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Geometry and dynamics in the moduli space of meromorphic differentials

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Content

Introduction

The work is dedicated to the study of moduli spaces of meromorphic differentials on algebraic curves. In the first part we study spaces of holomorphic differentials. Using the methods developed in this part we extend them further to study spaces of differentials with poles.

Moduli spaces of holomorphic differentials. Elements of the space H_g of holomorphic differentials on genus g Riemann surfaces, are referred to as translation surfaces or flat surfaces. They arise naturally in the study of various basic dynamical systems. Defining a holomorphic 1-form ω on a compact Riemann surface X is the same as giving a collection of charts on X such that the transition maps are translations; the charts are allowed to be ramified at finitely many points corresponding to the zeros of ω and are given locally, in a neighborhood of a point $z_0 \in X$, by $z \mapsto \int_{z_0}^{z} \omega$. These special charts of (X, ω) have an echo on strata of the moduli space $H_g(\kappa)$ consisting of genus g Riemann surfaces endowed with holomorphic 1-forms having zeros of given multiplicities $\kappa = (k_1, \ldots, k_n)$. The strata are themselves locally modeled on complex vector spaces, with transition functions between charts being linear functions, called "period coordinates". Local coordinates of the period atlas are: integrals of the differential along closed loops on the surface punctured at the poles (absolute periods) together with integrals of the differential along paths joining distinct zeroes (relative periods).

Isoperiodic foliation. Fixing the absolute periods defines on each stratum $H_g(\kappa)$ the isoperiodic foliation, also known in the literature as Rel or Kernel foliation. Isoperiodic foliation was studied for example in [17], [7], [29].

$GL_2^+(\mathbb{R})$ **action.** The group $GL_2^+(\mathbb{R})$ acts on the space of differentials. This action preserves the stratification of the space of differentials by the orders of their zeroes. The action of $GL_2^+(\mathbb{R})$ is locally given in period coordinates by a diagonal action on a product of copies of $\mathbb{C} \cong \mathbb{R}^2$, or explicitly in terms of the real and imaginary parts of the holomorphic 1-form as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Re(\omega) \\ Im(\omega) \end{pmatrix} = \begin{pmatrix} aRe(\omega) + bIm(\omega) \\ cRe(\omega) + dIm(\omega) \end{pmatrix}.$$

Orbits and orbit closures. Problems of classification of closed orbits, orbit closures and the classification of finite measures with respect to this action lead to many interesting results. Masur [26] and Veech [40] proved independently that there is a natural finite probability measure of Lebesgue class on the subset of unit area flat surfaces, known today as Masur-Veech measure. The action of the subgroup $SL_2(\mathbb{R})$ preserves this measure. Following the Hopf argument in ergodic theory, Masur and Veech deduced that the action of the diagonal subgroup of $SL_2(\mathbb{R})$ is ergodic with respect to this measure. Hence most orbits are dense.

The first $SL_2(\mathbb{R})$-closed orbits were discovered by Veech [42]. Those were surfaces such that their stabilizer is a lattice in $SL_2(\mathbb{R})$, for the proof, see, for example, [41]. For more results relating the properties of the $SL_2(\mathbb{R})$-orbits and the geometry of a given flat surface see, for example, Masur [25], Minsky and Weiss [31], Smillie and Weiss [36].

The fact that the stabilizer of a closed orbit is a lattice implies that the projection of a closed orbit in $H_g(\kappa)$ to the moduli space of Riemann surfaces M_g is an algebraic curve. In fact all isometrically immersed curves in M_g (known as Teichmüller curves) are projections of closed orbits up to a "double covering" relating quadratic differentials to Abelian differentials.

One family of examples of closed orbits comes from square-tiled surfaces, which are finite-sheet tori coverings. The problem of constructing more closed orbits appeared to be very difficult, since computing the stabilizer explicitly for a given translation surface is usually hard, for a known algorithm, see the work of Mukamel [35]. The other known set of examples of such orbits was discovered by studying $GL_2^+(\mathbb{R})$ orbit closures.

Affine-invariant submanifolds. From the work of Eskin, Mirzakhani and Mohammadi [11] it is known that $GL_2^+(\mathbb{R})$-invariant subsets are finite unions of affine-invariant submanifolds. The latter is the image of a proper immersion of an open connected manifold to a stratum $H_g(\kappa)$ such that the image of each point together with a small neighbourhood is determined by linear equations in period coordinates with real coefficients and the vanishing constant term.

On the other side, any affine-invariant submanifold is $GL_2^+(\mathbb{R})$-invariant, this is a much easier observation, for the proof see i.e. [44]. There are only countably many affine invariant submanifolds [11], [43].

Form the above, closed $GL_2^+(\mathbb{R})$ orbits are the same as 2-dimensional affine-invariant submanifolds.

