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Grekov Andrei

Elliptic integrable systems: eigenfunctions and dualities

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Contents

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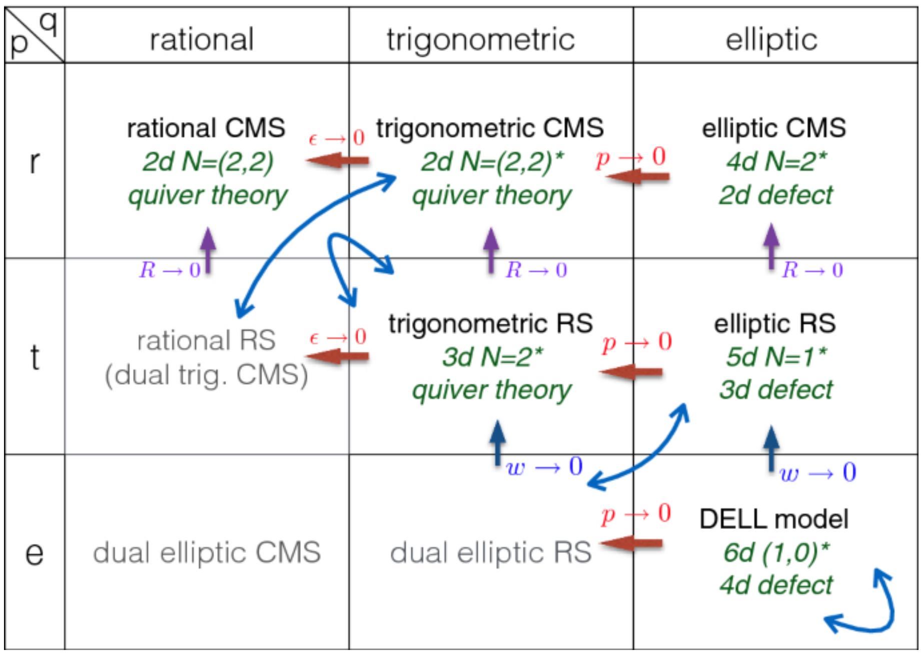
Introduction

This dissertation is devoted to the topic of dualities in many-particle integrable systems. The central player in it is a series of Calogero-Moser-Sutherland / Ruijsenaars-Schneider systems and their degenerations/generalizations. In canonical coordinates, these systems are characterized by the dependence of the Hamiltonians on the positions and momenta of the particles. Both of these dependencies can be rational, trigonometric, or elliptic, respectively. Thus, we get a 3x3 table of 9 systems.([119])



These systems are connected by dualities both with each other and with models such as spin chains and Gaudin systems. We will look at just a few of them. The first part of the dissertation is devoted to systems with a rational dependence on momentum - Calogero-Moser systems (first row of the table). They are labeled by a choice of a simple Lie algebra. Their classical integrability is guaranteed by the existence of a Lax pair. In its minimal version, this approach leads to embedding the dynamics of the system into the coadjoint orbit of the corresponding simple Lie group. However, in a recent work, A. Levin, M. Olshanetsky, and A. Zotov ([96]) constructed a new type of Lax matrix for the system associated with the algebra sl(N). Its design uses the concept of a quantum R-matrix, which emerges as a basic building block in spin chain systems. In the thesis, we study its generalizations to the case of other root systems, as well as its application to the construction of a new system of the Haldane-Shastri spin chain class. The second part concerns systems with a trigonometric dependence on momentum, namely Ruijsenaars-Schneider systems (second row of the table). It is devoted to a generalization of the recent work of A. Zotov and A. Zabrodin ([3]) on the connection of these systems with the quantum Knizhnik Zamolodchikov equation, which in turn generalizes even earlier results on this topic, starting with the work of A. Matsuo ([4]). Duality works as follows. Solutions of the quantum Knizhnik-Zamolodchikov equation are numbered by the value of the total spin. Summing up the components of the solution in the sector with a fixed spin, we obtain the wave function of the Ruijsenaars-Schneider system. This correspondence is a deformation of the quantum-classical duality between Lagrangian submanifolds in the phase space of the classical Ruijsenaars-Schneider system and the solutions of the Bethe equations of the corresponding spin chain. The dissertation examines the generalization of this duality to spin chains constructed from R-matrices associated with supersymmetric algebras.

The third and fourth parts of the dissertation study systems with an elliptic dependence on momentum (last row of the table). Two of these systems are related by hypothetical p-q duality to the elliptic model of Calogero and Ruijsenaars, respectively. The third, doubly elliptic system, is the most general of the entire series being studied. To consider it even at the classical level, several independent approaches were developed: the first - by A. Mironov, A. Morozov, and G. Aminov, and the second - by G. Braden and T. Hollowood. Recently, P. Koroteev and Sh. Shakirov proposed a quantum version of a double elliptic integrable system. Their model is a quantization of Braden's and Hollowood's approach. In the limit with trigonometric dependence on coordinates, the eigenfunctions hypothetically look like a generalization of Macdonald polynomials with coefficients elliptically depending on q and t. In this thesis, I will present a generalization of the Sekiguchi-Debiard-McDonald determinant for these functions. I will show how it can be used to prove the formula for the eigenvalues of the Hamiltonians in this limit. Partial progress has also been made in constructing Cherednik operators for this system. Unfortunately, this progress was not enough to prove the commutativity of the Hamiltonians, but it was still useful for the construction of an infinite number of particles limit, which turned out to be related to the representation theory of elliptic quantum toroidal algebra. Some of the above results are also true in the completely non-degenerate double elliptic case.

1 R-matrix valued Lax pairs

In this part of the thesis we consider the Calogero-Moser models [15] and their generalizations of different types. The Hamiltonian of the elliptic classical sl_N model

$$H=\sum_{i=1}^{N}\frac{p_i^2}{2}-\nu^2\sum_{i>j}^{N}\wp(q_i-q_j) \tag{1.1}$$

together with the canonical Poisson brackets

$$\{p_i,q_j\}=\delta_{ij}\,,\quad \{p_i,p_j\}=\{q_i,q_j\}=0\,. \tag{1.2}$$

provides equations of motion for N-particle dynamics:

$$\dot q_i=p_i\,,\quad \ddot q_i=\nu^2\sum_{k:k\neq i}^{N}\wp'(q_{ik})\,. \tag{1.3}$$

All variables and the coupling constant ν are assumed to be complex numbers. Equations (1.3) can be written in the Lax form. The Krichever's Lax pair with spectral parameter [34] reads as follows[1]:

$$L(z)=\sum_{i,j=1}^{N}E_{ij}\,L_{ij}(z)\,,\quad L_{ij}(z)=\delta_{ij}p_i+\nu(1-\delta_{ij})\phi(z,q_{ij})\,,\ q_{ij}=q_i-q_j\,, \tag{1.4}$$

$$M_{ij}(z)=\nu d_i\delta_{ij}+\nu(1-\delta_{ij})f(z,q_{ij})\,,\quad d_i=\sum_{k:k\neq i}^{N}E_2(q_{ik})=-\sum_{k:k\neq i}^{N}f(0,q_{ik})\,, \tag{1.5}$$

i.e. the Lax equations

$$\dot L(z)\equiv\{H,L(z)\}=[L(z),M(z)] \tag{1.6}$$

are equivalent to (1.3) identically in z. The definitions and properties of elliptic functions entering (1.1)-(1.5) are given in the Appendix. The proof is based on the identity written as

$$\phi(z,q_{ab})f(z,q_{bc})-f(z,q_{ab})\phi(z,q_{bc})=\phi(z,q_{ac})(f(0,q_{bc})-f(0,q_{ab}))\,. \tag{1.7}$$

and

$$\phi(z,q_{ab})f(z,q_{ba})-f(z,q_{ab})\phi(z,q_{ba})=\wp'(q_{ab})\,. \tag{1.8}$$

[1] $\{E_{ij}\in\mathrm{Mat}(N),\ i,j=1...N\}$ – is the standard basis in $\mathrm{Mat}(N)$: $(E_{ij})_{kl}=\delta_{ik}\delta_{jl}$.

These are particular cases of the genus one Fay identity

$$\phi(z,q_{ab})\phi(w,q_{bc}) = \phi(w,q_{ac})\phi(z-w,q_{ab}) + \phi(w-z,q_{bc})\phi(z,q_{ac})\,. \tag{1.9}$$

The model (1.1)-(1.3) is included into a wide class of Calogero-Moser models associated with root systems [42]. The corresponding Lax pairs with spectral parameters were found in [19, 13]. In particular, for the BC_N root system described by the Hamiltonian

$$H = \frac{1}{2}\sum_{a=1}^{N} p_a^2 - \nu^2 \sum_{a<b}^{N}(\wp(q_a-q_b)+\wp(q_a+q_b)) - \mu^2\sum_{a=1}^{N}\wp(2q_a) - g^2\sum_{a=1}^{N}\wp(q_a) \tag{1.10}$$

there exists the Lax pair with a spectral parameter of size $(2N+1)\times(2N+1)$ if (as in [42])

$$g(g^2-2\nu^2+\nu\mu)=0\,. \tag{1.11}$$

Let us remark that the Lax pairs of size $3N\times 3N$ [30] or $2N\times 2N$ [20] corresponding to the general case (all constants are arbitrary) are not considered in this thesis.

