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Non-standard models of mathematical physics related to systems of quasi-linear conservation laws

SUMMARY of the thesis

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INTRODUCTION

1. General description of the research area

The proposed thesis examines a number of non-standard mathematical models that can be expressed in the form of a system of conservation laws in the presence of both dissipative terms and source members. Namely, the corresponding systems of partial differential equations have the form

$$\frac{\partial}{\partial t}\mathbf{T}\left(\boldsymbol{U}\left(t,\boldsymbol{x}\right)\right)+\sum_{j=1}^{m}\frac{\partial}{\partial x_j}\left(\boldsymbol{F}_j\left(\boldsymbol{U}\left(t,\boldsymbol{x}\right)\right)\right)=\boldsymbol{D}^2\left(\boldsymbol{U}\left(t,\boldsymbol{x}\right)\right)+\boldsymbol{R}\left(\boldsymbol{U}\left(t,\boldsymbol{x}\right)\right), \qquad (1.1)$$

where $\left(t,\boldsymbol{x}\right)\equiv\left(t,x_1,...,x_m\right)$, $\boldsymbol{U}\left(t,\boldsymbol{x}\right):\mathbb{R}^+\times\mathbb{R}^m\to\mathbb{R}^n$, $\boldsymbol{U}\left(t,x\right)=\left(u_1\left(t,\boldsymbol{x}\right),...,u_n\left(t,\boldsymbol{x}\right)\right)$; $\left(t,\boldsymbol{x}\right)\in\Pi_T\equiv\left\{\left(t,\boldsymbol{x}\right):\left(t,\boldsymbol{x}\right)\in\left[0,T\right]\times\Omega,\Omega\subseteq\mathbb{R}^m\right\}$; $\mathbf{T}=\left(\mathrm{T}_1,...,\mathrm{T}_n\right),\boldsymbol{F}_j=\left(f_{1j},...,f_{nj}\right)$ are sufficiently smooth (at least $\mathbf{T},\boldsymbol{F}_j\in C^1\left(\mathbb{R}^n\right)$) vector functions of variables $\left(u_1,...,u_n\right)$. Further, $\boldsymbol{D}^2\left(\boldsymbol{U}\left(t,\boldsymbol{x}\right)\right)$ is, generally speaking, a nonlinear second-order operator, but $\boldsymbol{R}\left(\boldsymbol{U}\left(t,\boldsymbol{x}\right)\right)$ is nonlinear right-hand side. At the same time, the nonlinearities included in these expressions are also assumed to be smooth. The specific type of expressions of type (1.1) under consideration and the corresponding conditions for their constituent functions will be given in the relevant sections of the work. Here and further, vector quantities will be indicated in bold in the formulas.

The general problems considered in the thesis are mostly related to systems of conservation laws of type (1.1), where the operator $\boldsymbol{D}^2\boldsymbol{U}$, the function $\mathbf{T}$ and the right part of $\boldsymbol{R}$ are absent or have an auxiliary character, changing the specific form of solutions, but retaining their features characteristic of traditional systems of conservation laws

$$\frac{\partial}{\partial t}\boldsymbol{U}\left(t,\boldsymbol{x}\right)+\sum_{j=1}^{m}\frac{\partial}{\partial x_j}\left(\boldsymbol{F}_j\left(\boldsymbol{U}\left(t,\boldsymbol{x}\right)\right)\right)=0. \qquad (1.2)$$

The main property of solutions (1.2) is that even with smooth initial/boundary data, these solutions, in general, turn out to have singularities of different types. The most characteristic are discontinuous solutions. This circumstance makes it difficult to choose the main functional spaces and, accordingly, the need for a sufficiently detailed consideration of particular solutions. The study of quasi-linear hyperbolic systems of the form (1.2) has a long history, but the construction of a general theory has encountered difficulties, which will be discussed in more detail below and which have not yet been overcome. Therefore, the consideration of systems of the type (1.2) requires the search for non-standard approaches. Also, a number of physical processes lead to the consideration of non-standard systems and equations of the form (1.1), the solutions of which reveal specific behavior requiring theoretical understanding. However, it turns out that these solutions of non-standard systems and equations of the form (1.1) have properties characteristic of systems of conservation laws of the type (1.2) and can be naturally considered from the standpoint of the theory of systems of conservation laws. The proposed thesis contains a more specific and detailed study of these problems in the case of one ($m=1$) or two ($m=2$) spatial variables.

The theory of quasi-linear systems of conservation laws in the modern version began to develop in the second half of the last century. However, despite a number of impressive achievements, a fairly complete theory, including the multidimensional case, was built for only one conservation law, see, for example, [Vo], [Kr]. In the case of systems, fairly general results are obtained for only one spatial variable and, as a rule, assuming a small domain of change for at least unknown functions, see, for example, the fundamental works [L1, G, Gl] and a more complete presentation in [B]. Intermediate results of works in the field of conservation laws theory can be found, for example, in books [S1], [S2], [BGS]. A modern presentation of the fundamentals of the theory of conservation laws, including physical applications and numerical methods, can be found, for example, in the monographs [Da], [He], [LTP]. The need to work with, generally speaking, discontinuous functions in constructing the theory of systems of conservation laws led to the use of the concept of a generalized solution, i.e. a solution understood in the sense of satisfying a given integral identity for (1.2) (more on this will be discussed in subsequent sections). However, when trying to move on to the case of not a small, but an arbitrary limited, domain of change of unknown functions, it became necessary to expand the concept of solution and consider stronger singularities than discontinuities. A new concept of measure-valued solutions [DP] has appeared, and along the way it was possible to find a proof of fairly general theorems of the existence of generalized solutions to systems of two conservation laws (one spatial variable) using the principle of compensated compactness based on the vanishing viscosity method. However, the developed technique as a whole could not be extended even to systems of three conservation laws with one spatial variable. Nevertheless, interest in the concept of measure-valued solutions remains due to computational aspects for multidimensional systems of conservation laws, see, for example, [FST], as well as obtaining some additional a priori estimates [S3].