Prym eigenforms. Möller proved that over the 2-dimensional orbit closures, the zeroes of the holomorphic 1-forms must map to torsion points on (a factor of) the Jacobian [32], [33]. Filip generalized the result for higher dimensions in [13], [14], showing that affine invariant submanifolds are quasiprojective varieties. In particular, this means that any $GL_2^+(\mathbb{R})$ orbit closure can be completely defined in terms of algebraic conditions on the Jacobian.

The first examples of affine-invariant submanifolds are due to McMullen in [28], where he described orbit-closures for some flat surfaces of genus 2. A complete classification in genus 2 was obtained by the same author in [30]. He proved that if an orbit is neither closed nor dense, then it is a Prym eigenform. The latter are surfaces admitting a special kind of involution, such that the Prym subvariety of the Jacobian defined with respect to this involution admits real multiplication by some quadratic order (see sect. 1.4 for the exact definitions).

In [27], infinite families of Prym eigenforms, called Prym eigenform loci, are constructed in genus up to 5. It is shown that they cannot exist for higher genera. Each locus is proven to be a $GL_2^+(\mathbb{R})$ invariant subset. The construction depends on the discriminant of the quadratic order and only allows to construct loci whose connected components are orbit closures. In the case of genus 2, the number of connected components of the loci for both strata were computed in [30].

The question of deducing the individual orbit-closures in Prym eigenform loci in genera $3, 4$ was solved for some strata in the series of papers by Lannaeu and Nguyen [21], [23], [24], [22].

In this work we solve the problem for the highest possible genus 5, for the stratum $H_5(4, 4)$, where a 1-form has two zeroes of order 4. We show that each Prym eigenform locus is a single orbit closure, see 1.9.

The methods used in our work extend the approach developed by Lanneau and Nguyen. The major tool that we use is isoperiodic transformations of Prym eigenforms.

An isoperiodic deformation of a translation surface $(X, \omega) \in H_g(\kappa)$ is a path $(X_t, \omega_t), t \in [a, b]$, within the stratum $H_g(\kappa)$ such that for any absolute homology class γ, the value of $\int_\gamma \omega_t$ is constant. Surfaces obtained by isoperiodic transformations from Prym eigenforms also belong to the Prym eigenform loci.

Moduli spaces of meromorphic differentials. The notion of a translation surface may be generalized to the case of meromorphic 1-forms. A Riemann surface X endowed with a non-zero meromorphic 1-form ζ on it is referred to as a *flat surface with poles* or a *non-compact translation surface*. Let $h = (h_1, \cdots, h_n)$ be a tuple of positive integers such that $\sum h_i > 1$, and let $M_{g,n}(h)$ denote the moduli space of pairs (X, ζ) where X is a compact Riemann surface of genus g and ζ is a meromorphic differential with poles of order h_i at the marked points $P_1, \ldots, P_n$.

The space is stratified by the multiplicities of the zeroes. Period coordinates on the strata may be defined using the relative cohomology group of the punctured surface $X \setminus \{P_1, \ldots, P_n\}$.

The connected components of the strata were classified by Boissy [5]. There is a natural $GL_2^+(\mathbb{R})$-action on the moduli space, and the moduli space can be endowed with an analogue of the Masur–Veech measure; however its total volume is infinite. While the $GL_2^+(\mathbb{R})$-action may have positive-dimensional stabilizers in this case, the notion of Teichmüller curves can still be introduced, and a classification of such objects is given in [34]. In [37], [38] the geometry of the connected components is studied. The strata decompose into chambers, which are separated by the locus called the discriminant. The discriminant is known to be a $GL_2^+(\mathbb{R})$-invariant codimension 1 hypersurface.

The fibration of the strata by fibers consisting of 1-forms with prescribed residues at the poles is studied in [15].

The existence of local period coordinates allows one to define the isoperiodic foliation on the strata similar to the case of holomorphic differentials. The properties of the foliation were recently studied in [12] for the case of genus 1 surfaces, where the meromorphic 1-form has a single order two pole and two simple zeroes.

The isoperiodic foliation on the subspace of the moduli space of meromorphic differentials with all periods being real was studied in [20]. Such meromorphic 1-forms are referred to as real normalized differentials. They are central objects in the Whitham perturbation theory of algebraic-geometrical solutions of the integrable systems. In [16], it was shown that certain structures and constructions of the Whitham theory can be instrumental in understanding the geometry of the moduli spaces of Riemann surfaces with marked points. In particular, a new proof of Diaz' bound on the dimension of complete subvarieties of the moduli spaces was obtained. In [19], real normalized differentials were used for a proof of Arbarello's conjecture [10]. Another context, where normalized differentials arise, is the asymptotic analysis of complex-orthogonal polynomials, see e.g. [9], [3], [2]. Very recently real normalized differentials were used as a tool to study spaces of solutions of a given degree of the complex Pell-Abel equations in [4].

In this work, we further investigate properties of the isoperiodic foliation on the space of real-normalized differentials with a single pole of order two for genus g curves. This space (denoted R_g) is stratified by the multiplicities of the zeroes and the strata are denoted $R_g(\kappa)$.