The Lax pair (1.4)-(1.5) of the sl_N model (1.1)-(1.3) has the following generalization [96] (the R-matrix-valued Lax pair):[2]

$$\mathcal{L}(z) = \sum_{i,j=1}^{N} E_{ij}\otimes\mathcal{L}_{ij}(z)\,, \qquad \mathcal{L}_{ij}(z) = 1_{\tilde N}^{\otimes N}\,\delta_{ij}p_i + \nu(1-\delta_{ij})R_{ij}^{z}(q_{ij}) \tag{1.12}$$

$$\mathcal{M}_{ij}(z) = \nu d_i\delta_{ij} + \nu(1-\delta_{ij})F_{ij}^{z}(q_{ij}) + \nu\delta_{ij}\,\mathcal{F}^0\,, \quad d_i = -\sum_{k:k\neq i}^{N} F_{ik}^{0}(q_{ik})\,, \tag{1.13}$$

where $F_{ij}^{z}(q) = \partial_q R_{ij}^{z}(q)$, $F_{ij}^{0}(q) = F_{ij}^{z}(q)|_{z=0} = F_{ji}^{0}(-q)$ and

$$\mathcal{F}^0 = \sum_{k>m}^{N} F_{km}^{0}(q_{km}) = \frac{1}{2}\sum_{k,m=1}^{N} F_{km}^{0}(q_{km})\,. \tag{1.14}$$

It has block-matrix structure[3]. The blocks are enumerated by $i,j=1...N$ as matrix elements in (1.4). Each block of $\mathcal{L}(z)$ is some $\mathrm{GL}(\tilde N)$-valued R-matrix in fundamental representation, acting on the N-th tensor power of $\tilde N$-dimensional vector space $\mathcal{H} = (\mathbb{C}^{\tilde N})^{\otimes N}$. So the size of each block is $\dim\mathcal{H}\times\dim\mathcal{H}$, and $\dim\mathcal{H} = \tilde N^N$, i.e. $\mathcal{L}(z)\in \mathrm{Mat}_N\otimes\mathrm{Mat}_{\tilde N}^{\otimes N}$. We will refer to Mat_N component as auxiliary space, and to $\mathrm{Mat}_{\tilde N}^{\otimes N}\cong\mathcal{H}^{\otimes 2}$ – as "quantum" space since $\mathcal{H}$ is the Hilbert space of $\mathrm{GL}(\tilde N)$ spin chain (in fundamental representation) on N sites.

An R-matrix R_{ij} acts trivially in all tensor components except i,j. It is normalized in a way that for $\tilde N=1$ it is reduced to the Kronecker function $\phi(z,q_{ij})$ (see the main text for definition) [112]. For instance, in one of the simplest examples R_{ij} is the Yang's R-matrix [74]:

$$R_{12}^{\eta}(q) = \frac{1\otimes 1}{\eta} + \frac{\tilde N P_{12}}{q}\,, \tag{1.15}$$

where P_{12} is the permutation operator . In general (and as a default) case R_{ij} is the Baxter-Belavin [81, 9] elliptic R-matrix . The properties of this R-matrix are very similar to those of the function $\phi(z,q)$. The key equation for R_{ij} (which is needed for existence of the R-matrix-valued Lax pair) is the associative Yang-Baxter equation [23]

$$R_{ab}^{z}R_{bc}^{w} = R_{ac}^{w}R_{ab}^{z-w} + R_{bc}^{w-z}R_{ac}^{z}\,, \quad R_{ab}^{z} = R_{ab}^{z}(q_a-q_b)\,. \tag{1.16}$$

[2] Equations of motion following from (1.12)-(1.13) contain the coupling constant $\tilde N\nu$ instead of ν in (1.1), (1.3).

[3] The operator-valued Lax pairs with a similar structure are known [31, 27, 28, 10, 32]. We discuss it below.

It is a matrix generalization of the Fay identity (1.9), and it is fulfilled by the Baxter-Belavin R-matrix [44]. The degeneration of (1.16) similar to (1.7) is of the form:

$$R^z_{ab}F^z_{bc} - F^z_{ab}R^z_{bc} = F^0_{bc}R^z_{ac} - R^z_{ac}F^0_{ab}\,. \tag{1.17}$$

It underlies the Lax equations for the Lax pair (1.12)-(1.13). The last term $\mathcal{F}^0$ in (1.13) is not needed in (1.5) since for $\tilde{N}=1$ it is proportional to the identity $N\times N$ matrix. But it is important for $\tilde{N}>1$ since it changes the order of R and F^0 in the r.h.s. of (1.17). Namely,

$$[R^z_{ac},\mathcal{F}^0] + \sum_{b\neq a,c} R^z_{ab}F^z_{bc} - F^z_{ab}R^z_{bc} = \sum_{b\neq c} R^z_{ac}F^0_{bc} - \sum_{b\neq a} F^0_{ab}R^z_{ac}\,, \quad \forall\ a\neq c. \tag{1.18}$$

This identity provides the cancellation of non-diagonal blocks in the Lax equations. See [44, 45, 37, 38, 56] for different properties and applications of R-matrices of the described type. Here we need two more important properties. These are the unitarity

$$R^z_{12}(q_{12})R^z_{21}(q_{21}) = 1\otimes 1\,\tilde{N}^2(\wp(\tilde{N}z)-\wp(q_{12})) \tag{1.19}$$

and the skew-symmetry

$$R^z_{ab}(q) = -R^{-z}_{ba}(-q)\,. \tag{1.20}$$

On the one hand, these properties are needed for the Lax equations since they lead to

$$F^0_{ab}(q) = F^0_{ba}(-q) \tag{1.21}$$

and to the analogue of (1.8) (obtained by differentiating the identity (1.19))

$$R^z_{ab}F^z_{ba} - F^z_{ab}R^z_{ba} = \tilde{N}^2\wp'(q_{ab})\,, \tag{1.22}$$

which provides equations of motion in each diagonal block in the Lax equations.

On the other hand, together with (1.19) and (1.20) the associative Yang-Baxter equation leads to the quantum Yang-Baxter equation

$$R^\eta_{ab}R^\eta_{ac}R^\eta_{bc} = R^\eta_{bc}R^\eta_{ac}R^\eta_{ab}. \tag{1.23}$$

In this respect, we deal with the quantum R-matrices satisfying (1.16), (1.19), (1.20), and the Planck constant of R-matrix plays the role of the spectral parameter for the Lax pair (1.12)-(1.13). In trigonometric case the R-matrices satisfying the requirements include the standard $\mathrm{GL}(\tilde{N})$ XXZ R-matrix [35] and its deformation [122, 5] ($\mathrm{GL}(\tilde{N})$ extension of the 7-vertex R-matrix). In the rational case the set of the R-matrices includes the Yang's one (1.15) and its deformations [122, 50] ($\mathrm{GL}(\tilde{N})$ extension of the 11-vertex R-matrix).

The aim of this part of the thesis is to clarify the origin of the R-matrix-valued Lax pairs and examine some known constructions, which work for the ordinary Lax pairs.

First, we study extensions of (1.12)-(1.13) to other root systems. More precisely, we propose R-matrix-valued extensions of the D'Hoker-Phong Lax pairs for (untwisted) Calogero-Moser models associated with classical root systems and BC_N (1.10). The auxiliary space in these cases is given by Mat_{2N} or Mat_{2N+1} because such root systems are obtained from sl_{2N} or sl_{2N+1} cases by discrete reduction. There are two natural possibilities for arranging tensor components of the quantum spaces. The first one is to keep $2N+1$ (or $2N$) components of the quantum spaces in the reduced root system. The second – is to leave only N (or $N+1$) components. We study both cases.

Next, we proceed to quantum Calogero-Moser models [15, 43, 16]. To some extent, they are described by a quantum analog of the Lax equations (1.6) [51, 14]:

$$[\hat{H},\hat{L}(z)] = \hbar\,[\hat{L}(z),M(z)]\,, \tag{1.24}$$

where $\hat{H}$ is the quantum Hamiltonian (it is scalar in the auxiliary space), $\hat{L}(z)$ is the quantum Lax matrix and $\hbar$ is the Planck constant. The operators $\hat{H}$ and $\hat{L}(z)$ are obtained from the classical (1.1) and (1.4) by replacing momenta p_i with $\hbar\,\partial_{q_i}$, and the coupling constant in the Hamiltonian acquires the quantum correction. We verify if the obtained R-matrix-valued Lax pairs are generalized to quantum case in a similar way. It appears that (besides the sl_N case) only models associated with SO type root systems are generalized. As a result, we show[4]

Proposition 1.1. *The D'Hoker-Phong Lax pairs for (untwisted) classical Calogero-Moser models associated with classical root systems and* BC_N *admit* R*-matrix-valued extensions with additional constraints:*

– for the coupling constants in C_N *and* BC_N *cases;*

– for the size of R*-matrix* ($\tilde N = 2$) *in* B_N *and* D_N *cases.*

The latter cases are directly generalized to quantum Lax equations, while the C_N *and* BC_N *cases are not. The* A_N *Lax pair is generalized to the quantum case straightforwardly without any restrictions.*

The Calogero-Moser models [15, 43] possess also spin generalizations [24]. Its Lax description is known at classical [12] and quantum levels [31, 27, 28, 10, 32]. It is important to note that for the quantum Calogero-Moser models with spin, the quantum Lax pairs have the same operator-valued (tensor) structure as in (1.12)-(1.13). The term analogues to $\mathcal{F}^0$ (1.14) is treated as a part of the quantum Hamiltonian, describing the interaction of spins. We explain how the R-matrix-valued Lax pairs generalize (and reproduce) the previously known results.

Finally, we discuss the origin of the R-matrix-valued Lax pairs (for sl_N case (1.12)-(1.13) with $\mathrm{GL}_{\tilde N}$ R-matrices) by relating them to Hitchin systems on $\mathrm{SL}(N\tilde N)$-bundles over an elliptic curve. Originally systems of this type were derived by A. Polychronakos from matrix models [47] and later were described as Hitchin systems with nontrivial characteristic classes [55, 39]. It is also known as the model of interacting tops since it is Hamiltonian (or equations of motion) are treated as interaction of N $\mathrm{SL}(\tilde N)$-valued elliptic tops.

