NATIONAL RESEARCH UNIVERSITY
HIGHER SCHOOL OF ECONOMICS

Faculty of Mathematics

as a manuscript

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Nonlinear Kantorovich transportation problems

Summary of the PhD thesis
for the purpose of obtaining academic degree
Doctor of Philosophy in Mathematics

Academic Supervisor:
Doctor of Science, Professor
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MOSCOW – 2024

Introduction

The central objects of the dissertation are several new modifications of the classical Kantorovich transportation problem that have appeared in the last decade:

- nonlinear Kantorovich transportation problems, in particular, the Kantorovich transportation problem with conditional measures,
- nonlinear Kantorovich transportation problems with density constraints,
- transportation problems with fixed barycenters.

Approbation of the dissertation research results. The results of the dissertation were presented by the author at the following scientific conferences:

1. 6-th St. Petersburg Youth Conference in Probability and Mathematical Physics

2. III International Conference "Mathematical Physics, Dynamical Systems, Infinite-Dimensional Analysis", dedicated to the 100th anniversary of V.S. Vladimirov, the 100th anniversary of L.D. Kudryavtsev and the 85th anniversary of O.G. Smolyanov

Articles. The results of this dissertation are published in two articles:

- V.I. Bogachev, A.V. Rezbaev, Existence of solutions to the nonlinear Kantorovich problem of optimal transportation, Mathematical Notes, 112:3 (2022), 369–377.
- V. I. Bogachev, S. N. Popova, A. V. Rezbaev, On nonlinear Kantorovich problems with density constraints, Moscow Mathematical Journal, 23:3 (2023), 285–307.

Structure and scope of the dissertation. The dissertation consists of the introduction, five chapters, the conclusion and the bibliography of 58 titles. The total size of the dissertation is 76 pages.

Kantorovich transportation problem

Let us recall the modern formulation of the classical Kantorovich transportation problem (for the original formulation and some economic premises of the problem, see, for example, the original works of L.V. Kantorovich and L.V. Kantorovich in collaboration with G.Sh. Rubinshtein [1], [2], [3], [4], [5], and [6]). Let $(X, \mathcal{B}_X, \mu)$ and $(X, \mathcal{B}_Y, \nu)$ be two abstract probability spaces and let h be a nonnegative $\mathcal{B}_X \otimes \mathcal{B}_Y$-measurable function, called a cost function. The set of all Radon probability measures on the product $(X \times Y, \mathcal{B}_X \otimes \mathcal{B}_Y)$ with projections μ and ν on the factors (the measures in this set are called Kantorovich plans or transport plans) is denoted by $\Pi(\mu, \nu)$:

$$\sigma(A \times Y) = \mu(A), A \in \mathcal{B}_X, \quad \sigma(X \times B) = \nu(B), B \in \mathcal{B}_Y.$$

The measures μ and ν are called *marginals* or *marginal distributions*. The Kantorovich problem consists in minimization of the integral

$$\int\limits_{X \times Y} h(x, y)\sigma(dx\, dy)$$

over the measures $\sigma \in \Pi(\mu, \nu)$. If there is a measure at which the minimum is attained, then the Kantorovich problem is said to have a solution, and this measure is called *an optimal plan*. A number of very broad sufficient conditions are known for the existence of solutions of the Kantorovich problem For example, a solution exists if X and Y are completely regular topological spaces, the marginal distributions are Radon measures, and the cost function h is bounded and lower semicontinuous (see [10]). In the general case, there exists the infimum

$$K_h(\mu, \nu) = \inf_{\sigma \in \Pi(\mu,\nu)} \int\limits_{X \times Y} h(x, y)\sigma(dx\, dy).$$

1 Nonlinear Kantorovich transportation problems

This chapter of the dissertation is devoted to nonlinear Kantorovich transportation problems, and the Kantorovich problem with conditional measures in particular. In the first section, a formulation of a nonlinear transport problem is given and a theorem is proved that provides sufficient conditions for the existence of a solution to such a problem. The central object of the second section is the Kantorovich problem with conditional measures, the specificity of which lies in the special form of dependence of the cost function on the transport plan (through conditional measures). There we also prove an existence theorem for such a problem. Both theorems are proven for a fairly wide class of cost functions and spaces X and Y. The case of Polish spaces is considered in the work [8].

