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Locally nilpotent derivations, additive actions and algebraic monoids

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Introduction

Description of the research area. We study affine algebraic varieties over an algebraically closed field $\mathbb{K}$ of characteristic zero and their regular automorphisms. It is known that the automorphism group $\mathrm{Aut}(X)$ of an affine variety X need not be a (finite-dimensional) algebraic group, and it is an important problem to describe algebraic subgroups of $\mathrm{Aut}(X)$. A subgroup G in $\mathrm{Aut}(X)$ is said to be algebraic if G has a structure of an algebraic group such that the action $G \times X \to X$ is a morphism of algebraic varieties.

A normal algebraic variety is called toric if it admits an action of an algebraic torus with an open orbit. The theory of toric varieties has deep connections with combinatorics, commutative algebra and convex geometry; see for example monographs [Fu93, CLS11]. In particular, any toric variety is given by its fan consisting of rational polyhedral cones, and a lot of geometric questions on toric varieties have answers in combinatorial terms.

Any affine algebraic group G has a unique maximal torus T up to conjugation. If G is connected, then it is generated by its maximal torus T and one-parameter unipotent subgroups $\mathbb{G}_a = (\mathbb{K}, +)$ normalized by T, which are called root subgroups with respect to T; see [Hu75]. One may apply this to describe the automorphism group $\mathrm{Aut}(X)$ of a complete toric variety X, which is known to be an affine algebraic group; see also [Br21]. In his seminal work [De70], Demazure described $\mathrm{Aut}(X)$ and introduced special elements in the character lattice of the acting torus, which are in bijection with root subgroups. Nowadays, these elements are called Demazure roots.

In [Cox95], Cox suggested another method to describe the automorphism group of a complete toric variety. He defined an important invariant of a variety called the homogeneous coordinate ring or the Cox ring; see also [Bat93]. In the toric case, the Cox ring is a graded polynomial ring, and the description of the automorphism group of a complete toric variety can be reduced to the description of $\mathbb{G}_a$-actions on an affine space normalized by the diagonal torus action and centralized by a certain quasitorus.

An important tool to study $\mathbb{G}_a$-actions are locally nilpotent derivations. A $\mathbb{K}$-linear map $\delta\colon R \to R$ is called a derivation of the algebra R if $\delta(fg) = \delta(f)g + f\delta(g)$ for any $f, g \in R$. A derivation δ is said to be locally nilpotent if for any $f \in R$ there exists $k \in \mathbb{Z}_{>0}$ such that $\delta^k(f) = 0$. If R is graded by some abelian group, then a derivation δ of R is called homogeneous if it maps homogeneous elements of R to homogeneous ones. In other terms, a torus action on an affine variety X is given by a grading of the algebra of regular functions $\mathbb{K}[X]$ by the character lattice of the acting torus. In turn, regular actions of one-parameter unipotent subgroups $\mathbb{G}_a$ on X are in bijection with locally nilpotent derivations on $\mathbb{K}[X]$. Further, a $\mathbb{G}_a$-action is normalized by a torus if and only if the corresponding locally nilpotent derivation is homogeneous with respect to the grading defined by this torus. This technique is used in many works in order to describe automorphisms and to study the geometry of affine varieties, see e.g. [FZ05, AH06, Li10, AKZ12, AG17, Sh17, Ar18, GS19, Ga21, LRU22]. Moreover, lifting automorphisms to the spectrum of the Cox ring, one can reduce the study of automorphisms of certain projective varieties to the study of homogeneous automorphisms of affine varieties equipped with an action of the so-called Neron-Severi quasitorus, see [Cox95, HK00, BH03] for the original approach and [Ga08, AG10, AHHL14, APS14, AK15, ADHL15] for further developments. This method opens a wide area of applications and motivates the study of graded affine algebras and homogeneous locally nilpotent derivations.

Recall that the complexity of a torus action is the codimension of a generic orbit. Toric varieties are precisely varieties with a torus action of complexity zero. Their natural generalization are varieties with torus actions of complexity one. Any affine toric variety is given by binomials, see e.g. [St96, Chapter 4]. At the same time, the study of varieties with torus action of complexity one is related to some specific relations called trinomials.