The relation between the R-matrix-valued Lax pairs and the interacting tops come from rewriting the Lax equation for (1.12)-(1.13) in the form

$$\{H,\mathcal{L}\} + [\nu\mathcal{F}^0,\mathcal{L}(z)] = [\mathcal{L}(z),\bar{\mathcal{M}}(z)]\,, \tag{1.25}$$

where in contrast to (1.13) $\bar{\mathcal{M}}$ does not include the $\mathcal{F}^0$ term (1.14). In this respect the R-matrix-valued Lax pair is "half-quantum": the spin variables are quantized in the fundamental representation, while the positions and momenta of particles remain classical. The $\mathcal{F}^0$ term in this treatment is the (anisotropic) spin exchange operator. We will show that the classical analog for such spin exchange operator appears in the above-mentioned Hitchin systems. Alternatively, the result is formulated as follows.

Proposition 1.2. *The quantum Hamiltonian* $\hat H^{\rm tops}$ *of the model of* N *interacting* $\mathrm{SL}(\tilde N)$ *elliptic tops (with spin variables being quantized in the fundamental representation) coincides with the sum of the quantum Calogero-Moser Hamiltonian (1.24) and* $\mathcal{F}^0$*-term (1.14)*

$$\hat H^{\rm tops} = \hat H^{\rm CM} + \hbar\nu\mathcal{F}^0 + 1_{\tilde N}^{\otimes N}\,const \tag{1.26}$$

up to a constant proportional to identity matrix in $\mathrm{End}(\mathcal{H})$ *and redefinition of the coupling constants.*

2 Supersymmetric qKZ-Ruijsenaars correspondence

The KZ-Calogero and qKZ-Ruijsenaars correspondences are the Matsuo-Cherednik type constructions [69, 67, 75, 76] for solutions of the Calogero-Moser-Sutherland [121] and Ruijsenaars-Schneider [135] quantum problems by means of solutions of the Knizhnik-Zamolodchikov (KZ) [65] and quantum Knizhnik-Zamolodchikov (qKZ) equations [68] respectively. Consider, for example, the qKZ equations[5] related to the Lie group $GL(K)$:

$$e^{\eta\hbar\partial_{x_i}}\Big|\Phi\Big\rangle = \mathbf{K}_i^{(\hbar)}\Big|\Phi\Big\rangle, \qquad i=1,\ldots,n, \tag{2.27}$$

[4] Some more details are given in the Conclusion.

[5] The quantum R-matrices entering (2.28) are assumed to be unitary: $\mathbf{R}_{ij}(x)\mathbf{R}_{ji}(-x) = \mathrm{id}$.

$$\mathbf{K}_i^{(\hbar)} = \mathbf{R}_{i\,i\text{-}1}(x_i - x_{i-1} + \eta\hbar) \dots \mathbf{R}_{i1}(x_i - x_1 + \eta\hbar)\mathbf{g}^{(i)}\mathbf{R}_{in}(x_i - x_n) \dots \mathbf{R}_{i\,i+1}(x_i - x_{i+1}), \tag{2.28}$$

where $\mathbf{g} = \mathrm{diag}(g_1, \dots, g_K)$ is a diagonal $K \times K$ (twist) matrix, and $\mathbf{g}^{(i)}$ acts by $\mathbf{g}$ multiplication in the i-th tensor component of the Hilbert space $\mathcal{V} = (\mathbb{C}^K)^{\otimes n}$. The quantum R-matrices $\mathbf{R}_{ij}$ are in the fundamental representation of $GL(K)$. They act in the i-th and j-th tensor components of $\mathcal{V}$ and satisfy the quantum Yang-Baxter equation, which guarantees compatibility of equations (2.27). The twist matrix $\mathbf{g}$ is the symmetry of $\mathbf{R}_{ij}$: $\mathbf{g}^{(i)}\mathbf{g}^{(j)}\mathbf{R}_{ij} = \mathbf{R}_{ij}\mathbf{g}^{(i)}\mathbf{g}^{(j)}$. In the rational case, we deal with the Yang's R-matrix [74]:

$$\mathbf{R}_{ij}(x) = \frac{x\mathbf{I} + \eta\mathbf{P}_{ij}}{x + \eta}, \tag{2.29}$$

where $\mathbf{I}$ is identity operator in $\mathrm{End}(\mathcal{V})$, and $\mathbf{P}_{ij}$ is the permutation operator, which interchanges the i-th and j-th tensor components in $\mathcal{V}$. The operators[6]

$$\mathbf{M}_a = \sum_{l=1}^{n} e_{aa}^{(l)} \tag{2.30}$$

commute with $\mathbf{K}_i^{(\hbar)}$ and provide the weight decomposition of the Hilbert space $\mathcal{V}$ into the direct sum

$$\mathcal{V} = V^{\otimes n} = \bigoplus_{M_1,\dots,M_K} \mathcal{V}(\{M_a\}) \tag{2.31}$$

of eigenspaces of operators $\mathbf{M}_a$ with the eigenvalues $M_a \in \mathbb{Z}_{\geq 0}$, $a = 1, \dots, K$: $M_1 + \dots + M_K = n$. Using the standard basis $\{e_a\}$ in $\mathbb{C}^K$ introduce the basis vectors in $\mathcal{V}(\{M_a\})$ as the vectors

$$\Big| J \Big\rangle = e_{j_1} \otimes e_{j_2} \otimes \dots \otimes e_{j_n}, \tag{2.32}$$

where the number of indices j_k such that $j_k = a$ is equal to M_a for all $a = 1, \dots, K$. The dual vectors $\Big\langle J \Big|$ are defined in so that $\Big\langle J \Big| J' \Big\rangle = \delta_{J,J'}$.

Then the statement of the qKZ-Ruijsenaars correspondence is as follows [76]. For any solution of the qKZ equations (2.27) $\Big|\Phi\Big\rangle = \sum_J \Phi_J \Big| J \Big\rangle$ from the weight subspace $\mathcal{V}(\{M_a\})$ the function

$$\Psi = \sum_J \Phi_J, \quad \Phi_J = \Phi_J(x_1, ..., x_n) \tag{2.33}$$

or

$$\Psi = \Big\langle \Omega \Big| \Phi \Big\rangle, \qquad \Big\langle \Omega \Big| = \sum_{J:\, \big| J \big\rangle \in \mathcal{V}(\{M_a\})} \Big\langle J \Big| \tag{2.34}$$

with the property

$$\Big\langle \Omega \Big| \mathbf{P}_{ij} = \Big\langle \Omega \Big| \tag{2.35}$$

is an eigenfunction of the Macdonald difference operator:

$$\sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{x_i - x_j + \eta}{x_i - x_j} \Psi(x_1, \dots, x_i + \eta\hbar, \dots, x_n) = E\Psi(x_1, \dots, x_n), \quad E = \sum_{a=1}^{K} M_a g_a. \tag{2.36}$$

The eigenvalues of the higher rational Macdonald-Ruijsenaars Hamiltonians

$$\hat{\mathcal{H}}_d = \sum_{I \subset \{1,\dots,n\}, |I| = d} \Big(\prod_{s \in I, r \notin I} \frac{x_s - x_r + \eta}{x_s - x_r} \Big) \prod_{i \in I} e^{\eta\hbar\partial_{x_i}} \tag{2.37}$$

[6] The set $\{e_{ab} \,|\, a, b = 1 \dots K\}$ is the standard basis in $\mathrm{Mat}(K, \mathbb{C})$: $(e_{ab})_{ij} = \delta_{ia}\delta_{jb}$.

are given by the elementary symmetric polynomial of n variables
$e_d(\underbrace{g_1,\ldots,g_1}_{M_1},\ldots\underbrace{g_N,\ldots,g_K}_{M_K})$.

QC-duality. Using the asymptotics of solutions to the (q)KZ equations [72] it was also argued in [75, 76] that the qKZ-Ruijsenaars correspondence can be viewed as a quantization of the quantum-classical duality [58, 64, 59] (see also [70, 62]), which relates the generalized inhomogeneous quantum spin chains and the classical Ruijsenaars-Schneider model. Consider the classical K-body Ruijsenaars-Schneider model, where the positions of particles $\{x_i\}$ are identified with the inhomogeneity parameters of the spin chain which is described by its transfer matrix

$$\mathbf{T}(x) = \mathrm{tr}_0\Big(\widetilde{\mathbf{R}}_{0n}(x-x_n)\ldots\widetilde{\mathbf{R}}_{02}(x-x_2)\widetilde{\mathbf{R}}_{01}(x-x_1)(\mathbf{g}\otimes\mathbf{I})\Big) \tag{2.38}$$

with the R-matrix

$$\widetilde{\mathbf{R}}(x) = \frac{x+\eta}{x}\,\mathbf{R}(x) = \mathbf{I} + \frac{\eta}{x}\,\mathbf{P}. \tag{2.39}$$

The quantum spin chain Hamiltonians are defined as follows:

$$\mathbf{H}_i = \operatorname*{Res}_{x=x_i}\mathbf{T}(x) = \widetilde{\mathbf{R}}_{i\,i-1}(x_i-x_{i-1})\ldots\widetilde{\mathbf{R}}_{i1}(x_i-x_1)\mathbf{g}^{(i)}\widetilde{\mathbf{R}}_{in}(x_i-x_n)\ldots\widetilde{\mathbf{R}}_{i\,i+1}(x_i-x_{i+1}). \tag{2.40}$$

Therefore,

$$\mathbf{H}_i = \mathbf{K}_i^{(0)}\prod_{j\neq i}^{n}\frac{x_i-x_j+\eta}{x_i-x_j}\,, \quad \mathbf{K}_i^{(0)} = \mathbf{K}_i^{(\hbar)}\,|_{\hbar=0}\,. \tag{2.41}$$

Identify also the generalized velocities $\{\dot{x}_i\}$ with the eigenvalues of (2.40). Then the action variables $\{I_i|\,i=1,...,K\}$ of the classical model (eigenvalues of the Lax matrix) are given by the values of $g_1,...,g_K$ with multiplicities $M_1,...,M_K$:

$$\{I_i|\,i=1,...,K\} = \Big\{\underbrace{g_1,\ldots,g_1}_{M_1},\ldots\underbrace{g_N,\ldots,g_K}_{M_K}\Big\}. \tag{2.42}$$

See details in [64], where this statement was proved using the algebraic Bethe ansatz technique.