1.1 Nonlinear Kantorovich problems

In its most general form, the nonlinear Kantorovich transportation problem is formulated as follows. Let $(X, \mathcal{B}_X, \mu)$ and $(X, \mathcal{B}_Y, \nu)$ be two abstract probability spaces. The set of all Radon probability measures on the product $(X \times Y, \mathcal{B}_X \otimes \mathcal{B}_Y)$ with projections μ and ν on the factors is denoted by $\Pi(\mu, \nu)$:

$$\sigma(A \times Y) = \mu(A), A \in \mathcal{B}_X, \quad \sigma(X \times B) = \nu(B), B \in \mathcal{B}_Y.$$

Suppose that for every $\sigma \in \Pi(\mu, \nu)$ there is a function $h : X \times Y \times \Pi(\mu, \nu) \to \mathbb{R}_+$ measurable with respect to σ-algebras $\mathcal{B}_X \otimes \mathcal{B}_Y$ and $\mathcal{B}(\mathbb{R})$. The problem

$$\int\limits_{X\times Y} h(x, y, \sigma)\sigma(dx\, dy) \to \min, \quad \sigma \in \Pi(\mu, \nu).$$

is called the **nonlinear Kantorovich transportation problem**. The measures μ and ν are called *marginals* or *marginal distributions.* If there is a measure σ_0 at which the minimum is attained, then the nonlinear Kantorovich problem is said to have a solution, and this measure is called *an optimal plan.* In the general case, there exists the infimum

$$K_h(\mu, \nu) = \inf_{\sigma\in\Pi(\mu,\nu)} \int\limits_{X\times Y} h(x, y, \sigma)\sigma(dx\, dy).$$

For a completely regular space X we denote by $\mathcal{P}_r(X)$ the space of Radon probability measures on X, i.e., Borel measures μ such that for every Borel set B and every $\varepsilon > 0$ there exists a compact set $K \subset B$ such that $\mu(B \backslash K) \leq \varepsilon$. Let $\mathcal{B}(X)$ denote the Borel σ-algebra of a topological space X and let $\mathcal{B}a(X)$ denote the Baire σ-algebra that is generated by all continuous functions on X. In the case of a completely regular Souslin space the equality $\mathcal{B}a(X) = \mathcal{B}(X)$ holds, see [9, Theorem 6.7.7]. We equip $\mathcal{P}_r(X)$ with the weak topology, which on the whole space of signed measures is generated by the family of seminorms

$$p_f(\mu) = \left| \int_X f\, d\mu \right|,$$

where f is a continuous bounded function on X.

A family of measures $M \subset \mathcal{P}_r(X)$ is called uniformly tight if, for every $\varepsilon > 0$, there exists a compact set $K \subset B$ such that $\mu(B \backslash K) \leq \varepsilon$ for all $\mu \in M$.

1.2 Existence theorem for the nonlinear Kantorovich problem

This section is devoted to the proof of the theorem on the existence of a solution to the nonlinear Kantorovich transportation problem for wide classes of probability spaces and cost functions. The proof of the main theorem is based on the following lemma.

Lemma 1. Let X be a completely regular space, let Π be a uniformly tight compact subset in $\mathcal{P}_r(X)$, and let a function $h\colon X \times \Pi \to [0, +\infty)$ be lower semicontinuous on all sets of the form $K \times \Pi$, where K is compact in X. Then, the following function is lower semicontinuous:

$$J_h(\sigma) = \int_X h\, d\sigma, \quad \Pi \to [0, +\infty].$$

Existence theorem. Let X and Y be completely regular spaces and let μ and ν be Radon probabilty measures on X and Y respectively. Suppose that the function h is lower semicontinuous on all sets of the form $K \times \Pi(\mu, \nu)$, where K is a compact set in $X \times Y$. Then there exists an optimal plan.