By a trinomial we mean a polynomial of the form $g = T_0^{l_0} + T_1^{l_1} + T_2^{l_2}$ such that each variable appears in at most one monomial $T_i^{l_i}$. While the Cox ring of a toric variety is a polynomial ring, the Cox ring of a variety with a torus action of complexity one is a factor-algebra of a polynomial ring by an ideal generated by trinomials; see [HS10, HHS11, HH13, AHHL14, HW17]. This motivates us to study homogeneous locally nilpotent derivations on trinomial algebras; see Section 1.

In parallel to the theory of algebraic groups, the theory of algebraic monoids has been developed. An algebraic variety X with an associative multiplication $X \times X \to X$ is called an algebraic monoid if the multiplication is a morphism of algebraic varieties and has a unity. The group of invertible elements of an algebraic monoid X is an algebraic group, which is Zariski open in X, see [Ri98, Theorem 1] and [Ri07, Theorem 5].

By a group embedding we mean an irreducible affine variety X with an open embedding $G \hookrightarrow X$ of an affine algebraic group G such that the action of the group $G \times G$ by left and right multiplications on G can be extended to the action of $G \times G$ on X. It appears that for an affine algebraic group G there is a natural correspondence between group embeddings of G and monoid structures with group of invertible elements G; see [Vi95, Theorem 1] for characteristic zero and [Ri98, Proposition 1] for the general case. The theory of affine algebraic monoids and group embeddings is a rich area of mathematics lying at the intersection of algebra, algebraic geometry, combinatorics and representation theory; see [Pu88, Vi95, Ri98, Re05] for general presentations.

An affine algebraic monoid is called reductive if its group of invertible elements is a reductive affine algebraic group. The theory of reductive monoids is the most developed, see e.g. the combinatorial classification of reductive monoids in [Vi95, Ri98]. It is based on the representation theory of reductive groups, i.e., the highest weight theory.

The next possible aim is a classification for other classes of monoids, for example, solvable or commutative. A monoid is both reductive and commutative if and only if it is a toric variety with a canonical multiplication. It is important to find all monoid structures on a fixed variety, for example, on an affine space. It is also interesting to obtain explicit formulas for multiplications in monoids; see Section 2.

Let us focus on affine varieties with big automorphism groups. The most interesting is the transitive case. The classical examples here are homogeneous spaces of affine algebraic groups. It is natural to ask whether there are other varieties with the transitive action of the automorphism group. Such example can be found among Danielewski surfaces and Danilov-Gizatullin surfaces, see [Gi70, GD77, ML01, Du04] and Section 3.

Let us recall the notion of flexibility, which is close to that of homogeneity. The subgroup of the automorphism group $\mathrm{Aut}(X)$ of a variety X generated by all $\mathbb{G}_a$-subgroups in $\mathrm{Aut}(X)$ is called the special automorphism group $\mathrm{SAut}(X)$. A smooth point x of a variety X is called flexible if the tangent space to X at the point x is generated by tangents to orbits of $\mathbb{G}_a$-subgroups passing through the point x. A variety X is called flexible if any smooth point of X is flexible. In [AFKKZ13, Theorem 0.1], it is proved that the following conditions are equivalent for an irreducible affine variety X:

(a) the variety X is flexible;

(b) the group $\mathrm{SAut}(X)$ acts on the set of smooth points of X transitively.

Moreover, if the variety X has dimension at least 2, then these conditions are equivalent to

(c) the group $\mathrm{SAut}(X)$ acts on the set of smooth points of X infinitely transitive.

There are many interesting examples of flexible varieties. One useful construction here is a suspension $\mathrm{Susp}(X, f) = \{uv = f(x)\} \subseteq \mathbb{A}^2 \times X$ over an affine variety X. If X is a flexible irreducible affine variety of positive dimension, then any suspension over X is flexible as well; see [AKZ12] for an algebraically closed field of characteristic zero and [KM12] for the case of the ground field $\mathbb{R}$. In the context of automorphism groups suspensions were considered for the first time in [KZ99].

