{D}\subset IntQ\subset\mathbb{C}$, and let $G=T(\overline{D}),G\subset E\subset\mathbb{C}^2$, where the transformation $T(z)$ is defined in (e). If some holomorphic in G function $F:G\to\mathbb{C}$ can be approximated by a sequence of polynomials $P_n(\zeta,\eta),degP_n\leq n$, in two variables such that for some constant $C(F,G)$ and arbitrary $n\in\mathbb{N}$ the inequalities hold:*

$$|F(\zeta,\eta)-P_n(\zeta,\eta)|\le C(F,G)\delta_n^r(\zeta,\eta)\omega(\delta_n(\zeta,\eta))$$

for $(\zeta,\eta)\in\partial G$, *then* $F\in H^{\omega+r}(G)$.

Theorem 2.1 follows from the following result.

Theorem 2.2. *Let* D *be a simply connected domain (defined in parts (b) and (c) on page 5),* $\overline{D}\subset IntQ,\Gamma=\partial D$. *Let* $f:\overline{D}\to\mathbb{C}$. *If there is a sequence of polynomials of two variables* $P_n(\zeta,\eta), degP_n\le n$ *such that for some constant* $C(F,D)$ *independent of* n *the inequalities are satisfied*

$$|f(z)-P_n(\mathfrak{P}(z),\mathfrak{P}'(z))|\le C(F,D)\delta_n^r(z)\omega(\delta_n(z)), \qquad (f)$$

for $z\in\Gamma$, *then the function* f *belongs to the class* $H^{\omega+r}(\overline{D})$.

The proof of Theorem 2.2 requires establishing a version of Bernstein's inequality.

Theorem 2.3. *Let* D *be the domain described above,* $\overline{D}\subset IntQ,\Gamma=\partial D$. *For an arbitrary polynomial of two variables* $q_n(\zeta,\eta), degq_n\le n$, *for which the inequality holds*

$$|q_n(\mathfrak{P}(z),\mathfrak{P}'(z))|\le\delta_n^r(z)\omega(\delta_n(z)),$$

for $z\in\Gamma$, *the inequality is also true*

$$|q_n'(\mathfrak{P}(z),\mathfrak{P}'(z))|\le C(D)\delta_n^{r-1}(z)\omega(\delta_n(z)),$$

at $z\in\Gamma$.

Consider the Weierstrass sigma function $\sigma(u)$ defined by the expression

$$log\frac{\sigma(z)}{z}=-\int_0^z\left(\int_0^v\left(\mathfrak{P}(\eta)-\frac{1}{\eta^2}\right)d\eta\right)dv$$

According to [2, Chapter 1], $\sigma(z)$ is an entire function that has simple zeros at the vertices of the period grid (i.e. at the points $2\boldsymbol{\omega}_1n_1+2\boldsymbol{\omega}_2n_2$, where $n_1,n_2\in\mathbb{Z}$), for which expression g) is valid.

Let us choose a natural m and integers k_1^0,k_2^0 so that $\omega^0\stackrel{\text{def}}{=}\frac{2k_1^0\omega_1+2k_2^0\omega_2}{m}\in D$.

Consider the function

$$\mathfrak{I}(z)=\frac{\sigma(z-\boldsymbol{\omega}^0)^m}{\sigma(z)^{m-1}\sigma(z+2k_1^0\boldsymbol{\omega}_1+2k_2^0\boldsymbol{\omega}_2)},$$

where

$$\sigma(z+2\boldsymbol{\omega}_i)=-e^{2\eta_i(z+\boldsymbol{\omega}_i)}\sigma(z), i=1,2, \qquad (g)$$

quantities $\eta_i, i=1,2$ are the parameters of the Weierstrass function.