1.3 Kantorovich problems with conditional measures

In this section we continue to consider nonlinear Kantorovich transportation problems, but all attention will be focused on the case of cost functions of the special form

$$h\colon X \times Y \times \mathcal{P}(X \times Y) \to \mathbb{R}, \quad h(x, y, \sigma) = h(x, \sigma^x).$$

We call nonlinear transportation problems with cost functions of this type **Kantorovich transport problems with conditional measures**. Since now the cost function h depends on transport plans σ through conditional measures, which can violate its continuity on sets of the form $K \times \Pi(\mu, \nu)$, where K is compact in $X \times Y$, in order to prove the existence of solutions to such problems we have to impose more restrictive conditions on the function h.

The existence of conditional measures for σ means that σ has the form

$$\sigma(dx\, dy) = \sigma^x(dy)\mu(dx),$$

the function $x \mapsto \sigma^x(B)$ is measurable with respect to μ for all $B \in \mathcal{B}(Y)$, and for every bounded function f on $X \times Y$, measurable with respect to $\mathcal{B}(X) \otimes \mathcal{B}(Y)$, the equality

$$\int_{X \times Y} f\, d\sigma = \int_X \int_Y f(x, y)\, \sigma^x(dy)\, \mu(dx)$$

holds. It is clear that it suffices to have this equality for all functions of the form $I_A(x) I_B(y)$, where $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$. Conditional measures exist under rather broad assumptions, for example, in case of Souslin spaces, i.e., images of complete separable metric spaces under continuous mappings.

Since we consider general completely regular topological spaces X and Y (in this case there is no guarantee that conditional measures exist), we must always require in advance that conditional measures exist, otherwise the statement of the problem itself will be meaningless.

The proof of the main result is based on the following two lemmas.

Lemma 2. (i) Suppose that E is a completely regular space, a net $P_\alpha \in \mathcal{P}_r(E)$ is uniformly tight and converges weakly to a measure $P \in \mathcal{P}_r(E)$, and $H\colon E \to [0,1]$ is a function such that for each $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset E$ for which $P_\alpha(E\backslash K_\varepsilon) < \varepsilon$ for all α and the restriction of H to K_ε is lower semicontinuous. Then

$$\int_E H\,dP \le \liminf_\alpha \int_E H\,dP_\alpha. \tag{1}$$

(ii) Suppose that Y is a completely regular space, a measure $Q \in \mathcal{P}_r(\mathcal{P}_r(Y))$ is concentrated on a countable union of uniformly tight sets and a bounded function H on $\mathcal{P}_r(Y)$ in convex and lower semicontinuous on uniformly tight sets. Then

$$H\biggl(\int_{\mathcal{P}_r(Y)} p\,Q(dp)\biggr) \le \int_{\mathcal{P}_r(Y)} H(p)\,Q(dp).$$

Lemma 3. Suppose that a function $H\colon X \times \mathcal{P}_r(Y) \to [0,+\infty)$ is measurable with respect to $\mathcal{B}(X)\otimes\mathcal{B}a(\mathcal{P}_r(Y))$, and lower semicontinuous on the sets of the form $K\times S$, where K is a compact set in X and $S \subset \mathcal{P}_r(Y)$ is uniformly tight and convex in the second argument. Then the function

$$J_H(\sigma) = \int_X H(x,\sigma^x)\,\mu(dx)$$

is lower semicontinuous on $\Pi(\mu,\nu)$.

The following main result follows directly from the previous two lemmas and the weak compactness of the set of plans $\Pi(\mu,\nu)$.