The last subject we are interested in is an additive analogue of toric varieties. The idea is to replace the multiplicative group of the ground field by an additive one and consider a commutative unipotent group $\mathbb{G}_a^n$. By an additive action on a variety we mean an action of the group $\mathbb{G}_a^n$ with an open orbit. In other words, we consider open equivariant embeddings of vector groups into algebraic varieties. The affine case is trivial here since any orbit of a unipotent group on an affine variety is closed. For projective varieties, the theory is nontrivial even for a projective space. In [HT99], Hassett and Tschinkel establish a correspondence between finite-dimensional commutative local unital algebras and additive actions on projective spaces; see also [KL84]. It appears that there are infinite families of pairwise non-equivalent additive actions on $\mathbb{P}^n$ starting from $n = 6$.

Similar approach may be applied to the study of additive actions on projective hypersurfaces. This time we need an additional data: a hyperplane U in the maximal ideal $\mathfrak{m}$ of the algebra A. It is known that the degree of the hypersurface X corresponding to a pair (A, U) equals the maximal exponent d with $\mathfrak{m}^d \not\subseteq U$, see [AS11]. An additive action on a non-degenerate quadric is unique [AS11], and (infinitely many) induced additive actions on degenerate quadrics of corank one are described in [AP14]. In [Baz13], the case of cubic hypersurfaces is studied; in particular, it turns out that an induced additive action on a non-degenerate cubic hypersurface is also unique. The next step is to study both non-degenerate and degenerate hypersurfaces of arbitrary degree; see Section 4.

Main results. Main results of the thesis are as follows.

1. All homogeneous locally nilpotent derivations on trinomial algebras are elementary.
2. Classifications of commutative monoid structures on $\mathbb{A}^3$ and of monoid structures of corank one on an arbitrary normal affine variety.
3. A classification of Danielewski surfaces that are homogeneous varieties but not homogeneous spaces.
4. The uniqueness of an induced additive action on a non-degenerate projective hypersurface.

Publications. The results of the thesis are published in 7 articles:

[Z19] Yulia Zaitseva. Homogeneous locally nilpotent derivations of non-factorial trinomial algebras. Mathematical Notes 105 (2019), no. 6, 818-830

[GZ19] Sergey Gaifullin and Yulia Zaitseva. On homogeneous locally nilpotent derivations of trinomial algebras. Journal of Algebra and Its Applications 18 (2019), no. 10, article 1950196, 1-19

[ABZ20] Ivan Arzhantsev, Sergey Bragin, and Yulia Zaitseva. Commutative algebraic monoid structures on affine spaces. Communications in Contemporary Mathematics 22 (2020), no. 8, article 1950064

[DZ21] Sergey Dzhunusov and Yulia Zaitseva. Commutative algebraic monoid structures on affine surfaces. Forum Mathematicum 33 (2021), no. 1, 177-191

[AZ22] Ivan Arzhantsev and Yulia Zaitseva. Equivariant completions of affine spaces. Russian Mathematical Surveys 77 (2022), no. 4, 571-650

[AZ24] Ivan Arzhantsev and Yulia Zaitseva. Affine homogeneous varieties and suspensions. Research in the Mathematical Sciences 11 (2024), no. 2, article 27, 1-13

[Z24] Yulia Zaitseva. Affine monoids of corank one. `https://arxiv.org/abs/2312.08316`, 15 pages, accepted to Results in Mathematics

Approbation. The results of the thesis were presented in the following talks.