Combinatorial model. A combinatorial model of the principal stratum (where all the zeroes have order 1) of the space of real-normalized differentials with a single pole of order two suggested in [20] describes isoperiodic transformations using tools from the theory of Vassiliev knot invariants. In this work we use this description to characterize the leaves of the isoperiodic foliation on this stratum.

An additive group isomorphic to $\mathbb{Z}^{2g}$ together with its homomorphism to $\mathbb{C}$, endowed with a symplectic form, is called a polarized module [29].

We show that for a given group of rank $2g$, the corresponding polarized modules enumerate the leaves of the isoperiodic foliation on the principal stratum, see 2.2.

In the next sections we provide the general background and state the obtained results. The second section serves to explain the notion of translation surfaces, the moduli spaces of translation surfaces and the Prym eigenform loci. The third section is dedicated to the moduli spaces of meromorphic differentials and of real-normalized differentials.

Personal contribution

All the results presented in this theses were obtained by the author.

Published articles

The main results of the thesis are published in 3 papers:

Nenasheva M., *About the isoperiodic foliation on the stratum of codimension 1 in the space of real-normalized differentials*, 2024, Algebra and Analysis, 36:2, pp. 93-107

Nenasheva M., *Principal stratum in the moduli space of real-normalized differentials with a single pole,* 2024, accepted by Functional Analysis and Its Applications

Nenasheva M., *Connectedness of Prym Eigenform Loci in Genus 5,* 2023, Dokl. Math., 108, pp. 486–489

Approbation of the results

The results of the thesis were presented at the following conferences:

International conference “Invariance and integrability 2”, Pushkin, September 2024 .

Talk: “Isoperiodic foliation on the space of meromorphic real-normalized differentials”

International “Conference Constructive Methods of the Theory of Riemann Surfaces and Applications”, Sirius University, Sochi, November 2023.

Talk: “Isoperiodic foliation on the space of meromorphic real-normalized differentials”

Research School and Conference “Invariants and Integrability”, Repino, October-November 2023.

Talk: “Isoperiodic foliation on the space of meromorphic real-normalized differentials”

International conference Ergodic Theory and Related Topics, Moscow, November 2022.

Talk: “On the connected component of Prym eigenform loci in genus 5”

2nd Conference of Mathematical Centers of Russia, Moscow, November 2022.

Talk: “On the connected component of Prym eigenform loci in genus 5”

Reliability of results

All results of the dissertation are justified by strict mathematical proofs. The findings of the theses were presented at several conferences and scientific seminars.

1 Translation surfaces

An Abelian differential ω on a Riemann surface X is a global holomorphic section of the cotangent bundle of X. The complex vector space of Abelian differentials on a Riemann surface X is denoted $H^{1,0}(X)$. For a genus g surface X, $\dim_{\mathbb{C}} H^{1,0}(X) = g$. Since every nonzero Abelian differential on X is a closed but not an exact 1-form on X, the space $H^{1,0}(X)$ is naturally identified with a subspace of $H^1(X, \mathbb{C})$, the first cohomology group of X.

A holomorphic Abelian differential ω on a genus $g \geq 1$ surface has $2g-2$ zeroes, taking multiplicities into account. Let Σ denote the set of zeroes of ω. Then there is an atlas of charts on $X\backslash\Sigma$ such that all the transition maps are translations. At any regular point $p_0 \in X\backslash\Sigma$ there is a local coordinate z such that $\omega = dz$. This choice is unique up to translation, because for any holomorphic function f, if $df = dz$, then $z = f + C$ for some constant C. At a point where ω has a zero of order k, there exists a local coordinate z such that $\omega = z^k dz$. Indeed, let w be a local coordinate on X and suppose ω vanishes to order k at $w = 0$. Then locally $\omega = w^k g(w)dw$, where g is a holomorphic function non-vanishing at $w = 0$. Define the new local coordinate z by taking a $(k+1)$-st root of the following: $z^{k+1} = (k+1)\int_0^w g(t)t^k dt$.

A pair (X, ω) is called a *translation surface*. It is also called a *flat surface*, since the 1-form ω induces a flat metric on $X\backslash\Sigma$. In the chosen coordinate z such that $\omega = dz$, if we put $z = x + iy$, the metric reads $\omega \otimes \bar{\omega} = dx^{\otimes 2} + dy^{\otimes 2}$. The metric does not extend to the set of zeroes Σ, it is assumed to have *singularities* at these points. Points at which ω vanishes to order k are also called singularities of order k.

1.1 Moduli space of translation surfaces

If (X, ω) and (X', ω') are such that there is a biholomorphism $f : X \to X'$ with $f_*\omega' = \omega$, then f is an isometry for the metrics defined by ω and ω'. Moreover, for the local coordinates defined by ω, ω', the map f is in fact a translation.