Existence theorem. Let X and Y be completely regular spaces and let μ and ν be Radon probability measures on X and Y respectively. Suppose that the cost function $H\colon X\times\mathcal{P}_r(Y) \to [0,+\infty)$ is measurable with respect to $\mathcal{B}a(X)\otimes\mathcal{B}a(\mathcal{P}_r(Y))$, lower semicontinuous on all sets of the form $K\times S$, where K is a compact set in X and $S\subset\mathcal{P}_r(Y)$ is uniformly tight, and convex in the second argument. Then

$$\inf_{\sigma\in\Pi(\mu,\nu)} \int_X H(x,\sigma^x)\,\mu(dx)$$

is attained, that is, an optimal plan exists.

Remark 2. (i) The statements obtained above remain valid in the situation where the functions H and h take values in $[0, +\infty]$. It suffices to apply the established facts to the functions $\min(h, N)$ and $\min(H, N)$.

(ii) It is clear from the proof that it suffices to impose the measurability condition on the function H on the sets of the form $K \times S$ with compact factors. For a broad class of spaces, the lower semicontinuity on such sets implies the $\mathcal{B}(K) \otimes \mathcal{B}a(S)$-measurability. For example, this is true if Y is Souslin and the Borel and Baire σ-algebras coincide on compact sets in X. In the case of general spaces (even Souslin spaces), the condition of lower semicontinuity on compact sets is weaker than the global lower semicontinuity (in the theorem, the condition is even slightly weaker, since we are speaking of uniformly tight compact sets). In completely regular Souslin spaces, compact sets are metrizable, therefore, this condition can be verified by using countable sequences. Moreover, for such spaces, the lower semicontinuity on compact sets implies the Borel measurability on compact sets, which coincides with the $\mathcal{B}a(X) \otimes \mathcal{B}a(\mathcal{P}_r(Y))$-measurability, therefore, it need not be required additionally. On some sequential properties of spaces of measures, see [?].

We return to Kantorovich problems with conditional measures in Chapter 4, where, due to a special form of cost functions, we can somewhat weaken the convexity condition.

2 Nonlinear Kantorovich problems with density constraints

This chapter deals with another new modification of the Kantorovich transportation problem. While in the previous chapter the novelty of the problem was to change the type of the cost function (dependence on transport plans appeared), now the change will affect both the cost function itself and the domain where the functional is defined. The main results of this chapter are two existence theorems related to nonlinear Kantorovich problems with density constraints and to Kantorovich problems with conditional measures and density constraints. Both theorems provide sufficient conditions for the existence of solutions to such problems. Problems with density constraints were first considered in the works [11], [12], [13], and [14].

2.1 Nonlinear Kantorovich problems with density constraints

Let $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ be two abstract probability spaces and let λ be a probability measure on the measurable space $(X_1 \times X_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$. Suppose that Φ is a nonnegative $\mathcal{B}_1 \otimes \mathcal{B}_2$-measurable function integrable with respect to λ. As above, the set $\Pi_\Phi(\mu_1, \mu_2)$ consists of all probability measures σ on $(X_1 \times X_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$ with projections μ_1 and μ_2 on the factors such that σ is absolutely continuous with respect to λ and for the corresponding Radon–Nikodym density we have

$$\varrho_\sigma = \frac{d\sigma}{d\lambda} \leq \Phi \quad \lambda\text{-a.e.}$$

We assume that $\Pi_\Phi(\mu_1, \mu_2)$ is not empty. This assumption is fulfilled if, for example, $\lambda = \mu_1 \otimes \mu_2$ and $\Phi \geq 1$ (in this case $\lambda \in \Pi_\Phi(\mu_1, \mu_2)$).

The set of measures $\Pi_\Phi(\mu_1, \mu_2)$ can be identified with the set of their densities with respect to λ and regarded as a subset of $L^1(\lambda)$.

Let $\mathcal{P}_\lambda$ be the set of all probability densities in $L^1(\lambda)$. Let $\mathcal{B}(\mathcal{P}_\lambda)$ be the Borel σ-algebra with respect to the norm of $L^1(\lambda)$.

Suppose also that there is a function

$$h\colon X \times Y \times \mathcal{P}_\lambda \to [0, +\infty).$$

that is $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{B}(\mathcal{P}_\lambda)$-measurable.