- Seminar of Department of Mathematics and Computer Science, 11 April 2024, Saint Petersburg
- The Third Conference of Mathematical Centres of Russia, 10-15 October 2023, Maykop
- Conference “Mathematics in Contemporary World”, 19-23 September 2023, Vologda
- Conference of Small Research Groups, 26-30 June 2023, Saint Petersburg
- Spring School-conference in Algebra in Euler Institute, 29 April - 3 May 2023, Saint Petersburg
- Workshop “Affine Spaces, Algebraic Group Actions, and LNDs”, 13-17 March 2023, Kolkata, India
- Shafarevich Seminar, Steklov Institute of Mathematics, 21 February 2023, Moscow
- The Tenth School-conference on Lie Algebras, Algebraic Groups and Invariant Theory, 28 January - 2 February 2023, Moscow
- The Second Conference of Mathematical Centres of Russia, 7-11 November 2022, Moscow
- Conference “Algebraic Groups: the White Nights Season - II”, 4-8 July 2022, Saint Petersburg
- The Ninth School-conference on Lie Algebras, Algebraic Groups and Invariant Theory, 21-26 August 2021, Samara
- The First Conference of Mathematical Centres of Russia, 9-13 August 2021, Sochi
- Mini-workshop “Algebraic Groups: the White Nights Season”, 12-16 July 2021, Saint Petersburg
- Iskovskih Seminar, Steklov Institute of Mathematics, 12 March 2020, Moscow
- Conference “Algebraic Transformation Groups: the Mathematical Legacy of Domingo Luna”, 28-30 October 2019, Rome, Italy, poster
- Conference “Department of Higher Algebra becomes 90”, 28-31 May 2019, Moscow
- Seminar on Lie Groups and Invariant Theory, 31 October 2018, Moscow
- The Seventh School-conference on Lie Algebras, Algebraic Groups and Invariant Theory, 18-26 August 2018, Samara

Main definitions and results of the thesis are formulated in further sections. All results are new and relevant to the subject of the dissertation. They may be used in the further studies of algebraic transformation groups and automorphisms of varieties.

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1. Locally nilpotent derivations on trinomial algebras

A trinomial is a polynomial $g = T_0^{l_0} + T_1^{l_1} + T_2^{l_2} \in \mathbb{K}[T_{ij},\, 0 \leqslant i \leqslant 2,\, 1 \leqslant j \leqslant n_i]$, where $n_0, n_1, n_2 \in \mathbb{Z}_{>0}$, $n = n_0 + n_1 + n_2$ and $T_i^{l_i} = T_{i1}^{l_{i1}} \dots T_{in_i}^{l_{in_i}}$ for $0 \leqslant i \leqslant 2$. Consider a trinomial algebra $R(g)=\mathbb{K}[T_{ij}]/(g)$. Denote by K the factor group $K=\mathbb{Z}^n/\operatorname{Im}L^*$, where

$$L^* = \begin{pmatrix} -l_0 & -l_0 \\ l_1 & 0 \\ 0 & l_2 \end{pmatrix}$$

is an $(n\times 2)$-matrix L^* corresponding to the trinomial g, and by $Q\colon \mathbb{Z}^n \to K$ the canonical projection. Then the equations $\deg T_{ij} = Q(e_{ij})$ define a K-grading on the algebra $R(g)$.

The construction of elementary derivations is given in [AHHL14] and is described below in the case of trinomial hypersurfaces. Let us define a derivation $\delta_{C,\beta}$ of $R(g)$, where the input data are a sequence $C=(c_0,c_1,c_2)$, $c_i\in\mathbb{Z}$, $1\leqslant c_i\leqslant n_i$, and a vector $\beta=(\beta_0,\beta_1,\beta_2)$, $\beta_i\in\mathbb{K}$, $\beta_0+\beta_1+\beta_2=0$. If $\beta_i\neq 0$ for all $i=0,1,2$ and there is at most one i_1 with $l_{i_1c_{i_1}}>1$, then we set

$$\delta_{C,\beta}(T_{ij}) = \begin{cases} \beta_i \prod\limits_{k\neq i} \frac{\partial T_k^{l_k}}{\partial T_{kc_k}}, & j=c_i, \\ 0, & j\neq c_i. \end{cases}$$

If $\beta_{i_0}=0$ for a unique i_0 and there is at most one i_1 with $i_1\neq i_0$ and $l_{i_1c_{i_1}}>1$, then

$$\delta_{C,\beta}(T_{ij}) = \begin{cases} \beta_i \prod\limits_{k\neq i,i_0} \frac{\partial T_k^{l_k}}{\partial T_{kc_k}}, & j=c_i, \\ 0, & j\neq c_i. \end{cases}$$

These assignments define a derivation $\delta_{C,\beta}$ on the algebra $R(g)$, which is homogeneous and locally nilpotent. If $h\in R(g)$ is a homogeneous element in the kernel of $\delta_{C,\beta}$, then $h\delta_{C,\beta}$ is also a homogeneous locally nilpotent derivation. Such derivations are called elementary.