Consider the moduli space of pairs (X, ω), where X is a Riemann surface of genus g and ω a holomorphic one-form on X (an Abelian differential). Two pairs are equivalent, $(X, \omega) \sim (X', \omega')$, if there is a biholomorphism $f : X \to X'$ such that $f_*\omega' = \omega$. Denote this space by H_g. It is a $\mathbb{C}^g$-vector bundle over the moduli space M_g of Riemann surfaces.

The space H_g is stratified by the strata $H_g(\kappa)$ corresponding to unordered partitions $\kappa \vdash 2g-2$, $\kappa = (\kappa_1, \ldots, \kappa_n)$, It is known that $H_g(\kappa)$ is an algebraic variety and a complex orbifold of dimension $2g + n - 1$. In general, it is not a fiber bundle over M_g. Not all the $H_g(\kappa)$ appear to be connected, this fact was first observed by W. Veech, see [39].

1.2 $GL_2^+(\mathbb{R})$ action

The group $GL_2^+(\mathbb{R})$ of 2×2-matrices over reals with positive determinants naturally acts on the complex line $\mathbb{C}$ if we represent the latter as the real plane $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$. This action induces the action of $GL_2^+(\mathbb{R})$ on the space $\widetilde{H}_g(\kappa)$ defined by

$$\forall G \in GL_2^+(\mathbb{R}) \qquad \Phi(G \circ (X, \omega, f)) = G \circ \Phi(X, \omega, f).$$

Here Φ is the period map, defined in 1.5. The action descends to an action on $H_g(\kappa)$ in a way making the canonical projection $GL_2^+(\mathbb{R})$-equivariant.

1.3 Affine-invariant orbifolds

Definition 1. *[Affine invariant orbifold] An affine invariant orbifold is a closed connected subset M of $H_g(\kappa)$ obtained as the image of a complex orbifold $\mathfrak{M}$ under a proper immersion $i : \mathfrak{M} \to H_g(\kappa)$ that satisfies the following property: for any $x \in \mathfrak{M}$, there is an open set U around x, an orbifold chart (V, ϕ) around $i(x)$ and an $\mathbb{R}$-linear vector subspace W of $H^1(X, \Sigma, \mathbb{R})$ such that*

$$i(U) \cap \phi(V) = \phi(V \cap W \otimes \mathbb{C}). \tag{1}$$

Affine invariant orbifolds are invariant under the action of $GL_2^+(\mathbb{R})$ and the converse was also shown to be true. Unions of affine invariant orbifolds are called *affine invariant loci.*

1.4 Prym eigenform loci

Examples of affine invariant loci in genera ≤ 5, consisting of surfaces with specific symmetry, are known today as Prym eigenform loci. Let X be a compact Riemann surface equipped with an automorphism $\rho : X \to X$ of order two.

The action of ρ determines a splitting $\Omega(X) = \Omega(X)^- \oplus \Omega(X)^+$ of the space of Abelian differentials on X into even and odd 1-forms: $\omega \in \Omega(X)^\pm$ iff $\rho^*(\omega) = \pm\omega$. The sublattices $H_1(X, \mathbb{Z})^\pm \subset H_1(X, \mathbb{Z})$ consisting of even and odd cycles, respectively, are defined in a similar way. Let $(\Omega(X)^-)^*$ denote the

$\mathbb{C}$-dual vector space to $(\Omega(X)^-)$.

Definition 2. *(Prym variety) The Prym variety* $P = Prym(X, \rho) = (\Omega(X)^-)^*/H_1(X, \mathbb{Z})^-$ *is the Abelian subvariety of* $Jac(X) = \Omega(X)^*/H_1(X, \mathbb{Z})$ *consisting of the* 1*-forms that are odd with respect to* ρ*.*

We refer to $\Omega(X)^-$ as the space of Prym forms on (X, ρ). Note that $\Omega(X)^+$ is canonically isomorphic to $\Omega(Y)$, $Y = X/\rho$, and thus dim $P = g(X) - g(Y)$. The variety P is canonically polarized by the intersection pairing on $H_1(X, \mathbb{Z})^-$.

Let $End(P)$ denote the endomorphism ring of a polarized Abelian variety $P \simeq \mathbb{C}^g/L$. We can regard elements of $End(P)$ as complex-linear maps T: $\mathbb{C}^g \to \mathbb{C}^g$ such that $T(L) = L$. An endomorphism is called *self-adjoint* if it satisfies $\langle Tx, y\rangle = \langle x, Ty\rangle$ with respect to the symplectic pairing of $x, y \in P$. Let $D > 0$ be an integer congruent to 0 or 1 mod 4, and let $O_D \simeq \mathbb{Z}[t]/(t^2 + bt + c), D = b^2 - 4c$, for integers b and c, be a real quadratic order of discriminant D. The variety P is said to admit *real multiplication* by O_D if $dim_{\mathbb{C}} P = 2$, and there is a proper sub-ring $R \simeq O_D \subset End(L)$ generated by a self-adjoint endomorphism $T \in End(L)$. (Here "proper" means that if $U \in End(L)$, and $nU \neq 0$ belongs to R, then $U \in R$).