Definition. The problem

$$J_h(p) := \int\limits_{X\times Y} h(x, y, p)p(x, y)\, \lambda(dx\, dy) \to \min, \quad p \in \Pi_\Phi(\mu, \nu)$$

is called the **nonlinear Kantorovich problem with density constraints**. If there is a measure $p \in \Pi_\Phi(\mu, \nu)$ at which the minimum is attained, then the problem is said to have a solution,

Theorem 3. Let for λ-a.e. $(x_1, x_2) \in X_1 \times X_2$ the function $p \mapsto h(x_1, x_2, p)$ is lower semicontinuous on $\Pi_\Phi(\mu_1, \mu_2)$ with respect to the norm of $L^1(\lambda)$ and the function J_h is convex. Then the nonlinear Kantorovich problem with density constraints

$$J_h(p) := \int\limits_{X\times Y} h(x, y, p)p(x, y)\,\lambda(dx\,dy) \to \min, \quad p \in \Pi_\Phi(\mu, \nu)$$

has a solution. Note that the condition of continuity of h in p with respect to the norm is much weaker than the condition of continuity with respect to the weak topology

The convexity condition on J_h has an obvious drawback: it does not follow from the convexity of h in p.

2.2 Kantorovich problems with conditional measures and density constraints

We now turn to the case where the convexity of J_h follows from the convexity of h in p, but can hold without the latter. Let the cost function have the form

$$h\colon X \times Y \times \mathcal{P}_\lambda \to [0, +\infty), \quad h(x, y, p) = h(x, p^x),$$

where p^x are conditional measures of p with respect to μ. We will call nonlinear transportation problems with density constraints and with cost functions of this type **Kantorovich transportation problems with conditional measures and density constraints**.

Of course, in order to define such functions it is necessary to assume that every measure $p \in \Pi_\Phi(\mu, \nu)$ has conditional measures on Y with respect to its projection on X. This is fulfilled automatically for all Borel measures on Souslin spaces (see [9, Chapter 10]).

If $\lambda = \mu \otimes \nu$ and we identify measures in $\Pi_\Phi(\mu, \nu)$ with their densities with respect to λ, then the conditional measures for such measures $p \cdot \lambda$, where p is a $\mathcal{B}_X \otimes \mathcal{B}_Y$-measurable version of the density, can be defined by the formula

$$p^x = p(x, \cdot) \cdot \nu.$$

Once we deal with conditional measures on $\mathcal{B}_Y$, it is reasonable to equip the space $\mathcal{M}(Y)$ of all bounded measures on $\mathcal{B}_Y$ with the σ-algebra $\mathcal{E}(\mathcal{M}(Y))$ generated by all functions $\nu \mapsto \nu(B)$, $B \in \mathcal{B}_Y$. One can show that this σ-algebra is countably generated if Y is Souslin.

In the next theorem we assume that the function h is measurable with respect to the σ-algebra $\mathcal{B}_X \otimes \mathcal{E}(\mathcal{M}(Y))$. Then the function

$$x \mapsto h(x, p^x)$$

is $\mathcal{B}_X$-measurable provided the mapping $x \mapsto p^x$ is $(\mathcal{B}_X, \mathcal{E}(\mathcal{M}(Y)))$-measurable, because the mapping $x \mapsto (x, p^x)$ is measurable with respect to the pair of σ-algebras $\mathcal{B}_X$ and $\mathcal{B}_X \otimes \mathcal{E}(\mathcal{M}(Y))$.

Transportation problems of this type can be written as

$$\int_X h(x, p^x)\, \mu(dx) \to \min, \quad p \in \Pi_\Phi(\mu, \nu),\ p(dx\, dy) = p^x(dy)\mu(dx).$$

Due to a special form of h the convexity of h implies the convexity of J_h. Hence the next theorem covers some cases not covered by the previous theorem, although the assumptions are the same (of course, the functional J_h is not the same and involves conditional measures).