In [Z19], homogeneous locally nilpotent derivations on a class of trinomial algebras $R(g)$ are described. This class includes all non-factorial trinomial algebras. More precisely, we assume that there is at most one monomial in g including a variable with exponent 1. The remaining case is done in [GZ19]. The result is obtained in inseparable collaboration with Sergey Gaifullin. Joining two cases, we obtain the following theorem.

Theorem 1 ([GZ19, Theorem 1], [Z19, Theorem 1]). *Every homogeneous locally nilpotent derivation of a trinomial algebra $R(g)$ is elementary.*

2. Affine algebraic monoids

An irreducible algebraic variety X with an associative multiplication $\mu\colon X\times X\to X$ is called an algebraic monoid if μ is a morphism and has a unity. By the rank of a monoid we mean the dimension of a maximal torus in the group of invertible elements. In [ABZ20], we study commutative algebraic monoid structures on $\mathbb{A}^n$. There are unique commutative monoid structures of ranks 0 and n on $\mathbb{A}^n$; the operation is isomorphic to a coordinatewise addition and multiplication in these cases. Using results of [AK15], we obtain a classification of commutative monoid structures of rank $n-1$ on $\mathbb{A}^n$ [ABZ20, Proposition 1]. This covers all commutative monoid structures on $\mathbb{A}^1$ and $\mathbb{A}^2$ [ABZ20, Proposition 2].

Let us formulate the result for $\mathbb{A}^3$. For $b,c\in\mathbb{Z}_{>0}$, $b\leqslant c$, denote by $Q_{b,c}$ the polynomial

$$Q_{b,c}(x_1,y_1,x_2,y_2) = \sum_{k=1}^{d} \binom{d+1}{k} x_1^{e+b(k-1)} y_1^{e+b(d-k)} x_2^{d-k+1} y_2^{k},$$

where $c=bd+e,\ d,e\in\mathbb{Z},\ 0\leqslant e<b$.

Theorem 2 ([ABZ20, Theorem 1]). *Every commutative monoid on $\mathbb{A}^3$ is isomorphic to one of the following monoids:*

rk	*Notation*	$(x_1, x_2, x_3) * (y_1, y_2, y_3)$
0	$3A$	$(x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3)$
1	$M \underset{b}{+} A \underset{c}{+} A$	$\big(x_1y_1,\ x_1^by_2 + y_1^bx_2,\ x_1^cy_3 + y_1^cx_3\big),\ b, c \in \mathbb{Z}_{\geqslant 0},\ b \leqslant c$
1	$M \underset{b}{+} A \underset{b,c}{+} A$	$\big(x_1y_1,\ x_1^by_2 + y_1^bx_2,\ x_1^cy_3 + y_1^cx_3 + Q_{b,c}(x_1, y_1, x_2, y_2)\big), b, c \in \mathbb{Z}_{>0}, b \leqslant c$
2	$M + M \underset{b,c}{+} A$	$(x_1y_1,\ x_2y_2,\ x_1^bx_2^cy_3 + y_1^by_2^cx_3),\ b, c \in \mathbb{Z}_{\geqslant 0},\ b \leqslant c$
3	$3M$	$(x_1y_1,\ x_2y_2,\ x_3y_3)$

Moreover, every two monoids of different types or of the same type with different values of parameters from this list are non-isomorphic.

The most difficult case is that with group of invertible elements $\mathbb{G}_m \times \mathbb{G}_a^2$, where $\mathbb{G}_m = (\mathbb{K}^\times, \times)$ is the multiplicative group of the ground field. Here we reduce the problem to the classification of pairs of commuting locally nilpotent derivations δ_1, δ_2 on the polynomial algebra $\mathbb{K}[x, y, z]$ which are homogeneous of degree zero with respect to the grading coming from the $\mathbb{G}_m$-action.