Note that $\mathfrak{I}(z)$ has a zero of multiplicity m at the point $\boldsymbol{\omega}^0$ and a pole of order $m-1$ at the point 0. Also from (g) it follows that

$$\mathfrak{I}(z+\boldsymbol{\omega}_i) = \mathfrak{I}e^{2\eta_i(m\boldsymbol{\omega}^0 - 2k_1^0\boldsymbol{\omega}_1 - 2k_2^0\boldsymbol{\omega}_2)} = \mathfrak{I}(z), i = 1, 2,$$

i.e. $\mathfrak{I}(z)$ is a doubly periodic function with periods $2\boldsymbol{\omega}_1, 2\boldsymbol{\omega}_2$. Let the set $\Omega = \Omega(D)$ be obtained from the domain D by transformations $z \to z + 2\boldsymbol{\omega}_1 n_1 + 2\boldsymbol{\omega}_2 n_2$:

$$\Omega = \{z : z = 2\boldsymbol{\omega}_1 n_1 + 2\boldsymbol{\omega}_2 n_2 + z_D, z_D \in D, n_1, n_2 \in \mathbb{Z}\}$$

The purpose of this section is to construct a function $V(z)$ harmonic in $\mathbb{C} \setminus \overline{\Omega}$, taking on $\partial\Omega$ values equal to

$$V_{\mathfrak{I}}(z) = -log|\mathfrak{I}(z)|.$$

Let us use the solution of the auxiliary Dirichlet problem. For an arbitrary natural number L, consider a parallelogram

$$Q_L = \{z : z = 2\boldsymbol{\omega}_1 k_1 + 2\boldsymbol{\omega}_2 k_2, k_1, k_2 \in \mathbb{R}, |k_1| \leq L, |k_2| \leq L\}.$$

Next, consider the function $\mathrm{w}_L(z)$, harmonic in the domain $Q_L \setminus \overline{\Omega}$ and continuous up to its boundary, satisfying the following boundary conditions:

$$\mathrm{w}_L(z) = \begin{cases} 1, & \mathrm{z} \in \partial Q_L, \\ 0, & \mathrm{z} \in \partial\Omega \bigcap Q_L. \end{cases}$$

Supporting conclusions and statements:

Lemma 6. *In an arbitrary bounded domain $\tilde{\Omega}, \tilde{\Omega} \subset \mathbb{C} \setminus \Omega$,*

$$\mathrm{w}_L(z) \rightrightarrows 0, L \to \infty.$$

Using the principle of maximum modulus and the properties of the function $\sigma(z)$ we arrive at the proof of Lemma 6.

$$V_{min} = \min_{z \in \partial D} V_{\mathfrak{I}}(z), V_{max} = \max_{z \in \partial D} V_{\mathfrak{I}}(z).$$

Let $V_L^+(z), V_L^-(z)$ be solutions to the Dirichlet problem in the domain $Q_L \setminus \overline{\Omega}$ with the following boundary conditions conditions:

$$V_L^+(z) = \begin{cases} V_{max}, & \mathrm{z} \in \partial Q_L. \\ V_{\mathfrak{I}}(z), & \mathrm{z} \in \partial\Omega. \end{cases}$$

$$V_L^-(z) = \begin{cases} V_{min}, & \text{z} \in \partial Q_L. \\ V_{\mathfrak{J}}(z), & \text{z} \in \partial\Omega. \end{cases}$$

From the maximum principle it follows that for $L \to \infty, V_L^+(z)$ is a decreasing family, and $V_L^-(z)$ is an increasing family functions.

And further, applying the results of Lemma 6, we come to the conclusion that there are limits

$$\lim_{x\to\infty} V_L^+(z) = \lim_{x\to\infty} V_L^-(z) = V(z),$$

and the function $V(z)$ is harmonic in $\mathbb{C} \setminus \Omega$.

Next, we establish some restrictions on the growth of polynomials in $\mathfrak{P}, \mathfrak{P}'$, near the poles of the functions $\mathfrak{P}, \mathfrak{P}'$.