QC-correspondence. On the other hand, the quantum-classical duality possesses a generalization to the so-called quantum-classical correspondence [73], where the classical Ruijsenaars-Schneider model is related not to a single spin chain but to the set of $K+1$ supersymmetric spin chains [66] associated with supergroups

$$GL(K|\,0)\,,\ GL(K-1|\,1)\,,\quad \ldots\quad ,GL(1|\,K-1)\,,\ GL(0|\,K)\,. \tag{2.43}$$

More precisely, it was shown in [73] that the previous statement (2.42) is valid for all supersymmetric chains with supergroups (2.43).

The aim we pursue in this thesis is to quantize the (supersymmetric) quantum-classical correspondence, that is to establish a supersymmetric version of the qKZ-Ruijsenaars correspondence for the qKZ equations related to the supergroups $GL(N|M)$. We construct generalizations of the vector $\Big\langle\Omega\Big|$ (2.34) and show that the quantum K-body Ruijsenaars-Schneider model follows from all $K+1$ qKZ systems of equations related to the supergroups $GL(N|M)$ with $N+M=K$ (2.43). The skew-symmetric vectors $\Big\langle\Omega_-\Big|$ with the property $\Big\langle\Omega_-\Big|\mathbf{P}_{ij} = -\Big\langle\Omega_-\Big|$ (instead of symmetric vector (2.35)) are described as well. They lead to the Ruijsenaars-Schneider model with different sign of the coupling constant η and $\hbar$.

This part of the thesis is organized as follows. For simplicity, we start with the rational KZ-Calogero correspondence. Then we proceed to the rational and trigonometric qKZ-Ruijsenaars relations. Most of the notations are borrowed from [75, 76, 73]. The Appendix briefly describes the notations and definitions related to the graded Lie algebras (groups).

3 Lax matrix and Sekiguchi determinant for the Dell System

The double elliptic (or Dell) model [119] is an integrable system with an elliptic dependence on both – positions of particles and their momenta. It extends the widely known Calogero-Moser-Sutherland [121, 95] and Ruijsenaars-Schneider [135] families of many-body integrable systems. Historically, the model was first derived as the elliptic self-dual system with respect to the Ruijsenaars (or equivalently, p-q or action-angle) duality interchanging positions of particles and action variables [134]. At the classical level, the original group-theoretical Ruijsenaars construction was not applicable to the elliptic case. Instead, a geometrical approach was used based on the studies of spectral curves and Seiberg-Witten differentials [90]. In this way the Dell Hamiltonians were proposed in terms of higher genus theta-functions with dynamical period matrices. For this reason, a definition of the standard set of algebraic tools for integrable systems (including Lax pairs, R-matrix structures, exchange relations, etc) appeared to be a complicated problem. The classical Poisson structures underlying the Dell model were studied in [83, 78].

An alternative version of the Dell Hamiltonians was suggested recently in [128]. The authors exploited the explicit form of the 6d Supersymmetric Yang-Mills partition functions with surface defects compactified on a torus, which are conjectured to serve as the wavefunctions for the corresponding Seiberg-Witten integrable systems [101, 102, 103, 77]. The exact correspondence of their results with the previous studies is an interesting open problem though the matching has been already verified in a few simple cases. In this thesis, we deal with the Koroteev-Shakirov version of the generating function for commuting Hamiltonians. Namely, for the N-body system consider the operator of complex variables:

$$\hat{\mathcal{O}}(\lambda) = \sum_{n_1,\dots,n_N\in\mathbb{Z}} \omega^{\sum_i \frac{n_i^2-n_i}{2}} (-\lambda)^{\sum_i n_i} \prod_{i<j}^{N} \frac{\theta_p(t^{n_i-n_j}\frac{x_i}{x_j})}{\theta_p(\frac{x_i}{x_j})} \prod_i^N q^{n_i x_i \partial_i} = \sum_{n\in\mathbb{Z}} \lambda^n \hat{\mathcal{O}}_n \,. \tag{3.44}$$

This is a definition of the infinite set of (non-commuting) operators $\hat{\mathcal{O}}_k$. The positions of particles q_i enter through $x_i = e^{q_i}$; $t = e^{\eta}$ – is exponent of the coupling constant η; $q = e^{\hbar}$ – is exponent of the Planck constant $\hbar$; and $\partial_i = \partial_{x_i}$, so that $\partial_{q_i} = x_i\partial_i$. The constant ω is the second modular parameter (controlling the ellipticity in momenta) and λ is the (spectral) parameter of the generating function. The definition of the theta-function $\theta_p(x)$ with the constant modular parameter τ ($p = e^{2\pi i\tau}$) (controlling the ellipticity in coordinates) is given in (appendix of the main text). The commuting Hamiltonians of the Dell system were conjectured and argued to be of the form:

$$\hat{H}_n = \hat{\mathcal{O}}_0^{-1}\hat{\mathcal{O}}_n \,, \qquad\qquad n = 1, \dots, N. \tag{3.45}$$

A solution to the eigenvalue problem for $\hat{H}_n$ was suggested in [114, 116] by extending the Shiraishi functions [108] – solutions to a non-stationary Macdonald-Ruijsenaars quantum problem.

Our study, on the contrary, does not appeal to the explicit form of the wavefunctions and is mostly focused on the generating function itself. It is based on the usage of the intertwining matrix $\Xi(z)$ of the IRF-Vertex correspondence (see main body for its explicit form) and the Hasegawa's factorization formula [92, 93].

$$\hat{L}^{\mathrm{RS}}(z, q, t) = g^{-1}(z) g(z - N\eta)\, q^{\mathrm{diag}(\partial_{q_1},\dots,\partial_{q_N})} \in \mathrm{Mat}(N, \mathbb{C}) \tag{3.46}$$

for the gl_N elliptic Ruijsenaars-Schneider Lax operator with spectral parameter z [135]

$$\hat{L}^{RS}_{ij}(z, \eta, \hbar) = \frac{\vartheta(-\eta)\vartheta(z + q_{ij} - \eta)}{\vartheta(z)\vartheta(q_{ij} - \eta)} \prod_{k\neq j} \frac{\vartheta(q_{jk} + \eta)}{\vartheta(q_{jk})}\, e^{\hbar\partial_{q_j}} \,. \quad q_{ij} = q_i - q_j \,. \tag{3.47}$$

The matrix $\Xi(z) = \Xi(z, x_1, \dots, x_N|p)$ enters the normalized intertwining matrix $g(z, \tau) = \Xi(z)D^{-1}$ from (3.46), where $D(x_1, \dots, x_N)$ is a diagonal matrix used for convenient normalization only, see (appendix of the main text). A key property of these matrices, which will be used, is that $\det \Xi$ is proportional to the Vandermonde determinant. These intertwining matrices are known from the IRF-Vertex correspondence at quantum and classical levels [82, 96, 97, 111]. The IRF-Vertex correspondence provides relation between dynamical and non-dynamical quantum (or classical) R-matrices as a special twisted gauge transformation with the matrix $g(z)$, thus relating the Lax operator (3.47) with the one of the Sklyanin type [109].

4 Towards Cherednik and Nazarov Sklyanin construction for dual elliptic Ruijsenaars

In the previous chapter we started discussing the double elliptic (Dell) integrable model, being a generalization of the Calogero-Ruijsenaars family of many-body systems [121, 135] to elliptic dependence on the particles momenta. There are two versions for this type of models. The first one was introduced and extensively studied by A. Mironov and A. Morozov [119]. Its derivation was based on the requirement for the model to be self-dual with respect to the Ruijsenaars (or action-angle or p-q) duality [134]. The Hamiltonians are rather complicated. They are given in terms of higher genus theta functions, and the period matrix depends on dynamical variables. Another version of the Dell model was suggested by P. Koroteev and Sh. Shakirov in [128]. It is close to the classical model introduced previously by H.W. Braden and T.J. Hollowood [118], though the precise relation between them needs further elucidation. The generating function of quantum Hamiltonians in this version is given by a relatively simple expression, where both modular parameters (for elliptic dependence on momenta and coordinate) are free constants. Another feature of the Koroteev-Shakirov formulation is that it admits some algebraic constructions, which are widely known for the Calogero-Ruijsenaars family of integrable systems. In particular, it was shown in the previous chapter that the generating function of the Hamiltonians has a determinant representation, and the classical L-operator satisfies the Manakov equation instead of the standard Lax representation. For both formulations, the commutativity of the Hamiltonians has not been proved yet, but verified numerically. To find a possible relation between two formulations of the Dell model is an interesting open problem.