Prym eigenforms. Now suppose $P = Prym(X, \rho)$ is a Prym variety with real multiplication by O_D. Then O_D also acts on $\Omega(P) \simeq \Omega(X)^-$, and the space of odd forms splits into a pair of one-dimensional eigen-spaces of this action. We say $\omega \in \Omega(X)^-$ is a Prym eigenform if $0 \neq O_D \cdot \omega \subset \mathbb{C}\omega$. The following proposition was proven by McMullen [27]:

Proposition 1. *The closure of the* $GL_2^+(\mathbb{R})$*-orbit of any Prym eigenform is a rank one affine invariant orbifold.*

The union of these orbit closures for the Prym eigenforms, for a given D, is denoted ΩE_D. It admits the stratification $\Omega E_D(\kappa)$ indexed by the multiplicities of the zeroes, which is induced by the stratification $\{H_g(\kappa)\}$ of the ambient space H_g.

1.5 Period coordinates

Denote by $\widetilde{H}_g(\kappa)$ the space of equivalence classes of *marked translation surfaces* (X, ω, f), where $f : S \to X$ is a homeomorphism from a fixed genus g surface S such that the preimage of the singularities of ω under f is a given subset $\Sigma \subset S$, and the orders of the singularities of ω are prescribed by κ. The mapping Φ defined for every $\gamma \in H_1(S, \Sigma; \mathbb{C})$ as

$$\Phi : \widetilde{H}(\kappa) \to H^1(S, \Sigma; \mathbb{C})$$
$$\Phi : (X, \omega, f) \mapsto \left(\gamma \mapsto \int_{f \circ \gamma} \omega\right)$$

is called a *period map.*

There is a complex structure on $\widetilde{H}(\kappa)$ that turns Φ into a local biholomorphism, and if $MCG(S, \Sigma)$ denotes the relative mapping class group of S that fixes Σ globally, then $MCG(S, \Sigma)$ acts almost freely on $\widetilde{H}(\kappa)$ by precomposition: $\phi : (X, \omega, f) \mapsto (X, \omega, f \circ \phi^{-1})$. The quotient with respect to this action is isomorphic to $H(\kappa)$ and the latter is endowed with the complex orbifold structure that turns the canonical projection π: $\widetilde{H}(\kappa) \to H(\kappa)$ into a local biholomorphism (in the orbifold sense).

1.6 Isoperiodic foliation

Let $(X, \omega, f) \in \widetilde{H}(\kappa)$ be a marked translation surface. Then the subset of the moduli space $\widetilde{H}(\kappa)$ consisting of surfaces whose absolute periods coincide with the image $\Phi(\omega)$ of the period map is well-defined. For any choice of the homology basis $\{\gamma_1, \ldots, \gamma_{2g}\} \subset H_1(X, \mathbb{Z})$, there are corresponding $2g$ values of $\Phi(\omega)$. The additive subgroup of $\mathbb{C}$ generated by these $2g$ complex numbers is referred to as the *group of absolute periods* of ω. The subsets in $\widetilde{H}(\kappa)$ corresponding to different groups of periods do not intersect, forming leaves of a foliation of $\widetilde{H}(\kappa)$. This foliation is called the *absolute isoperiodic foliation*, kernel foliation or Rel foliation.

Let $\rho : H^1(S, \Sigma; \mathbb{C}) \to H^1(S; \mathbb{C})$ denote the canonical restriction map. The map $\rho \circ \Phi$ is a $MCG(S, \Sigma)$-equivariant submersion. The isoperiodic foliation on the moduli space of non-marked translation surfaces $H(\kappa)$ is defined as the quotient foliation of the foliation on $\widetilde{H}(\kappa)$ by connected components of the level sets of $\rho \circ \Phi$.

If ω has n zeroes $\Sigma_1, \ldots, \Sigma_n$ on X, then there are $n-1$ degrees of freedom for perturbing (X, ω) to stay in the same leaf. The values $t_i(X, \omega) = \int_{\Sigma_1}^{\Sigma_i} \omega$ for $i = 2, .., n$, which are called the *relative periods*, provide coordinates on the leaf. We can move along an isoperiodic leaf of (X, ω) by changing t_i. Local changes in t_i can be constructed via isoperiodic transformations (see 1.8).

The complex dimension of the leaves in the principal stratum $H_g(1^{2g-2})$ equals $2g-3$ (for $g \geq 2$). The isoperiodic foliation is globally defined on $H_g = \bigcup_\kappa H_g(\kappa)$, where its leaves have dimension $2g-3$. For $\kappa = (g-1, g-1)$, the leaves have complex dimension 1 and admit a structure of a Riemann surface; this is true, in particular, for $g = 5$, $\kappa = (4, 4)$, the case we are interested in.