Theorem 4. Suppose that for μ-almost every x the function $p \mapsto h(x, p)$ is lower semicontinuous with respect to the total variation norm on $\mathcal{M}(Y)$. If, in addition, the function

$$C_h(p) = \int_X h(x, p^x)\, \mu(dx)$$

is convex, then it attains its minimum on $\Pi_\Phi(\mu, \nu)$ (in particular, this is true if h is convex with respect to the second argument), i.e. the Kantorovich transportation problem with conditional measures and density constraints has a solution.

3 Kantorovich problems with fixed barycenters

This chapter is devoted to the consideration of another new type of Kantorovich transportation problems. The specificity of this problem is that now the marginal distribution of ν begins to play a slightly different role. While in the classical Kantorovich problem the logic of the formulation itself required both marginal distributions to be considered as the corresponding projections of admissible transport plans, in the Kantorovich problem with a fixed barycenter the measure μ is still considered as a projection, and the measure ν now plays the role of the barycenters (averages) of measures from the set $\mathcal{P}(\mathcal{P}(Y))$, that is, we do not know exactly the second marginal distribution, but only its mean (barycenter).

Let us recall that for any Radon probability measure Q on the space of measures $\mathcal{P}(Y)$ with the weak topology the barycenter is defined by the formula

$$\beta_Q := \int_{\mathcal{P}(Y)} p\, Q(dp),$$

where this vector integral with values in the space of measures is understood as the equality

$$\beta_Q(A) = \int_{\mathcal{P}(Y)} p(A)\, Q(dp)$$

for all Borel sets $A \subset Y$. It is a well known fact that the function $p \mapsto p(A)$ is Borel measurable on $\mathcal{P}(Y)$ and the measure β_Q is τ-additive (see [9, Proposition 8.9.8 and Corollary 8.9.9]).

If P is a Radon measure on $X \times \mathcal{P}(Y)$, μ is its projection on X and there are conditional measures P^x on $\mathcal{P}(Y)$ with respect to μ, then the barycenter of the projection $P_{\mathcal{P}}$ of P on $\mathcal{P}(Y)$ is given by

$$\beta_{P_{\mathcal{P}}}(B) = \int_X \int_{\mathcal{P}(Y)} p(B)\, P^x(dp)\, \mu(dx).$$

We denote by $\Pi^\beta(\mu)$ the set of all Radon probability measures π on $X \times \mathcal{P}(Y)$ such that the projection π_X of π on X is μ and the barycenter of the projection $\pi_{\mathcal{P}}$ of π on $\mathcal{P}(Y)$ is a given measure $\beta \in \mathcal{P}(Y)$:

$$\Pi^\beta(\mu) := \{\pi \in \mathcal{P}(X \times \mathcal{P}(Y))\colon \pi_X = \mu, \quad \beta_{\pi_{\mathcal{P}}} = \beta\}.$$

Definition. For a given function $h : X \times \mathcal{P}(Y) \to \mathbb{R}_+$ measurable with respect to $\mathcal{B}(X) \otimes \mathcal{B}(\mathcal{P}(Y))$ and $\mathcal{B}(\mathbb{R})$ the problem

$$\int_{X\times\mathcal{P}(Y)} h(x,p)\, \pi(dx\, dp) \to \min, \quad \pi \in \Pi^\beta(\mu)$$

is called the **Kantorovich problem with fixed barycenter**.

Let us recall that a set $\Gamma \subset X \times Z$ is called h-cyclically monotone for a function h on $X \times Z$ if for all n one has

$$\sum_{i=1}^n h(x_i, z_i) \le \sum_{i=1}^n h(x_{i+1}, z_i)$$

for all pairs $(x_1, z_1), \dots, (x_n, z_n) \in \Gamma$, where $x_{n+1} := x_1$.