In this chapter, we deal with the Koroteev-Shakirov formulation, and our study is based on the assumption that the following Hamiltonians indeed commute:

$$\hat{H}_n = \hat{\mathcal{O}}_0^{-1}\hat{\mathcal{O}}_n \,, \tag{4.48}$$

where $\hat{\mathcal{O}}_n$ are defined through[7]

$$\hat{\mathcal{O}}(u) = \sum_{n_1,\dots,n_N\in\mathbb{Z}} \omega^{\sum_i \frac{n_i^2-n_i}{2}} (-u)^{\sum_i n_i} \prod_{i<j}^N \frac{\theta_p(t^{n_i-n_j}\frac{x_i}{x_j})}{\theta_p(\frac{x_i}{x_j})} \prod_i^N q^{n_i x_i\partial_i} = \sum_{n\in\mathbb{Z}} u^n \hat{\mathcal{O}}_n \,. \tag{4.49}$$

We mostly study the degeneration $p \to 0$ of (4.49), which is the system similar (in the Mironov-Morozov approach) to the model dual to elliptic Ruijsenaars-Schneider one, so that it is elliptic in momenta and trigonometric in coordinates (for simplicity, we will most of the time refer to this Dell ($p = 0$) case as just (ell, trig)-model). Together with the change t to t^{-1}, $q \leftrightarrow q^{-1}$ and conjugation by the function $\prod_{i<j} x_i x_j$, the limit $p \to 0$ in (4.49) yields

$$D_N(u) = D_N(u|x_1,\dots,x_N) = \sum_{n_1,\dots,n_N\in\mathbb{Z}} \omega^{\sum_i \frac{n_i^2-n_i}{2}} (-u)^{\sum_i n_i} \prod_{i<j}^N \frac{t^{n_j}x_i - t^{n_i}x_j}{x_i - x_j} \prod_i^N \gamma^{n_i} \,. \tag{4.50}$$

where we have introduced the notation

$$\gamma_i = q^{-x_i\partial_i} \,. \tag{4.51}$$

One more trigonometric limit $\omega \to 0$ being applied to (4.50) provides (the trigonometric) Macdonald-Ruijsenaars operators [129]. Then the generating function (4.50) is represented in the following form:

$$D_N(u)\Big|_{\omega=0} = \left(\det\left[x_i^{N-j}\right]_{i,j=1}^N\right)^{-1} \det\left[x_i^{N-j}(1-ut^{j-1}\gamma_i)\right]_{i,j=1}^N \,. \tag{4.52}$$

In the previous chapter different variants of determinant representations for (4.48)-(4.49) were proposed. Here we extend another set of algebraic constructions to the double-elliptic case (4.48). Our final goal is

[7] The notations in (4.49) are standard. They are given in the list of notations. In particular, ω and p are two free modular parameters responsible for elliptic dependence on momenta $q^{x_i\partial_i} = \exp(\hbar\partial_{q_i})$ and coordinates $x_i = e^{q_i}$ respectively.

to describe the large N limit for (ell, trig)-model. This limit is widely known for the Calogero-Moser and the Ruijsenaars-Schneider models [113, 131, 137, 132, 133, 125] including their spin generalizations [115]. Infinite particle limits of integrable systems are interesting to study because they could be related to the representation theory of infinite dimensional algebras. The Hamiltonians of an integrable system form its Cartan subalgebra. Thus studying them may give some clues on how the whole algebra looks like. The details are described in the Discussion section.

The purpose of this chapter is to describe $N \to \infty$ limit of the (ell, trig)-model by introducing double-elliptic version of the Dunkl-Cherednik approach [122] and by applying the Nazarov-Sklyanin construction for $N \to \infty$ limit, which was originally elaborated for the trigonometric Ruijsenaars-Schneider model [131]. For the latter model there exists a set of N commuting operators (the Cherednik operators)

$$C_i(t,q) = t^{i-1} R_{i,i+1}(t) ... R_{iN}(t) \gamma_i R_{1,i}(t)^{-1} ... R_{i-1,i}(t)^{-1}\,, \tag{4.53}$$

acting on $\mathbb{C}[x_1, ..., x_N]$, where the R-operators are of the form:

$$R_{ij}(t) = \frac{x_i - t x_j}{x_i - x_j} + \frac{(t-1)x_j}{x_i - x_j}\,\sigma_{ij}\,, \tag{4.54}$$

and σ_{ij} permutes the variables x_i and x_j. The commutativity of the Macdonald-Ruijsenaars operators (4.52) for different values of spectral parameter u follows from the commutativity of (4.53) and the following relation between $D_N(u)\Big|_{\omega=0}$ (4.52) and the Cherednik operators (4.53):

$$D_N(u)\Big|_{\omega=0} = \prod_{i=1}^{N}(1 - uC_i)\,\Big|_{\Lambda_N}\,, \tag{4.55}$$

where $\Lambda_N \subset \mathbb{C}[x_1, ..., x_N]$ is the space of symmetric functions in variables $x_1, ..., x_N$.

The generating function (4.52) is the one[8] considered in [131], where the authors derived $N \to \infty$ limit of the quantum Ruijsenaars-Schneider (or the Macdonald-Ruijsenaars) Hamiltonians. Let us recall the main steps of the Nazarov-Sklyanin construction since this chapter is organized as a straightforward generalization of their results to the (ell, trig)-case (4.50). First, one needs to express the generating function (4.52) through the covariant Cherednik operators acting on $\mathbb{C}(x_1, ..., x_N)$:

$$Z_i = \prod_{k\neq i}^{N} \frac{x_i - t x_k}{x_i - x_k}\,\gamma_i + \sum_{j\neq i}^{N} \frac{(t-1)x_i}{x_i - x_j} \prod_{k\neq i,j}^{N} \frac{x_j - t x_k}{x_j - x_k}\,\gamma_j \sigma_{ij}\,, \tag{4.56}$$

$$U_i = (t-1)\prod_{j\neq i} \frac{x_i - t x_j}{x_i - x_j}\gamma_i\,, \tag{4.57}$$

which satisfy the property

$$\sigma Z_i \sigma^{-1} = Z_{\sigma(i)}, \qquad \sigma U_i \sigma^{-1} = U_{\sigma(i)}, \qquad \sigma \in S_N, \tag{4.58}$$

where in the l.h.s. σ acts by permutation of variables $\{x_1, ..., x_N\}$. Then the generating function of the Macdonald-Ruijsenaars Hamiltonians (4.52) is represented in the form

$$D_N(tu) D_N(u)^{-1}\Big|_{\omega=0} = 1 - u\sum_{i=1}^{N} U_i\,\frac{1}{1 - uZ_i}\,\Big|_{\Lambda_N}\,, \tag{4.59}$$

The next step is to construct the inverse limits for the operators U_i and Z_i, where the inverse limit is the limit of the sequence

$$\Lambda_1 \leftarrow \Lambda_2 \leftarrow ... \tag{4.60}$$

[8]To match notations of [131] one should change u to $-u$.

with a natural homomorphism (below Λ is the space of symmetric functions in the infinite amount of variables)

$$\pi_N : \Lambda \to \Lambda_N \,, \tag{4.61}$$

sending the standard basis elements p_n from Λ to the power sum symmetric polynomials:

$$\pi_N(p_n) = \sum_{i=1}^{N} x_i^n \,. \tag{4.62}$$

Finally, using (4.59) one gets the inverse limit for $D_N(tu)D_N(u)^{-1}\Big|_{\omega=0}$.

Our strategy is to extend the above formulae to the (ell, trig)-case. Throughout the chapter, we use the following convenient notation. For any operator $A(q,t)$ set

$$\mathrm{P}\theta_\omega(uA)(q,t) = \sum_{n\in\mathbb{Z}} \omega^{\frac{n^2-n}{2}}(-u)^n A(q^n,t^n) = \sum_{n\in\mathbb{Z}} \omega^{\frac{n^2-n}{2}}(-u)^n A^{[n]}(q,t)\,, \tag{4.63}$$

at least formally[9]. Notation $A^{[n]}(q,t) = A(q^n,t^n)$ is also used. In particular, $A^{[1]} = A$.

5 Notations for elliptic functions

In addition to the standard basis in $\mathrm{Mat}_{\tilde N}$ we use the one [9]

$$T_a = T_{a_1a_2} = \exp\left(\frac{\pi\imath}{\tilde N}\, a_1a_2\right) Q^{a_1}\Lambda^{a_2}\,, \quad a = (a_1,a_2) \in \mathbb{Z}_{\tilde N}\times\mathbb{Z}_{\tilde N} \tag{5.64}$$

constructed by means of the finite dimensional representation of Heisenberg group

$$Q_{kl} = \delta_{kl}\exp(\frac{2\pi\imath}{\tilde N}k)\,, \quad \Lambda_{kl} = \delta_{k-l+1=0 \bmod \tilde N}\,, \quad Q^{\tilde N} = \Lambda^{\tilde N} = 1_{\tilde N\times\tilde N}\,. \tag{5.65}$$

The following relations hold

$$T_\alpha T_\beta = \kappa_{\alpha,\beta}T_{\alpha+\beta}\,, \quad \kappa_{\alpha,\beta} = \exp\left(\frac{\pi\imath}{\tilde N}(\beta_1\alpha_2 - \beta_2\alpha_1)\right)\,, \tag{5.66}$$

$$\mathrm{tr}(T_\alpha T_\beta) = \tilde N\delta_{\alpha,-\beta}\,, \tag{5.67}$$

where $\alpha+\beta = (\alpha_1+\beta_1, \alpha_2+\beta_2)$. The permutation operator takes the form

$$P_{12} = \sum_{i,j=1}^{\tilde N} \tilde E_{ij}\otimes\tilde E_{ji} = \frac{1}{\tilde N}\sum_{\alpha\in\mathbb{Z}_{\tilde N}\times\mathbb{Z}_{\tilde N}} T_\alpha\otimes T_{-\alpha}\,, \tag{5.68}$$

where $\tilde E_{ij}$ is the standard basis in $\mathrm{Mat}_{\tilde N}$.

The Kronecker function is defined in the rational, trigonometric (hyperbolic) and elliptic case as follows:

$$\phi(\eta,z) = \left\{\begin{array}{ll} 1/\eta + 1/z \quad - & \text{rational case}\,,\\ \coth(\eta)+\coth(z) \quad - & \text{trigonometric case}\,,\\ \frac{\vartheta'(0)\vartheta(\eta+z)}{\vartheta(\eta)\vartheta(z)} \quad - & \text{elliptic case}\,. \end{array}\right. \tag{5.69}$$

[9]The convergence of such series in this text is understood as in theta-function definition, i.e. we assume $\omega = e^{2\pi\imath\tilde\tau}$ and $\mathrm{Im}(\tilde\tau) > 0$.