1.7 Connectedness of $\Omega E_D(\kappa)$

Let F be the isoperiodic foliation in $H_g(\kappa)$ defined in 1.6 and let F_ω be the leaf which contains (X, ω). Let $M \subset H(\kappa)$ be an affine invariant orbifold and let $(X, \omega) \in M$. Denote by F_ω^M the connected component of $F_\omega \cap M$ which contains (X, ω). The following proposition was proven by Florent Ygouf (Prop. 2.3 in [45]) :

Proposition 2. *Let $i : \mathfrak{M} \to H(\kappa)$ be an immersion such that $i(\mathfrak{M}) = M$ as in Definition 1. There is a foliation $\mathfrak{F}$ on $\mathfrak{M}$ such that for any $(X, \omega) \in M$ and any $x \in \mathfrak{M}$ such that $i(x) = (X, \omega)$, there is a neighbourhood U of x in $\mathfrak{M}$ such that*

$$i(U \cap \mathfrak{F}_x) = i(U) \cap F_\omega^M. \tag{2}$$

The above proposition allows one to define the absolute period foliation restricted to an affine invariant locus in the following way.

As above, F_ω^M are the leaves of an immersed foliation, which will be referred to as the *M-isoperiodic foliation*. In other words, we now consider the subset of $H_g(\kappa)$, the points in which have the same absolute periods, with relative periods varied, plus an extra restriction that these points belong to the given affine invariant manifold.

Note that for the case of Prym eigenform loci, the relative periods do not affect the property whether the surface belongs to the locus: if $(X, \omega) \in \Omega E_D(\kappa)$, then $F_\omega \subset \Omega E_D(\kappa)$.

Since the leaves are connected, each leaf is contained in a single connected component of $\Omega E_D(\kappa)$. An important numerical invariant of affine manifolds is it rank (rk), defined as half the dimension of its tangent space projected to absolute cohomology. Call an affine invariant orbifold M *non absolute* whenever $\dim_{\mathbb{C}}(M) > 2\mathrm{rk}(M)$. A non absolute M-isoperiodic foliation respects the action of $GL_2^+(\mathbb{R})$, as it is shown in Proposition 2.4 in [45]:

Proposition 3. *Let M be a non absolute affine invariant orbifold, let (X,ω) be a translation surface in M, and let $g \in GL_2^+(\mathbb{R})$. Then the following holds:*

$$g \cdot F_\omega^M = F_{g\cdot(X,\omega)}^M \tag{3}$$

$$GL_2^+(\mathbb{R}) \cdot F_\omega^M = M. \tag{4}$$

The following proposition was proven by Lanneau and Nguyen in ([22], Corollary 3.2) for the following partitions: $\kappa = (1,1), (3,3), (2,2,1,1), (2,2), (1,1,2), (1,1,4), (4,4), (2,2,2)$:

Proposition 4. *Let $(X',\omega') \in \Omega E_D(\kappa)$ be a point close enough to $(X,\omega) \in \Omega E_D(\kappa)$. Then there exists a unique pair (g,ω), where $g \in GL_2^+(\mathbb{R})$ close to id, and $w \in \mathbb{C}$, with $|\omega|$ small, such that $(X',\omega') = g\cdot((X,\omega)) + w$.*

The statement means that a small neighbourhood of a point in $\Omega E_D(\kappa)$ can be completely described using the $GL_2^+(\mathbb{R})$ action and a 1-parameter deformation of $(X,\omega)+w$. In the case of the strata with two zeroes, the deformation is the isoperiodic deformation, which preserves the absolute periods and changes all the relative periods by adding the same complex number to each of them.

1.8 Isoperiodic deformations

Schiffer described infinitesimal transformations of Riemann surfaces. They come in 1-parameter families depending on ϵ. A surface obtained by a Schiffer variation from a given flat surface (X,ω) admits a holomorphic 1-form ω^*, which is the pushforward of ω by the locally invertible map corresponding to the variation. After the variation, integrals of the induced 1-form ω^* along the new cycles, representing the absolute periods are not changed, that is, the transformation is isoperiodic.

We describe four isoperiodic deformations and study their properties. First we describe the classical transformations called shifting a zero, splitting a zero, collapsing a zero using local surgeries. Then we introduce the transformation called *plumbing*. If there is a holomorphic 1-form ω on a curve obtained by normalization of a singular complex curve, then the plumbing transformation results in a non-singular curve of higher genus with the induced holomorphic 1-form ω^* on it, with the property that the integrals of ω^* along the added absolute cycles are all equal to zero, and the non-zero absolute periods are equal to the ones of ω.

1.9 Results

We solve the problem of describing the connected components of Prym eigenform loci in the stratum $H_5(4,4)$ consisting of Riemann surfaces of the highest possible genus 5 endowed with 1-forms having two zeroes of order 4.