Theorem 5. (i) Let h be a bounded lower semicontinuous function on the product $X \times \mathcal{P}(Y)$. For all $\mu \in \mathcal{P}(X)$ and $\beta \in \mathcal{P}(Y)$ the Kantorovich problem

$$J_h(\pi) = \int_{X\times\mathcal{P}(Y)} h(x,p)\, \pi(dx\, dp) \to \min, \quad \pi \in \Pi^\beta(\mu) \tag{2}$$

with a fixed barycenter has a solution. Moreover, this is true for h with values in $[0, +\infty]$ if J_h is not identically $+\infty$.

(ii) Every optimal measure P for this problem is also optimal for the classical linear problem with the same cost function and marginals μ and $P_{\mathcal{P}}$, where $P_{\mathcal{P}}$ is the projection of P on $\mathcal{P}(Y)$.

(iii) Finally, if X and Y are Souslin spaces, then P is concentrated on an h-cyclically monotone set.

The results of this chapter are used in Chapter 4 to prove existence theorems for Kantorovich problems with conditional measures under less restrictive assumptions than in Chapter 1, but at the expense of a special form of cost functions h.

4 Kantorovich problems with conditional measures. Continuation

In this section we continue our consideration of the Kantorovich problem with conditional measures, which we began in the first chapter. In Section 3 of Chapter 1, we saw that an important sufficient condition for the existence of a solution to such problems is the convexity in the second argument of the cost function h, and in Chapter 5 it is shown that in the absence of such convexity the problem may fail to have a solution. In this chapter, we show that for cost functions of some special form the convexity condition can be slightly weakened (or completely abandoned) while maintaining the solvability of the problem. The main results of this chapter are two existence theorems for Kantorovich problems with conditional measures.

Since all Borel measures on Suslin spaces are Radon, instead of the notation $\mathcal{P}_r(T)$ we will use the shorter notation $\mathcal{P}(T)$.

The first result is devoted to the case where the cost function decomposes into a product of two functions of one variable.

Theorem 6. Let X and Y be Souslin spaces. Suppose that

$$h(x, \sigma) = f(x)g(\sigma),$$

where $f\colon X \to \mathbb{R}$ and $g\colon \mathcal{P}(Y) \to \mathbb{R}$ are bounded continuous functions and the sets $\{p \in \mathcal{P}(Y)\colon g(p) \le c\}$ are convex for all $c \in \mathbb{R}$. Then, for any atomless measure $\mu \in \mathcal{P}(X)$ and any measure $\nu \in \mathcal{P}(Y)$ the infimum

$$\inf_{\sigma\in\Pi(\mu,\nu)} \int_X h(x, \sigma^x)\,\mu(dx)$$

is attained.

The following theorem shows that in the case where the cost function does not depend on x the requirement of convexity in the existence theorem can be completely abandoned.

Theorem 7. Let X and Y be Souslin spaces and let h be a bounded continuous function on $\mathcal{P}(Y)$ with the weak topology. Then, for any measures

$\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$, where μ has no atoms, the problem

$$\int_X h(\sigma^x)\,\mu(dx) \to \min, \quad \sigma \in \Pi(\mu,\nu),\ \sigma(dx\,dy) = \sigma^x(dy)\,\mu(dx)$$

has a solution.

5 Counter-examples

In this chapter we provide various counterexamples to some of the assertions proven in previous chapters. In particular, Example 1 shows that the statement of Theorem 2 about the existence of a solution in the Kantorovich problem with conditional measures loses its validity if the condition of convexity of the function h with respect to the second argument is abandoned. Similarly, Example 2 demonstrates that even the weakened convexity condition in Theorem 6, which was achieved by simplifying the form of the function h, cannot be abandoned if we require that the problem have a solution. Example 3 concerns the Kantorovich problem with conditional measures and with density constraints. It follows from this that the convexity condition in Theorem 4 is also essential and, in the general case, cannot be discarded without losing the existence of a solution to such a problem. Example 4 shows, firstly, that since the dependence of the cost function h on the plan σ through conditional measures, as in the Kantorovich problem with conditional measures, is of a more complex nature (than, say, in the simply nonlinear Kantorovich problem), in such problems, generally speaking, lower semicontinuity of h does not imply lower semicontinuity of J_h. Secondly, the condition of lower semicontinuity of the function J_h is not necessary for the existence of a solution to the Kantorovich problem with conditional measures.