Let $\delta_n(z)$ be defined above and the function $\mathrm{w}(z) : \Gamma \to \mathbb{R}$ satisfy the following conditions:

1. $\mathrm{w}(z) > A_1 n^{-A_2}, A_1 = A_1(D) > 0, A_2 = A_2(D) > 0$;
2. For points $z_1, z_2 \in \Gamma$, where $|z_1 - z_2| \leq \delta_n(z_1)$, the relation $\mathrm{w}(z_2) \asymp \mathrm{w}(z_1)$ is valid, and the constant commensurabilities depend only on the region D, but not on the points z_1, z_2;
3. For some positive constants $k_1 = k_1(D), C_1 = C_1(D)$ and constant points $z_1, z_2 \in \Gamma$ the following inequality holds:

$$\mathrm{w}(z_2) \leq C_1 \mathrm{w}(z_1) \left(\frac{|z_2 - z_1|}{\delta_n(z_1)} + 1 \right)^{k_1}.$$

Let us formulate the following lemma.

Lemma 7. *Let the function* $\mathrm{w}(z)$ *satisfy conditions (1)-(3). Then there is some number* $\varepsilon_0 > 0$ *such that for an arbitrary polynomial of two variables* $q_n(\zeta, \eta), deg q_n \leq n$ *satisfying the inequality*

$$|q_n(\mathfrak{P}(z), \mathfrak{P}'(z))| \leq \mathrm{w}(z),$$

for $z \in \Gamma$ *the inequality will be true:*

$$|q_n(\mathfrak{P}(z), \mathfrak{P}'(z))| \leq C_2 e^{C_3 n},$$

at $z \in Q_{\varepsilon_0}$, *where* $Q_{\varepsilon_0} = \mathbb{C} \setminus (\Omega \cup \bigcup_{n_1,n_2\in\mathbb{Z}} \{|z - 2\boldsymbol{\omega}_1 n_1 - 2\boldsymbol{\omega}_2 n_2| < \varepsilon_0\}), C_2 = C_2(D), C_3 = C_3(D)$.

Lemma 8. ([4], P.M. Tamrazov) *Let a bounded domain* $J \subseteq D$ *be given. Then there exists a constant* $C_4 = C_4(J)$ *such that for arbitrary positive constants* k, a, b *and for* *an arbitrary subharmonic function* $h(z)$ *in the domain* J *satisfying the inequality*

$$h(\zeta) \leq klog(a|\zeta - z_0| + b), \zeta \in \partial J,$$

the inequality is true

$$h(\zeta) \leq k[log(a|\zeta - z_0| + b) + C_4], \zeta \in J.$$

Lemma 9. *Let the function* $\mathrm{w}(z)$ *satisfy conditions 1)-3), and the polynomial* $q_n(\zeta, \eta)$ *satisfy the conditions of Lemma 7. Then there is a constant* $C_5 = C_5(D)$ *such that for an arbitrary point* $z_0 \in \Gamma$ *the following inequality holds:*

$$|q_n(\mathfrak{P}(\zeta), \mathfrak{P}'(\zeta))| \leq C_5 \mathrm{w}(z_0),$$

if $|\zeta - z_0| = \delta_n(z_0)$.

Theorem $2.3'$**.** *Let* D *be the domain defined in p.1, the function* $\mathrm{w}(z)$ *satisfies conditions 1)-3). For an arbitrary polynomial of two variables* $q_n(\zeta, \eta), degq_n \leq n$, *for which the inequality holds*

$$|q_n(\mathfrak{P}(z), \mathfrak{P}'(z))| \leq \mathrm{w}(z),$$

for $z \in \Gamma$, *the inequality also holds*

$$|q_n^{(k)}(\mathfrak{P}(z), \mathfrak{P}'(z))| \leq C_k \frac{\mathrm{w}(z)}{\delta_n^k(z)},$$

at $z \in \Gamma$.

To complete the proof of Theorem 2.1 in accordance with Tamrazov's theorem, it is enough to prove the following inequality

$$|f^{(r)}(z_1) - f^{(r)}(z_2)| \leq C_f \omega(|z_1 - z_2|),$$

for $z_1, z_2 \in \Gamma$.

The proof is carried out using the conclusions of Theorem 2.2 and Theorem $2.3'$. We choose a natural L in a certain way and assume that $\Delta_n(z) = P_{L^{n+1}}(z) - P_{L^n}(z)$. After some reasoning, we obtain that the function

$$f^{(r)}(z) = P_L^{(r)}(z) + \sum_{n=1}^{\infty} \Delta_n^{(r)}(z),$$

continuous.