Theorem 1. *Prym eigenform loci $\Omega E_D(4,4)$ are non-empty and have a single connected component for $D \geq 4$.*

In case of the stratum $H_5(8)$ we show that $\Omega E_D(8)$ is empty.

2 Moduli spaces of meromorphic differentials

A Riemann surface X endowed with a non-zero meromorphic 1-form ζ on it is referred to as a *flat surface with poles* or a *non-compact translation surface.*

In a neighbourhood of a pole of order one, a meromorphic 1-form ζ in local coordinates up to rescaling may be written as $\zeta = \frac{1}{z}dz$. Then we may choose the coordinate z' such that $z = e^{z'}$ and $\zeta = dz'$. In this coordinate the neighbourhood of a pole of order one is an infinite cylinder. For a pole P_0 of order $k+2 \geq 2$, there are two options: either the residue of the 1-form ζ is zero, or non-zero at this point. In the first case an open coordinate neighbourhood (U,ζ) of the point P_0 is biholomorphic to a Euclidean cone of a cone angle $2(k+1)\pi$ at $z=0$.

In the second case, let $r \neq 0$ be the residue around P_0. Then the neighbourhood of P_0 looks like a cylinder with holonomy r with k copies of the Euclidean plane attached to it (see [8]). Hence for any meromorphic form ζ on X, a translation atlas may be defined on $X \backslash P(\zeta)$, where $P(\zeta)$ is the set of poles. The translation surface in such metric has infinite area.

For any set of positive integers $h = \{h_1, \ldots, h_n\}$ let $M_{g,n}(h)$ denote the moduli space of curves of genus g with n marked points $\{P_1, \ldots, P_n\}$ endowed with a meromorphic differential with poles of order exactly h_i at each P_i.

If (X, ζ) and (X_0, ζ_0) are such that there is a biholomorphism $f : X \to X_0$ with $f_* \zeta_0 = \zeta$, then f is an isometry for the metrics defined by ζ and ζ_0. As in the case of Abelian differentials, we consider the moduli space of meromorphic differentials, where $(X, \zeta) \sim (X_0, \zeta_0)$ if there is a biholomorphism $f \colon X \to X_0$ such that $f_* \zeta_0 = \zeta$. It is a complex orbifold, $\dim_{\mathbb{C}}(M_{g,n}(h)) = 3g - 3 + n + g - 1 + \sum_i^n h_i$.

Similarly to the case of holomorphic differentials, moduli spaces of flat surfaces with poles of fixed order may be stratified by the multiplicities of the zeroes. Strata are complex-analytic orbifolds (see Theorem 2.1 in [1]). The complex dimension of such a stratum is $2g + n + m - 2$, where m is the number of zeroes and n is the number of poles. Local coordinates are given by the set of residues, together with absolute and relative periods.

For surfaces of genus one, there are strata of meromorphic differentials with arbitrary number of connected components (see 1.1 in [5]), while for surfaces of higher genera the spaces may have up to three connected components (Theorem 1.2 in [5]). Some special cases of flat surfaces with poles were studied recently, see [5], [6], [12].

Each meromorphic differential ζ on a Riemann surface X determines a homomorphism $\chi : H_1(X \backslash P(\zeta), \mathbb{Z}) \to \mathbb{C}$, where $P(\zeta)$ is the set of poles of ζ. The map is defined as the integral $\chi(\gamma) = \int_\gamma \zeta \in \mathbb{C}$. If we label the zeroes of ζ as $\{z_i\}$ for $i = 1, .., n$ and we choose the z_1 as a base-point, the *relative periods* are then defined as $\int_{z_1}^{z_i} \omega$ for $i = 1, \ldots, n-1$. A foliation on the space, similar to the isoperiodic foliation defined on the spaces of holomorphic differentials is constructed as follows (see [16], [18]).

Denote $T_{g,n}(h)$ the moduli space of objects as in $M_{g,n}(h)$ together with a chosen symplectic basis $\{A_i, B_i\} \in H_1(X, \mathbb{Z})$. The absolute periods along the basis cycles are denoted $\alpha_j = \int_{A_j} \zeta, \beta_j = \int_{B_j} \zeta$, we denote the residues of the 1-form at the poles by ρ_i for $i = 1, \ldots, n$. Then the foliation L on $T_{g,n}(h)$ is defined by the subvarieties $L_{r,a,b}$ given by the prescribed values $\rho_i = r_i, \alpha_j = a_j, \beta_j =$

$b_j, i = 1, \ldots, n; j = 1, \ldots, g$. The leaves of L are permuted by the action of the mapping class group. Hence the family L of the subvarieties $L_{r,a,b}$ for all values of r, a, b defines a complex foliation on the space $M_{g,n}(h)$ (see Lemma 2.4 in [16]). The absolute periods $a_1, \ldots, a_g, b_1, \ldots, b_g$ generate an additive subgroup of $\mathbb{C}$, which, in the case all the periods are incommesurable, has rank 2g; such group is called the group of periods.