In [Bi22], the classification of non-commutative monoid structures on normal affine surfaces is given. In [DZ21] and [Z24], we obtain a classification of commutative and noncommutative monoid structures of rank $n-1$ on normal affine varieties of dimension n, respectively. Let X_σ be an affine toric variety given by a cone σ in a vector space $N_\mathbb{Q} = N \otimes_\mathbb{Z} \mathbb{Q}$, where N is the lattice of one-parameter subgroups of the acting torus $\mathbb{T}$. Denote by M the character lattice of $\mathbb{T}$. There is a natural pairing $\langle\,\cdot\,,\,\cdot\,\rangle \colon N_\mathbb{Q} \times M_\mathbb{Q} \to \mathbb{Q}$, where $M_\mathbb{Q} = M \otimes_\mathbb{Z} \mathbb{Q}$. Consider the polyhedral cone $\sigma^\vee$ dual to the cone σ with respect to this pairing:

$$\sigma^\vee = \{u \in M_\mathbb{Q} \mid \langle v, u\rangle \geqslant 0 \ \text{ for all } v \in \sigma\}.$$

The set $S_\sigma = M \cap \sigma^\vee$ is a finitely generated semigroup with $\mathbb{K}[S_\sigma] \cong \mathbb{K}[X_\sigma]$. For a lattice element $u \in M$, let $\chi^u \colon \mathbb{T} \to \mathbb{K}^\times$ be the corresponding character. Since $\mathbb{T}$ can be identified with the open orbit, any character χ^u is identified with a rational function on X_σ. Then the algebra of regular functions on X_σ admits a decomposition $\mathbb{K}[X_\sigma] = \bigoplus_{u \in S_\sigma} \mathbb{K}\chi^u$.

Let $p_i \in N,\ 1 \leqslant i \leqslant m$, be primitive vectors on the rays of the cone σ. For any $1 \leqslant i \leqslant m$, denote

$$\mathfrak{R}_i = \{e \in M \mid \langle p_i, e\rangle = -1,\ \langle p_j, e\rangle \geqslant 0 \ \text{ for all } j \neq i,\ 1 \leqslant j \leqslant m\}.$$

Elements of the set $\mathfrak{R} = \bigsqcup_{1 \leqslant i \leqslant m} \mathfrak{R}_i$ are called the Demazure roots of the toric variety X_σ.

Theorem 3 ([Z24, Theorem 1]). *Let X be an affine monoid of corank one. Then $X = X_\sigma$ is toric, and the comultiplication $\mathbb{K}[X_\sigma] \to \mathbb{K}[X_\sigma] \otimes \mathbb{K}[X_\sigma]$ has the form*

$$\chi^u \mapsto \chi^u \otimes \chi^u \, (1 \otimes \chi^{e_1} + \chi^{e_2} \otimes 1)^{\langle p, u\rangle}, \tag{1}$$

where p is the primitive vector on a ray of the cone σ and e_1, e_2 are Demazure roots corresponding to p. Conversely, for any affine toric variety X_σ, any primitive vector p on a ray of the cone σ, and any Demazure roots e_1, e_2 corresponding to the same p, formula (1) defines a monoid structure of corank one on X_σ.

Also, we describe the sets of invertible elements and idempotents of such a monoid X_σ. They depend on the mutual position of Demazure roots e_1, e_2 and the cone σ. Namely, denote by $O_\gamma \subseteq X_\sigma$ the toric orbit corresponding to the face γ of σ and by x_γ a special point given in O_γ by equations $\chi^u(x_\gamma) = 1$ for all $u \in \gamma^\perp$. The set of invertible elements appears to be equal to the union $O_\rho \cup O_0$, where ρ is the ray of σ with primitive vector p [Z24, Theorem 1]. Let $E(X_\sigma)$ be the set of idempotents in X_σ and $E_\gamma = E(X_\sigma) \cap O_\gamma$. Then the following holds [Z24, Theorem 3]:

(a) $E_\gamma = \{x_\gamma\}$ if ρ is a ray of γ;
(b) $E_\gamma = \varnothing$ if ρ is not a ray of γ and $e_1, e_2 \notin \gamma^\perp$;
(c) $E_\gamma = \varnothing$ if ρ is not a ray of γ and $e_1, e_2 \in \gamma^\perp$;
(d) $E_\gamma = O_\gamma \cap \{\chi^u = 1 \ \ \forall u \in \operatorname{cone}(\gamma, \rho)^\perp \cap S_\sigma\}$ otherwise.