We have

$$|f^{(r)}(z_1) - f^{(r)}(z_2)| \leq |P_L^{(r)}(z_1) - P_L^{(r)}(z_2)| + \left| \sum_{n=1}^{\infty} (\Delta_n^{(r)}(z_1) - \Delta_n^{(r)}(z_2)) \right| \qquad (h)$$

In view of the peculiarities of constructing the polynomial $P_L(z)$, we derive the following relation:

$$|P_L^{(r)}(z_1) - P_L^{(r)}(z_2)| \leq C'_{13}|z_1 - z_2| \leq C_{13}\omega(z_1 - z_2)$$

Next, we evaluate the second term from (h) and obtain the following representation:

$$\sum_{n=1}^{N_0} |\Delta_n^{(r)}(z_1) - \Delta_n^{(r)}(z_2)| \leq C_{15}|z_1 - z_2| \sum_{n=0}^{N_0} \frac{\omega(\delta_{L^n}(z_1))}{\delta_{L^n}(z_1)}.$$

Using the results of Theorem 2.3′, the geometric properties of the regions under consideration, as well as the definition of the modulus of continuity $\omega(t)$, we obtain

$$\left|\sum_{n=0}^{N_0}(\Delta_n^{(r)}(z_1) - \Delta_n^{(r)}(z_2))\right| \leq C_{16}\frac{\omega(\delta_{L^{N_0}}(z_1))}{\delta_{L^{N_0}}(z_1)}|z_1 - z_2| \leq C_{16}\omega(|z_1 - z_2|),$$

where $C_{16} = C_{16}(D)$.

And further

$$\left|\sum_{n=N_0+1}^{\infty} (\Delta_n^{(r)}(z_1) - \Delta_n^{(r)}(z_2))\right| \leq C_{17} \sum_{n=0}^{\infty} \frac{1}{r^{n\varepsilon_0}} \omega(\delta_L^{N_0+1}(z_1)) \ leq$$

$$\leq C_{18}\omega(\delta_L^{N_0+1}(z_1)) \leq C_{19}\omega(|z_1 - z_2|).$$

The results obtained, taking into account all previous arguments, prove the conclusion of Theorem 2.2.

In **the third chapter** we will consider the approximation by polynomials of doubly periodic Weierstrass functions for functions that are analytic in a domain and continuous in its closure. This problem is closely related to the approximation by holomorphic polynomials in two variables of a function that is holomorphic in a domain on an elliptic curve. We assume that the entire boundary of the domain D on the plane has an arc length commensurate with the chord length (see condition b) on page 4). This condition can also be transferred to the domain $T(D)$ on an elliptic curve. The possibility of obtaining an approximation estimate by polynomials of the functions $\mathfrak{P}(z), \mathfrak{P}'(z)$, which agrees with the so-called inverse theorem, i.e., with the restoration of the smoothness of a function by the rate of approximation, was obtained (in Chapter 1) for classes of functions analytic in the domain, for which in the closure of the domain the derivative of a given order has a modulus of continuity of Hölder type, and of order less than one. The approximation method used earlier did not make it possible to study classes of analytic functions whose derivative of some order is limited. In this chapter, we use a different approximation method to approximate by polynomials in doubly periodic Weierstrass functions

functions that are analytic in a domain and whose derivative of a given order is bounded. Let $\mathfrak{P}(z)$ be a doubly periodic Weierstrass function with periods $2\omega_1, 2\omega_2$, $Q = \{z \in \mathbb{C} : z = 2t_1\boldsymbol{\omega}_1 + 2t_2\boldsymbol{\omega}_2, 0 < t_1 < 1, 0 < t_2 < 1\}$ is a parallelogram of periods of the function $\mathfrak{P}(z)$. For $Q_{\mathbf{t}} = \omega_1 + \omega_2 + \mathbf{t}(Q - \omega_1 - \omega_2)$, but $0 < \mathbf{t} < 1$, $\mathbf{t}$ will be chosen so that $\overline{D} \subset Q_{\mathbf{t}}, \Gamma = \partial D$.