2.1 Moduli spaces of real-normalized differentials

Meromorphic differentials such that all their absolute periods are real are referred to as *real-normalized*. Note that in this case all the residues of the differential are necessarily imaginary. Denote by $M_{g,n}^{real}(h)$ the moduli space of genus g curves with n marked points, endowed with a meromorphic real-normalized differential with poles of the fixed orders $h_1, \ldots h_n$ at the marked points. Here, as above, the moduli space means the space of equivalence classes of pairs (X, ψ), a complex curve and a real-normalized differential on it. Two pairs $(X, \psi), (X_0, \psi_0)$ are equivalent if there is a biholomorphism $f : X \to X_0$, s.t. $f_*\psi_0 = \psi$.

The *principal part* of a meromorphic differential at a point P on a Riemann surface X is an equivalence class of meromorphic differentials ζ in a neighborhood of P, with the equivalence $\zeta \sim \zeta'$ if and only if $\zeta - \zeta'$ is holomorphic at P.

The power of the real-normalization is in the uniqueness of a real-normalized differential with prescribed principal parts at the marked points (written out in terms of the jets of local coordinates at the marked points). Thus the real-normalized differentials provide a section of the bundle of meromorphic differentials with prescribed pole orders over the moduli space of curves with marked points, endowed with jets of local coordinates at these points.

Proposition 5. *For any fixed principal parts of poles with purely imaginary residues, whose sum is zero, there exists a unique real-normalized meromorphic differential ζ, having prescribed principal parts at the marked points.*

The foliation L induces a foliation of $M_{g,n}^{real}(h)$. The space $M_{g,n}^{real}(h)$ is a real-analytic orbifold, hence the foliation L on it is real-analytic, however, each individual leaf carries a complex orbifold structure (see [16]). In this work the results refer to meromorphic real-normalized differentials with a single pole of order 2 (and thus with zero residue at the marked point), we denote the corresponding space $R_g = M^{real}_{g,n}(2)$. Stratification on the space of meromorphic differentials by the multiplicities of the zeroes induces stratification on the spaces of real-normalized differentials. Stratification on the moduli spaces of meromorphic differentials by the multiplicities of the zeroes induces a stratification on the space R_g, the strata are denoted $R(\kappa)$, where κ is a partition of $2g$. The foliation L restricted on $R(\kappa)$ is denoted $L(\kappa)$.The restriction of a submanifold $L_{0,a,b}$ on $R(\kappa)$ is denoted $L(G)(\kappa)$, where G – is the group of periods generated by $a_1, \dots, a_g, b_1, \dots, b_g$.

We describe the foliation on the principal stratum in the space of real-normalized differentials. First we review the combinatorial model describing the principal stratum $R(1^{2g})$ in the space of real-normalized differentials R_g, suggested in [20]. An arc diagram is associated to a set of points in $R(1^{2g})$, and the isoperiodic transformations on the points in $R(1^{2g})$ are determined by the transformations over the corresponding arc diagrams, known in the literature as second Vassiliev moves.

Then we extend the model for the stratum $R(2, 1^{2g-2})$. For this we introduce the notion of a generalized arc diagram associated ot a set of points in $R(2, 1^{2g-2})$. The isoperiodic transformations on the stratum are then described in terms of transformations over the generalized arc diagrams called the generalized Vassiliev moves.

2.2 Results

For the principal stratum $R_g(1^{2g})$, where the 1-form has $2g$ zeroes of order one, we obtain the following result.

Theorem 2. *For a group of periods G of rank $2g$, the stratum $L(G)(1^{2g})$ in the space of real-normalized differentials with a single pole of order 2 on curves of genus $g \geq 1$ has a single connected component.*

For the stratum of codimension one in the space of real-normalized differentials with a single pole of order two, all the other poles being simple, we show that the analogous statement holds.

Theorem 3. *For a group of periods G of rank $2g$, the stratum $L(G)(2, 1^{2g-2})$ in the space of real-normalized differentials with a single pole of order 2 on curves of genus $g \geq 2$ has a single connected* *component.*

3 Main results

The thesis is devoted to the study of geometry of moduli spaces of holomorphic and meromorphic differentials on complex algebraic curves and foliations of these moduli spaces. The major results obtained as part of this thesis are stated in the following theorems.

Theorem 1. *Prym eigenform loci $\Omega E_D(4,4)$ are non-empty and have a single connected component for $D \geq 4$.*

Theorem 2. *For a group of periods L of rank $2g$, the stratum $L(G)(1^{2g})$ in the space of real-normalized differentials with a single pole of order 2 on curves of genus $g \geq 1$ has a single connected component.*

Theorem 3. *For a group of periods G of rank $2g$, the stratum $L(G)(2,1^{2g-2})$ in the space of real-normalized differentials with a single pole of order 2 on curves of genus $g \geq 2$ has a single connected component.*

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