In the latter case, the theta-function is the odd one

$$\vartheta(z)=\sum_{k\in\mathbb{Z}}\exp\left(\pi\imath\tau(k+\frac{1}{2})^2+2\pi\imath(z+\frac{1}{2})(k+\frac{1}{2})\right). \tag{5.70}$$

Similarly, the first Eisenstein (odd) function and the Weierstrass (even) $\wp$-function:

$$E_1(z)=\left\{\begin{array}{l}1/z\,,\\ \coth(z)\,,\\ \vartheta'(z)/\vartheta(z)\,,\end{array}\right. \qquad \wp(z)=\left\{\begin{array}{l}1/z^2\,,\\ 1/\sinh^2(z)\,,\\ -\partial_z E_1(z)+\frac{1}{3}\frac{\vartheta'''(0)}{\vartheta'(0)}\,.\end{array}\right. \tag{5.71}$$

The derivative

$$E_2(z)=-\partial_z E_1(z) \tag{5.72}$$

is the second Eisenstein function. The derivative of the Kronecker function:

$$f(z,q)\equiv\partial_q\phi(z,q)=\phi(z,q)(E_1(z+q)-E_1(q))\,. \tag{5.73}$$

Due to the following behavior of $\phi(z,q)$ near $z=0$

$$\phi(z,q)=z^{-1}+E_1(q)+z\,(E_1^2(q)-\wp(q))/2+O(z^2)\,. \tag{5.74}$$

we also have

$$f(0,q)=-E_2(q)\,. \tag{5.75}$$

The Fay trisecant identity:

$$\phi(z,q)\phi(w,u)=\phi(z-w,q)\phi(w,q+u)+\phi(w-z,u)\phi(z,q+u). \tag{5.76}$$

For the Lax equations, the following degenerations of (5.76) are needed

$$\phi(z,x)f(z,y)-\phi(z,y)f(z,x)=\phi(z,x+y)(\wp(x)-\wp(y))\,, \tag{5.77}$$

$$\phi(\eta,z)\phi(\eta,-z)=\wp(\eta)-\wp(z)=E_2(\eta)-E_2(z)\,. \tag{5.78}$$

Also

$$\begin{aligned}\phi(z,q)\phi(w,q)&=\phi(z+w,q)(E_1(z)+E_1(w)+E_1(q)-E_1(z+w+q))=\\ &=\phi(z+w,q)(E_1(z)+E_1(w))-f(z+w,q)\,.\end{aligned} \tag{5.79}$$

The set of $\tilde{N}^2$ functions

$$\varphi_a^\eta(z)=\exp(2\pi\imath\frac{a_2}{\tilde{N}}\,z)\,\phi(z,\eta+\frac{a_1+a_2\tau}{\tilde{N}})\,,\quad a=(a_1,a_2)\in\mathbb{Z}_{\tilde{N}}\times\mathbb{Z}_{\tilde{N}} \tag{5.80}$$

is used in the definition of the Baxter-Belavin's [81, 9] elliptic R-matrix

$$R_{12}^\eta(z)=\sum_\alpha T_\alpha\otimes T_{-\alpha}\varphi_\alpha(z,\omega_\alpha+\eta)\,. \tag{5.81}$$

The classical limit (behavior near $\eta=0$)

$$R_{12}^\eta(z)=\frac{1\otimes 1}{\eta}+r_{12}(z)+\eta\,m_{12}(z)+O(\eta^2) \tag{5.82}$$

is similar to (5.74). The classical r-matrix

$$r_{12}(z)=1\otimes 1E_1(z)+\sum_{\alpha\neq 0}T_\alpha\otimes T_{-\alpha}\varphi_\alpha(z,\omega_\alpha) \tag{5.83}$$

is skew-symmetric due to (1.20) or (1.19). From (5.82) we conclude that

$$F^0_{12}(q) = \partial_q R^\eta_{12}(q)|_{\eta=0} = \partial_q r_{12}(q) = F^0_{21}(-q)\,. \tag{5.84}$$

The finite Fourier transformation for the set of functions (5.80) is as follows (see e.g. [57]):

$$\frac{1}{\tilde N}\sum_\alpha \kappa^2_{\alpha,\gamma}\,\varphi_\alpha(\tilde N\eta, \omega_\alpha + \frac{z}{\tilde N}) = \varphi_\gamma(z, \omega_\gamma+\eta)\,, \quad \forall\gamma\,. \tag{5.85}$$

It is generated by the arguments symmetry (similarly to $\phi(z,q)=\phi(q,z)$)

$$R^z_{12}(q)P_{12} = R^{q/\tilde N}_{12}(\tilde N z)\,. \tag{5.86}$$

In particular, (5.85) leads to

$$\sum_\alpha E_2(\omega_\alpha+\eta) = \tilde N^2 E_2(\tilde N\eta) \quad \text{or} \quad \sum_\alpha \wp(\omega_\alpha+\eta) = \tilde N^2 \wp(\tilde N\eta) \tag{5.87}$$

and for $\gamma \neq 0$

$$\sum_\alpha \kappa^2_{\alpha,\gamma} E_2(\omega_\alpha+\eta) = -\tilde N^2 \varphi_\gamma(\tilde N\eta, \omega_\gamma)(E_1(\tilde N\eta+\omega_\gamma) - E_1(\tilde N\eta) + 2\pi\imath\partial_\tau\omega_\gamma)\,. \tag{5.88}$$

6 Statement of the main results

The results of this dissertation are reflected in four articles:

A. Grekov, A. Zotov, “On R-matrix valued Lax pairs for Calogero–Moser models”, J. Phys. A, 51 (2018), 315202 , 26 pp.,

A. Grekov, A. Zabrodin, A. Zotov, ”Supersymmetric extension of qKZ-Ruijsenaars correspondence”, Nuclear Physics B, 2018 ,

Grekov, A. Zotov, ”Characteristic determinant and Manakov triple for the double elliptic integrable system”, SciPost Phys. 10, 055 (2021) • published 4 March 2021

A. Grekov, A. Zotov, ”On Cherednik and Nazarov-Sklyanin large N limit construction for double elliptic integrable system”, J. High Energ. Phys. 2021, 62 (2021)

6.1 First part

In the first part of the thesis (which follows the paper A. Grekov, A. Zotov, “On R-matrix valued Lax pairs for Calogero–Moser models”), we considered R-matrix-valued Lax pairs for N-body Calogero-Moser models. The one for A_{N-1} root system was previously known [96]. We proposed their extensions to other root systems. Namely, we studied generalizations of the D’Hoker-Phong Lax pairs [19] for the classical roots systems in the untwisted case. These Lax pairs are block-matrices of $2N\times 2N$ or $(2N+1)\times(2N+1)$ size, and each block is of the size $\tilde N^r \times \tilde N^r$, where r – is the number of quantum spaces (spin sites). Two possibilities were considered. The first one is to keep all $2N$ (or $2N+1$) quantum spaces in R-matrices. This leads to the Lax pairs for C_N and BC_N cases. The second possibility is to leave only half (N or $N+1$) quantum spaces. It results in constructing B_N and D_N models with GL_2 ($\tilde N = 2$) Baxter’s R-matrix. The summary of admissible values of the coupling constants and the number of quantum spaces in R-matrices are presented in the table below (horizontally are the numbers of quantum spaces).

	N	N+1	2N	2N+1
SO(2N)	$g=0,\ \mu=0$ $\tilde{N}=2$			
SO(2N+1)		$g=\pm\sqrt{2}\nu,\ \mu=0$ $\tilde{N}=2$		
Sp(2N)			$g=0,\ \mu=\nu$ $\tilde{N}=\text{any}$	
BC(N)				$g=\pm\nu,\ \mu=\nu$ $\tilde{N}=\text{any}$

Number of spin quantum spaces and values of coupling constants.

The ordinary Lax pairs were defined for the following values of the coupling constants:
– SO_{2N}: $\mu=0$, $g=0$;
– SO_{2N+1}: $\mu=0$, $g^2=2\nu^2$;
– Sp_{2N}: $g=0$;
– BC_N: $g(g^2-2\nu^2+\nu\mu)=0$.

In this respect our results are as follows: the R-matrix-valued ansatz generalizing the D'Hoker-Phong results works with additional constraints. For SO cases the additional condition is $\tilde{N}=2$, while for C_N and BC_N cases there is no restriction on $\tilde{N}$ but the constants should satisfy $\mu=\nu$ together with $g=0$ or $g=\pm\nu$ for C_N or BC_N root systems respectively.

Then we proceed to the quantum Lax pairs. A short summary is that the classical R-matrix-valued Lax pairs are generalized to quantum Lax pairs only for SO cases from the above table.

The quantum Lax pairs are naturally related to the spin Calogero-Moser models. The corresponding spin exchange operators $\mathcal{F}^0$ appear as a scalar parts of the R-matrix-valued M-matrices. On the other hand, the same operators can be derived from KZ or KZB equations. We demonstrate these relations for sl_N R-matrix-valued Lax pair. The link between the operator-valued Lax pairs and KZ equations comes from the Matsuo-Cherednik duality. Its quasi-classical version provides the so-called quantum-classical duality between the quantum spin chains (Gaudin models) and the classical many-body systems of Ruijsenaars-Schneider (Calogero-Moser) type [64]. In this chapter, we deal with another example of quantum-classical relation. We treat the Lax equations for the classical Calogero-Moser model (1.12)-(1.14) with R-matrix-valued Lax pairs as half-quantum model (1.25), which quantum part is described by the spin exchange operator known previously as the "noncommutative spin interactions" [47]. The spin variables are quantized in the fundamental representation, while the particles degrees of freedom remain classical. We show that the classical counterpart of the elliptic anisotropic spin exchange operator comes from the Hitchin type system on $\mathrm{SL}_{N\tilde{N}}$-bundle with nontrivial characteristic class over elliptic curve. See the Proposition 1.2.

It was shown in [48] that the spin exchange operator $\mathcal{F}^0$ for $\tilde{N}=2$ being reduced to the equilibrium position $q_j=j/N$ provides the Hamiltonian for anisotropic extension of the Inozemtsev elliptic long-range chain. In view of the relation of the R-matrix-valued Lax pairs and the Hitchin systems on $\mathrm{SL}(N\tilde{N})$-bundles we expect that these types of long-range integrable spin chains admit Lax representations of size $N\tilde{N}\times N\tilde{N}$ at both - classical and quantum levels. They are obtained from the one for interacting tops by the substitution $p_j=0$, $q_j=j/N$. Such a Lax pair allows us to calculate the higher Hamiltonians. These questions are discussed in [25].