We assume that Γ satisfies the following condition: there exists $b > 0$, independent of $z_1, z_2 \in \Gamma$, such that at least for one of the arcs Γ with ends z_1 and z_2, which we denote by $\Gamma(z_1, z_2)$, its length does not exceed magnesium $b \cdot |z_2 - z_1|$. In what follows, the length of the arc α will be denoted by $|\alpha|$.

Let r be a natural number. Let us denote by $C_A^r(D)$ the following functional class: $C_A^r(D) = \{f : f$ is analytic in D and there is a constant c_f such that $|f^{(r)}(z)| \leq c_f, z \in D\}$. It is known [29, Ch. 7] that for $f \in C_A^r(D)$ for almost all $z_0 \in \Gamma$ with respect to the length of the curve Γ there are non-tangential limits of the function $f^{(r)}(z)$ and they are limited by c_f.

Main result:

Theorem 3.1. *Let $f \in C_A^r(D)$. Then there is a constant c_f^*, independent of n and z, and for any $n = 1, 2, ...$ there is a polynomial $P_n(u, v)$ in two variables, $degP_n \leq n$, such that the relation is valid*

$$|f(z) - P_n(\mathfrak{P}(z), \mathfrak{P}'(z))| \leq c_f^* \rho_{\frac{1}{n}}^r(z), z \in \partial D.$$

Note 3. *The class of functions $C_A^r(D)$ can be interpreted as the class of functions f, analytic in D, for which $f^{(r-1)}$ has in $\overline{D}$ a modulus of continuity $\omega(\delta) = c \cdot \delta$. The proof of the direct theorem in Chapter 1 for such a modulus of continuity does not work; in this chapter we use a different method of approximation .*

2 Publications based on research results

The results of the dissertation are presented in three articles [29], [30], [31] in peer-reviewed scientific journals. Articles [29] and [31] are included in the Scopus and Web of Science abstract databases. Article [30] is included in list D of the list of journals recommended by the Higher School of Economics.

• Sintsova K. A., Shirokov N. A., Approximation by Polynomials Composed of Weierstrass Doubly Periodic Functions, Vestnik St. Petersburg University, Mathematics, 56, pages 46–56, 2023.

• Sintsova K. A., Inverse approximation theorem on subsets of domains with peaks, Proceedings of scientific seminars of POMI, 527, 204–220, 2023.

• Sintsova K. A., Shirokov N. A., Approximation by doubly periodic functions in the classes C^r_A, Vestnik St. Petersburg University, Mathematics, 57, pages 345-352, 2024. DOI: 10.1134/S1063454124700195

Approbation of work

The results of the work were presented at a seminar on the theory of functions and operators of POMI RAS and at the international conference "Herzen Readings"at the Russian State Pedagogical University named after. A.I. Herzen in April 2024.

Conclusion

The main results of the dissertation work are as follows:
1. For functions f from classes of analytic functions $H^{r+\omega}(\overline{D})$, where the modulus of continuity ω satisfies the Dini condition, and the boundary of the domain $D \subset C$ contains a finite number of angles external to D equal to 2π, polynomials are constructed in two variables P_n such that $|f(z) - P_n(\mathfrak{P}(z), \mathfrak{P}'(z))|$ for $z \in \partial D$ have a decrease, consistent with the converse statement. Here $\mathfrak{P}$ is a fixed doubly periodic Weierstrass function.
2. The converse statement to point 1 has been proven, namely, if the function f for $z \in \partial D$ is approximated at the rate mentioned in point 1, then $f \in H^{r+\omega}(\overline{D})$.
3. For domains D, the length of the arc whose boundary is commensurate with the length of the chord, and for a function f, analytic in D and satisfying the condition $|f^{(r)}(z)| \leq c_f, z \in D, r \geq 1$, a polynomial P_n is constructed such that $P_n(\mathfrak{P}(z), \mathfrak{P}'(z))$ approximates the function f for $z \in \partial D$ with the speed obtained in step 1 for $\omega(t) = t$ and $r - 1$ instead of r.

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