Geometrically, irreducible components of the subvariety of idempotents do not intersect, each of them is either a point or is isomorphic to the affine line [Z24, Proposition 3(b)]. The affine line appears here as the closure of a set E_γ from item (d); this closure is the union of E_γ and one point, which is an idempotent from item (a) [Z24, Proposition 3(a)]. Idempotents are also connected with the action of the group $G \times G$ by left and right multiplication, where G is the group of invertible elements in the monoid. More precisely, any irreducible component of $E(X_\sigma)$ is a subset of a $(G \times G)$-orbit, and any $(G \times G)$-orbit contains at most one irreducible component of $E(X_\sigma)$ [Z24, Proposition 3(c)]. One of possibly existing idempotents is the zero element, i.e., such an element $\mathbf{0} \in X_\sigma$ that $\mathbf{0} * x = x * \mathbf{0} = \mathbf{0}$ for any $x \in X_\sigma$. We show that the monoid X_σ has zero if and only if $\sigma^\perp = 0$ and $-e_1, -e_2 \notin \sigma^\vee$; in this case $\mathbf{0} = x_\sigma$ [Z24, Proposition 4].

The center of the monoid X_σ is described as well. Namely, it equals

$$\overline{O_\rho} \cap \{\chi^{u+e_1} = \chi^{u+e_2} \quad \forall u \in S_\sigma : \langle p, u\rangle = 1\}$$

if $e_1 \neq e_2$, i.e., if X_σ is non-commutative [Z24, Proposition 5]. It follows that the dimension of the center equals $\dim X_\sigma - 2$ [Z24, Corollary 3]. Moreover, irreducible components of $E(X_\sigma)$ that are isomorphic to the affine line do not intersect the center, and isolated points in $E(X_\sigma)$ lie in the center [Z24, Proposition 6].

3. Affine homogeneous varieties

Let us call an algebraic variety X homogeneous if the automorphism group $\operatorname{Aut}(X)$ acts on X transitively. Recall that X is a homogeneous space if there exists a transitive action of an algebraic group G on X; in this case, X is identified with the variety of left cosets G/H, where H is the stabilizer in G of a point in X.

Let us give a definition of a suspension.

Definition 1. Let Y be an affine variety and $f \in \mathbb{K}[Y]$ be a nonconstant regular function on Y. Then the hypersurface $\operatorname{Susp}(Y, f)$ that is given in the direct product $\mathbb{A}^2 \times Y$ by the equation $uv = f(y)$, where $y \in Y$ and $\mathbb{A}^2 = \operatorname{Spec} \mathbb{K}[u, v]$, is called a suspension over Y.

In [AZ24], we find a criterion of smoothness of a suspension. Namely, the suspension $\operatorname{Susp}(Y, f)$ over an affine variety Y with a nonconstant $f \in \mathbb{K}[Y]$ is smooth if and only if the variety Y and the scheme $\operatorname{Spec} \mathbb{K}[Y]/(f)$ are smooth [AZ24, Corollary 1]. This gives a criterion of smoothness of iterated suspensions and allows to construct many homogeneous varieties [AZ24, Corollaries 1–4].

To provide an explicit class of examples, we give criteria for a Danielewski surface to be a homogeneous variety and a homogeneous space. Let x, y, z be coordinates in $\mathbb{A}^3$. A Danielewski surface is a surface in $\mathbb{A}^3$ given by equation $xz^n = f(y)$, where $n \in \mathbb{Z}_{>0}$ and $f \in \mathbb{K}[y]$. Two Danielewski surfaces with parameters $n_1, n_2 \in \mathbb{Z}_{>0}$ and polynomials

$f_1(y), f_2(y)$ are isomorphic if and only if $n_1 = n_2$ and $f_1(y) = af_2(by+c)$ for some $a, b \in \mathbb{K}^\times$, $c \in \mathbb{K}$, see [Da04, Lemma 2.10].