6.2 Second part

In the second part of the thesis (which follows the paper A. Grekov, A. Zabrodin, A. Zotov, "Supersymmetric extension of qKZ-Ruijsenaars correspondence"), we have proved the supersymmetric version of the qKZ-Ruijsenaars correspondence. Consider the supergroup $GL(N|M)$, with $N+M=K$. Denote $\mathbb{C}^{N|M}$ its fundamental representation. And let: $\mathcal{V} := (\mathbb{C}^{N|M})^n$ The qKZ equations taking values in $\mathcal{V}$ have the form

$$e^{\eta\hbar\partial_{x_i}}\Big|\Phi\Big\rangle = \mathbf{K}_i^{(\hbar)}\Big|\Phi\Big\rangle, \qquad i = 1, \dots, n\,, \tag{6.89}$$

where the operators in the r.h.s

$$\mathbf{K}_i^{(\hbar)} = \mathbf{R}_{i\,i-1}(x_i - x_{i-1} + \eta\hbar)\dots\mathbf{R}_{i1}(x_i - x_1 + \eta\hbar)\mathbf{g}^{(i)}\mathbf{R}_{in}(x_i - x_n)\dots\mathbf{R}_{i\,i+1}(x_i - x_{i+1}) \tag{6.90}$$

are constructed by means of the quantum R-matrix $\mathbf{R}$, which is a (unitary) solution of the graded Yang-Baxter equation.
The operators $\mathbf{K}_i^{(\hbar)}$ commute with the set of operators:

$$\mathbf{M}_a = \sum_{i=1}^{n} e_{aa}^{(i)} \tag{6.91}$$

where e_{ab} is a standard basis in $\mathrm{End}(\mathbb{C}^{N|M})$. Hence from now on we will restrict our qKZ-equation to the subspace $\mathcal{V}(\{M_a\})$ with fixed eigenvalues M_a of the operators $\mathbf{M}_a$.
We start with a rational case:

Theorem 6.1. *Let*

$$\mathbf{R}_{ij}(x) = \frac{x\mathbf{I} + \eta\mathbf{P}_{ij}}{x + \eta}, \tag{6.92}$$

where $\mathbf{P}_{ij}$ is the graded permutation operator :

$$P_{12}(e_a \otimes e_b) = (-1)^{p(a)p(b)}(e_b \otimes e_a) \tag{6.93}$$

for 2 vectors e_a, e_b with parities $\mathsf{p}(a), \mathsf{p}(b)$.
Consider a covector $\Big\langle\Omega\Big| \in \mathcal{V}^*$, such that:

$$\Big\langle\Omega\Big|P_{ij} = \Big\langle\Omega\Big| \qquad \forall\, i, j = 1, ..., n \tag{6.94}$$

And define:

$$\Psi = \Big\langle\Omega\Big|\Phi\Big\rangle \tag{6.95}$$

Then:

$$\sum_{i=1}^{n}\left(\prod_{j\neq i}^{n}\frac{x_i - x_j + \eta}{x_i - x_j}\right)e^{\eta\hbar\partial_{x_i}}\Psi = \left(\sum_{a=1}^{N+M} g_a M_a\right)\Psi\,,$$

where:

$$\sum_{i=1}^{n}\mathbf{g}^{(i)} = \sum_{a=1}^{N+M} g_a\mathbf{M}_a, \tag{6.96}$$

In the trigonometric case we have the following:

Theorem 6.2. *Let:*

$$\mathbf{R}_{12}(x) = \mathbf{P}_{12} + \frac{\sinh x}{\sinh(x+\eta)}\Big(\mathbf{I} - \mathbf{P}_{12}^q\Big) + \mathbf{G}_{12}^+\,, \tag{6.97}$$

where $q = e^\eta$, and $\mathbf{P}_{12}^q$ is a quantum graded permutation operator:

$$\mathbf{P}_{12}^q = \sum_{a=1}^{N+M}(-1)^{\mathsf{p}(a)}e_{aa}\otimes e_{aa} + q\sum_{a>b}^{N+M}(-1)^{\mathsf{p}(b)}e_{ab}\otimes e_{ba} + q^{-1}\sum_{a<b}^{N+M}(-1)^{\mathsf{p}(b)}e_{ab}\otimes e_{ba} \tag{6.98}$$

and

$$\mathbf{G}^{+}_{12} = \sum_{a=1}^{N+M} \Big(\frac{\sinh(x+\eta-2\eta\mathsf{p}(a))}{\sinh(x+\eta)} - (-1)^{\mathsf{p}(a)} + \\ + \frac{\sinh(x)}{\sinh(x+\eta)} ((-1)^{\mathsf{p}(a)} - 1) \Big) e_{aa} \otimes e_{aa} \\ = 2 \sum_{a \in \mathfrak{F}} \frac{(\cosh\eta - 1)\sinh x}{\sinh(x+\eta)} e_{aa} \otimes e_{aa} \tag{6.99}$$

or

$$\mathbf{G}^{+}_{12} = \sum_{a=1}^{N+M} \mathbf{G}^{+}_{a} \, e_{aa} \otimes e_{aa} \,, \qquad \mathbf{G}^{+}_{a} = \frac{(1-(-1)^{\mathsf{p}(a)})(\cosh\eta - 1)\sinh x}{\sinh(x+\eta)} \,. \tag{6.100}$$

Define a covector $\Big\langle \Omega_q \Big| \in \mathcal{V}^*$*, such that:*

$$\Big\langle \Omega_q \Big| \mathbf{P}^{q}_{i,i-1} = \Big\langle \Omega_q \Big| \tag{6.101}$$

then for:

$$\Psi = \Big\langle \Omega_q \Big| \Phi \Big\rangle \tag{6.102}$$

one has:

$$\sum_{i=1}^{n} \left(\prod_{j \neq i}^{n} \frac{\sinh(x_i - x_j + \eta)}{\sinh(x_i - x_j)} \right) e^{\eta\hbar\partial_{x_i}} \Psi \;\; = \left(\sum_{a=1}^{N+M} g_a \frac{\sinh(\eta M_a)}{\sinh\eta} \right) \Psi \,. \tag{6.103}$$

6.3 Third part

In the third part of the thesis (which follows the paper Grekov, A. Zotov, "Characteristic determinant and Manakov triple for the double elliptic integrable system"), using the Hamiltonians (4.49), we construct a generalization of the Macdonald determinant operator for the Dell system and study its applications.

We use a slightly modified and extended version of the generating function $\hat{\mathcal{O}}'(z,\lambda)$ (4.49), which depends on the additional spectral parameter z, and generates an equivalent[10] set of operators $\hat{\mathcal{O}}'_k$:

$$\hat{\mathcal{O}}'(z,\lambda) = \sum_{k \in \mathbb{Z}} \frac{\vartheta(z-k\eta)}{\vartheta(z)} \lambda^k \hat{\mathcal{O}}'_k = \\ = \sum_{n_1,\dots,n_N \in \mathbb{Z}} \frac{\vartheta(z-\eta\sum_{i=1}^{N} n_i)}{\vartheta(z)} \omega^{\sum_i \frac{n_i^2 - n_i}{2}} (-\lambda)^{\sum_i^N n_i} \prod_{i<j}^{N} \frac{\vartheta(q_i - q_j + \eta(n_i - n_j))}{\vartheta(q_i - q_j)} \prod_{i}^{N} e^{n_i \hbar \partial_{q_i}} \,. \tag{6.104}$$

This part of the thesis is organized as follows.

In **Section 15** we derive the expression for the generalized Macdonald determinant:

$$\hat{\mathcal{O}}'(z - Nq_0, \lambda) = \frac{1}{\det \Xi(z)} \det_{1 \le i,j \le N} \Big\{ \sum_{n \in \mathbb{Z}} (-\lambda)^n \omega^{\frac{n^2-n}{2}} \Xi_i(q_j + n\eta, z) e^{n\hbar\partial_{q_j}} \Big\} \,, \tag{6.105}$$

where q_0 is the center of mass coordinate. The determinant is well defined as the columns of the matrix commute. For the precise form of the matrix $\Xi_{ij} = \Xi_i(q_j, z)$ see (appendix of the main text).

In **Section 16** we express the generating function (6.104) in terms of the Lax matrix of the Ruijsenaars-Schneider model:

$$\hat{\mathcal{O}}'(z,\lambda) =: \det_{1 \le i,j \le N} \Big\{ \hat{\mathcal{L}}^{\mathrm{Dell}}_{ij}(z, \lambda \,|\, q, t \,|\, \tau, \omega) \Big\} : \,, \tag{6.106}$$

[10]Details of the relation between $\hat{\mathcal{O}}'_k$ and $\hat{\mathcal{O}}_k$ are given in the appendix of the main text.

where

$$\hat{\mathcal{L}}^{\rm Dell}_{ij}(z,\lambda\,|\,q,t\,|\,\tau,\omega)=\sum_{k\in\mathbb{Z}}\omega^{\frac{k^2-k}{2}}(-\lambda)^k\hat{L}^{RS}_{ij}(z\,|k\eta,k\hbar\,|\tau) \tag{6.107}$$

and the normal ordering is defined in the main body of the manuscript. The trigonometric and rational limits (for coordinate dependence) of (6.104)-(6.107) are described as well.

In **Section 17** we study the eigenvalue problem for the operator $\hat{\mathcal{O}}(u)$ (4.49) in the (coordinate) trigonometric limit $p=0$, which corresponds to the dual to elliptic Ruijsenaars model[11], and compare our results to the known in the literature [128, 114].