In [7, Examples 3.2 and 3.3] one can find examples of non-existence of minimum in the nonlinear problem with conditional measures, but with marginal distributions that are not absolutely continuous. In our examples both marginals coincide with Lebesgue measure on the unit interval

Let us recall that the standard Kantorovich–Rubinshtein norm on Radon measures on a bounded metric space X defined by the formula

$$\|\sigma\|_{KR} = \sup_{f\in \mathrm{Lip}_1,\, |f|\le 1} \int_X f\, d\sigma,$$

where Lip_1 is the class of all 1-Lipschitz functions on X. This norm generates the weak topology on the subset of nonnegative measures (see [9, Theorem 8.3.2]).

Example 1. Let $X = Y = [0,1]$ and let $\mu = \nu = \lambda$ be Lebesgue measure on the interval $[0,1]$. There is a bounded continuous function h on $X\times\mathcal{P}(Y)$, which is Lipschitz when $\mathcal{P}(Y)$ is considered with the Kantorovich–Rubinshtein norm, such that the nonlinear problem

$$\int_X h(x,\sigma^x)\mu(dx) \to \inf, \quad \sigma\in\Pi(\mu,\nu),\ \sigma(dxdy) = \sigma^x(dy)\mu(dx)$$

has no minimizer.

Our next example is similar, but the cost function breaks down into a product of functions of one variable.

Example 2. Let $X = Y = [0,1]$ and let $\mu = \nu = \lambda$ be Lebesgue measure on $[0,1]$. There are bounded continuous functions $f\colon X\to\mathbb{R}$ and $g\colon \mathcal{P}(Y)\to\mathbb{R}$, where $\mathcal{P}(Y)$ is considered with the weak topology, for which there is no minimum in the nonlinear Kantorovich problem

$$J(\sigma) = \int_X f(x)g(\sigma^x)\,\mu(dx) \to \inf, \quad \sigma\in\Pi(\mu,\nu).$$

This example demonstrates that in Theorem 6 we cannot abandon the convexity condition for the sets $\{p\in\mathcal{P}(Y)\colon g(p)\le c\}$ for all $c\in\mathbb{R}$.

Example 3. Let $X = Y = [0,1]$ and let $\mu = \nu = \lambda$ be the standard Lebesgue measure on $[0,1]$. We consider the Kantorovich problem with conditional measures and density constraints

$$J_h(\varrho) = \int_0^1 h(x,\varrho(x,\cdot))\,dx \to \inf,$$

$$\varrho(x,y)\le 4 \ \ \forall x,y, \quad \int_0^1 \varrho(x,y)\,dy = 1 \ \ \forall x, \quad \int_0^1 \varrho(x,y)\,dx = 1 \ \ \forall y.$$

There exists a bounded continuous function $h\colon X \times L^1[0,1] \to \mathbb{R}$, where the space $L^1[0,1]$ is equipped with the weak topology, such that there is no minimizer.

Example 4. Let $X = Y = [-1/2, 1/2]$ and let $\mu = \nu = \lambda$ be Lebesgue measure on $[-1/2, 1/2]$. Let

$$h(p) = \left(\int_{-1/2}^{1/2} \varphi(y)\, p(dy) \right)^k,$$

where φ is a non-constant continuous function, $k \in \mathbb{N}$, $k \geq 2$. We can assume that the integral of φ is zero, for example, $\varphi(y) = y$ is suitable. Then h is bounded and continuous on $\mathcal{P}(Y)$, the functional

$$J_h(\sigma) = \int_{-1/2}^{1/2} h(\sigma^x)\, dx, \quad \sigma(dxdy) = \sigma^x(dy)\, dx$$

is not lower semicontinuous on $\Pi(\mu, \nu)$, but it attains a minimum.

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