It is known that for $n \neq 1$ the Danielewski surface $xz^n = f(y)$ is not homogeneous; for example, this follows from the description of its automorphism group given in [ML01]. It is easy to check that X is smooth if and only if the polynomial f has no multiple roots. Let X be given by the equation $xz = f(y)$, where f has no multiple roots. Note that X is homogeneous. Indeed, X is a suspension over the affine line, so it is flexible [AKZ12]. According to [AFKKZ13], the action of $\mathrm{SAut}(X)$ is transitive on the set of regular points in X, which coincides with X since X is smooth.

Theorem 4 ([AZ24, Theorem 3])**.** *Let X be an affine surface given in $\mathbb{A}^3$ by equation $xz = f(y)$, where f is a nonconstant polynomial with no multiple roots. Then X is a homogeneous variety but not homogeneous space if and only if* $\deg f \geqslant 3$.

To prove this theorem, we use the classifications of surfaces admitting an action of an algebraic group with an open orbit such that the complement to this orbit is finite; see [Gi71, Po73]. For $\deg f = 1$ and $\deg f = 2$ the surface X is isomorphic to the affine space $\mathbb{A}^2$ and the homogeneous space SL_2 / T respectively, where T is a maximal torus in SL_2.

4. Additive actions on projective hypersurfaces

In [AZ22], we study additive actions, that is, effective regular actions of the group $\mathbb{G}_a^n$ with an open orbit. We apply the Hassett-Tschinkel correspondence to the study of induced additive actions on projective hypersurfaces, i.e., additive actions that can be extended to an action on the ambient projective space. More precisely, there is a bijection between induced additive actions on hypersurfaces in $\mathbb{P}^{n+1}$ that are not hyperplanes and pairs (A, U), where A is a local commutative associative unital algebra of dimension $n+2$ with maximal ideal $\mathfrak{m}$ and $U \subseteq \mathfrak{m}$ is a subspace of dimension n generating the algebra A.

Definition 2. Suppose a projective hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree d is given by the equation $f(z_0, \ldots, z_{n+1}) = 0$. A hypersurface X is called non-degenerate if there is no linear change of variables such that the number of variables in f after this change is at most $n+1$.

A finite-dimensional commutative associative algebra is called Gorenstein if the dimension of the socle $\mathrm{Soc}\, A = \{x \in A \mid \mathfrak{m}x = 0\}$ equals 1. It turns out that the case of non-degenerate hypersurfaces corresponds to Gorenstein local algebras. More precisely, the induced additive actions on non-degenerate hypersurfaces X of degree d in $\mathbb{P}^{n+1}$ are in bijection with the pairs (A, U), where A is a Gorenstein algebra of dimension $n+2$ with socle $\mathfrak{m}^d$, and $\mathfrak{m} = U \oplus \mathfrak{m}^d$ [AZ22, Theorem 2.30].

Several results on additive actions may be proved using this technique. In particular, we prove that an induced additive action on a non-degenerate projective hypersurface is unique if it exists.

Theorem 5 ([AZ22, Theorem 2.32])**.** *Let $X \subseteq \mathbb{P}^{n+1}$ be a non-degenerate hypersurface. Then there is at most one induced additive action on X up to equivalence.*

In order to prove the theorem, we consider the d-linear form F corresponding to the equation of X. Denote

$$\mathrm{Ker}\, F = \{x \in A \mid F(x, z^{(2)}, \ldots, z^{(d)}) = 0 \ \ \forall z^{(2)}, \ldots, z^{(d)} \in A\}.$$

We show that $\operatorname{Ker} F$ is the maximal ideal in A that is contained in U, where (A, U) is the pair corresponding to the hypersurface X; see [AZ22, Lemma 2.19(b)]. The condition that X is non-degenerate means that $\operatorname{Ker} F = 0$, so there is no non-zero ideal of A that is contained in U. This is a key point in the proof of the uniqueness of the pair (A, U).

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