The main statement here is the following: The operators $\hat{\mathcal{O}}(u)$ in the limit $p=0$ for different u could be simultaneously brought to the upper triangular form in some basis, their eigenvalues are labelled by Young diagrams $\lambda=(\lambda_1,...,\lambda_N)$, and equal to:

$$E(u)_\lambda=\prod_{i=1}^{N}\theta_\omega(ut^{N-i}q^{\lambda_i})\,. \tag{6.108}$$

In Section 18 we study the classical limit of the Dell system. Using the classical analogue $\mathcal{L}(z,\lambda)$ of (6.107) we show that the L-matrix

$$L(z,\lambda)=\mathcal{L}(z,1)^{-1}\mathcal{L}(z,\lambda)\in{\rm Mat}(N,\mathbb{C}) \tag{6.109}$$

satisfies the Manakov triple representation [99, 124] (instead of the Lax equation):

$$\dot{L}=[L,A]+BL\,,\quad {\rm tr}B=0\,. \tag{6.110}$$

The conservation laws are generated by the function det $L(z,\lambda)$ only. It reduces to expression for the spectral curve of the Ruijsenaars-Schneider model in the $\omega\to0$ limit.

In **Section 19** we describe the factorized structure for the L-matrix (6.109) $L(z,\lambda)$. Up to an inessential modification it is presented in the form, which is similar to the elliptic Kronecker function[12] (see appendix of the main text):

$$\begin{gathered}\check{L}(z,\lambda|\tau,\tilde\tau)=\Phi[G(z,\tau),u|\tilde\tau]:=\\=\frac{\vartheta'(0|\tilde\tau)}{\vartheta(u|\tilde\tau)}\Big[\vartheta(-\,{\rm ad}_{N\eta\partial_z}|\tilde\tau)\,G(z,\tau)\Big]^{-1}\vartheta(u-\,{\rm ad}_{N\eta\partial_z}|\tilde\tau)\,G(z,\tau)\,,\\ u=\log(\lambda)\,,\quad G(z,\tau)=g(z,\tau)\exp\Big(\frac{z}{Nc\eta}\,{\rm diag}(p_1,...,p_N)\Big),\end{gathered} \tag{6.111}$$

thus generalizing the classical version of the factorization (3.46) to the double elliptic case. The elliptic moduli $\tilde\tau$ appears as $\omega=e^{2\pi\imath\tilde\tau}$. It is responsible for the ellipticity in momenta, while τ controls the ellipticity in positions of particles.

We also describe connection of the L-matrix with the Sklyanin Lax operators, and propose its quantization in terms of the elliptic quantum R-matrix in the fundamental representation of ${\rm GL}_N$.

Possible applications of the obtained results and future plans are discussed at the end of the chapter. Appendices contain the elliptic functions definitions and properties, description of the intertwining matrices Ξ, computations of ${\rm GL}_2$ examples, and relations between different forms of the generating functions.

[11] The terminology like dual to elliptic Ruijsenaars (or Calogero) model comes from the Mironov-Morozov description of the Dell model based on the p-q duality. Here and in what follows we use it meaning the trigonometric (or rational) $p=0$ limit of (4.49), though its relation to p-q duality needs to be clarified.

[12] It is used in the widely known Lax pairs with spectral parameter [95, 135] in many-body systems.

6.4 Fourth part

In the fourth part of the thesis (which follows the paper A. Grekov, A. Zotov, "On Cherednik and Nazarov-Sklyanin large N limit construction for double elliptic integrable system"), we study the infinite particle limit of the (ell, trig) member of the Calogero-Ruijsenaars many-body systems family. The chapter is organized as follows.

In **Section 26** we introduce the (ell, trig) version of the Cherednik operators (4.53), acting on the space $\mathbb{C}[x_1,...,x_N]$:

$$\mathrm{P}\theta_\omega(uC_i) = \sum_{n\in\mathbb{Z}} \omega^{\frac{n^2-n}{2}}(-u)^n t^{n(i-1)} R_{i,i+1}(t^n)...R_{iN}(t^n)\gamma_i^n R_{1,i}(t^n)^{-1}\dots R_{i-1,i}(t^n)^{-1}\,, \tag{6.112}$$

where $R_{ij}(t)$ is given by (4.54), and u is a spectral parameter. These operators do not commute with each other. However, we prove the following relation between (6.112) and $D_N(u)$ (4.50):

$$D_N(u) = \prod_{i=1}^{N} \mathrm{P}\theta_\omega(uC_i)\Big|_{\Lambda_N} = \mathrm{P}\theta_\omega(uC_1)...\mathrm{P}\theta_\omega(uC_N)\Big|_{\Lambda_N}\,. \tag{6.113}$$

It is the (ell, trig) version of the relation (4.55). The order of operators in the above product is important. In what follows a product of non-commuting operators is understood as it is given in the r.h.s of (6.113). It is also mentioned in the list of notations.

In **Section 27**, using the covariant version of the Cherednik operators (4.56)

$$\mathrm{P}\theta_\omega(uZ_i) = \sum_{n\in\mathbb{Z}} \omega^{\frac{n^2-n}{2}}(-u)^n \left[\prod_{k\neq i}\frac{x_i-t^nx_k}{x_i-x_k}\gamma_i^n + \sum_{j\neq i}\frac{(t^n-1)x_i}{x_i-x_j}\prod_{k\neq i,j}\frac{x_j-t^nx_k}{x_j-x_k}\gamma_j^n\sigma_{ij}\right] \tag{6.114}$$

and the auxiliary covariant operators

$$\mathrm{P}\theta_\omega(uU_i) = \sum_{n\in\mathbb{Z}} \omega^{\frac{n^2-n}{2}}(-u)^n(t^n-1)\prod_{k\neq i}\frac{x_i-t^nx_k}{x_i-x_k}\gamma_i^n \tag{6.115}$$

we prove the following analog of (4.59):

$$I_N(u) := D_N(ut)D_N(u)^{-1} = 1 + \sum_{i=1}^{N}\mathrm{P}\theta_\omega(uU_i)\frac{1}{\mathrm{P}\theta_\omega(uZ_i)}\Big|_{\Lambda_N}\,. \tag{6.116}$$

In **Section 28** the matrix resolvent of the construction is presented. Namely, consider $N\times N$ matrix $\mathcal{Z}$ with elements

$$\mathcal{Z}_{ii} = \Big(\prod_{l\neq i}\frac{x_i-tx_l}{x_i-x_l}\Big)\gamma_i \tag{6.117}$$

$$\mathcal{Z}_{ij} = \frac{(t-1)x_j}{x_i-x_j}\Big(\prod_{l\neq i,j}\frac{x_i-tx_l}{x_i-x_l}\Big)\gamma_j \qquad \text{for } i\neq j\,. \tag{6.118}$$

It is the Lax matrix of the trigonometric quantum Ruijsenaars-Schneider model. Together with the column vector

$$\mathcal{E} = \begin{bmatrix}1\\1\\ \vdots\\1\end{bmatrix} \tag{6.119}$$

and the row vector

$$\mathrm{P}\theta_\omega(u\mathcal{U}) = \begin{bmatrix}\mathrm{P}\theta_\omega(uU_1) & \dots & \mathrm{P}\theta_\omega(uU_N)\end{bmatrix} \tag{6.120}$$

it provides the generating function of the (ell, trig)-model Hamiltonians in the following way:

$$I_N(u) = D_N(ut)D_N(u)^{-1} = 1 + \mathrm{P}\theta_\omega(u\mathcal{U})\mathrm{P}\theta_\omega(u\mathcal{Z})^{-1}\mathcal{E}\,\Big|_{\Lambda_N}\,. \tag{6.121}$$

In **Sections 29** and **30** we describe the generalization of the Nazarov-Sklyanin $N \to \infty$ limit construction for the (ell, trig)-model Hamiltonians and the covariant Cherednik operators.

Extend the homomorphism (4.61) to the space $\Lambda[w]$ of polynomials in a formal variable w with coefficients in Λ in the following way:

$$\begin{gathered}\tau_N(p_n) = \pi_N(p_n)\,, \quad \tau_N:\ \Lambda[w] \to \Lambda_N\,, \\ \tau_N(w) = t^N\,.\end{gathered} \tag{6.122}$$

Let $I(u)$ be the operator $\Lambda \to \Lambda[w]$, satisfying

$$I_N(u)\pi_N = \tau_N I(u)\,. \tag{6.123}$$

See (**??**) for details. Then the main result of these two Sections is as follows. The operator

$$\mathcal{I}(u) = \frac{\theta_\omega(u)}{\theta_\omega(uw)}I(u)$$

does not depend on w, thus mapping the space Λ to itself. It has the form:

$$\mathcal{I}(u) = \theta_\omega(u)\Big[\theta_\omega(u) + \mathrm{P}\theta_\omega(uY\beta) - \mathrm{P}\theta_\omega(uY\alpha)\mathrm{P}\theta_\omega(uX\alpha)^{-1}\mathrm{P}\theta_\omega(uX\beta)\Big]^{-1}, \tag{6.124}$$

where the operators $\alpha^{[n]}, \beta^{[n]}, X^{[n]}, Y^{[n]}$ are defined in the main body of the manuscript.

In **Section 31** the expressions for the operators $\alpha^{[n]}, \beta^{[n]}, X^{[n]}, Y^{[n]}$ are derived in a more explicit form. These operators yields the generating function of the $N \to \infty$ Hamiltonians. We prove, that these Hamiltonians commute as soon as the Shakirov-Koroteev Dell Hamiltonians commute[13].

In **Section 32** we write down the explicit form of the first few non-trivial $N \to \infty$ Hamiltonians to the first power in ω. The generating function equals:

$$\mathcal{I}(u) = \frac{1 - u + \omega(u^2 - u^{-1})}{1 - u - uJ(u) + \omega K(u)} + O(\omega^2)\,, \tag{6.125}$$

where $J(u)$ and $K(u)$ are given in the body of the main text. As well as the formulae for the first and the second Hamiltonians up to the first order in ω. In the limit $\omega = 0$, our answer (6.125) reproduces the Nazarov-Sklyanin result [131]:

$$\mathcal{I}(u) = \frac{1 - u}{1 - u - uJ(u)}\,. \tag{6.126}$$

In **Section 33** we also verify directly that the first and the second Hamiltonians commute with each other up to the first order in ω.

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