In the late 80s of the last century, it was also discovered that under certain conditions (namely, if Hugoniot's adiabata turns out to be a compact set), even for strictly hyperbolic, genuinly nonlinear systems of two equations with one spatial variable, the traditional solution of the Riemann problem (in the form, in general, of a discontinuous function) does not exist, and the solution, determined on the basis of the vanishing viscosity method, contains a delta-like singularity in the limit, [KK]. At the same time, difficulties arise in determining in what sense an object of the delta function type can satisfy a nonlinear equation. A concentrated presentation of the entire range of issues mentioned can be found, for example, in [K], [Se], see also, for example, [DS]. In addition, other extensions of the concept of a generalized solution have appeared in the Russian literature based on the introduction of specially constructed integral identities and other similar approaches, see, for example, [PSh], [Sh1]. In relatively recent works [MY1], [MY2], the concept of a weak* solution is proposed, which weakens the requirement of measurability and interprets solutions of systems of conservation laws as trajectories in a space conjugate to a suitable basic space, which requires generalization of the integration procedure. It was expected that this concept would be useful in the study of systems with many spatial variables.

However, it seems that these approaches have not led to any satisfactory solution to the accumulated problems in the theory of systems of conservation laws. The current situation was also noted by a well-known expert on systems of conservation laws, Peter Lax, in his book [L2, p. 165]. Apparently, the main tools of the theory of conservation law systems – the construction of a sequence of approximate solutions with further implementation of a limit transition; the vanishing viscosity method – are not sufficient to overcome the difficulties, especially in the case of several spatial variables. In addition, attempts to include delta-like singularities in the main solution space of nonlinear systems based on the construction of certain special integral identities create rather cumbersome constructions with limited possibilities for applying mathematical analysis. In a broader context, an attempt to construct a new (nonlinear) theory of generalized functions was made in [Co]. In this regard, it is important to note by Ya. G. Sinai that the study of cases of degenerate quasi-linear systems of equations may lead to new methodological approaches. From the point of view of the above, the thesis presents a new view on the structure of generalized solutions of quasi-linear systems of conservation laws based on a special representation for discontinuous generalized solutions.

Interest in the consideration of non-standard systems of the type (1.2), which from the point of view described can include systems of the type (1.1), arose both from the side of physical applications and from the side of theoretical research. As we will see below, perturbations or degenerations of the system (1.2) can lead to the use of new methods for constructing a solution, as well as to a better understanding of the nature of the emerging singularities. For greater clarity, we will introduce a number of definitions, and then point out the non-standard models considered in the thesis.

Definition 1.1. Let $m=1$ and the index j of the corresponding variables and functions be omitted. The system (1.2) is called *strictly hyperbolic* if the matrix $\boldsymbol{F}'(\boldsymbol{U})$ has exactly n different real eigenvalues $\lambda_1(\boldsymbol{U})<\ldots<\lambda_n(\boldsymbol{U})$ and, accordingly, a complete set of right $\boldsymbol{r}_1(\boldsymbol{U}),\ldots,\boldsymbol{r}_n(\boldsymbol{U})$ (and left $\boldsymbol{l}_1(\boldsymbol{U}),\ldots,\boldsymbol{l}_n(\boldsymbol{U})$) eigenvectors. The system (1.2) is called *genuinely nonlinear* if $\boldsymbol{r}_i\cdot\nabla\lambda_i\neq 0$, $i=1,\ldots,n$. In case some λ_i coincide for some values of $\boldsymbol{U}$, but the set of eigenvectors remains complete, the system (1.2) is called *non-strictly hyperbolic*.

Definition 1.2. Let $m=1$ and the index j of the corresponding variables and functions be omitted. Let's call the system (1.2) *degenerate non-strictly hyperbolic*, if the matrix $\boldsymbol{F}'(\boldsymbol{U})$ does not have a complete set of eigenvectors, but has the same Jordan form for all values of $\boldsymbol{U}$ under consideration.

For systems from Definition 1.1 it is possible to infer the characteristic form of system (1.2), $m=1$, by multiplying (1.2) from the left by any left eigenvector

$$\boldsymbol{l}_i\cdot\left(\frac{\partial}{\partial t}\boldsymbol{U}(t,x)+\lambda_i(\boldsymbol{U})\frac{\partial}{\partial x}\boldsymbol{U}(t,x)\right)=0. \tag{1.3}$$

There is a large volume of literature devoted to the study of various deviations from the structure of strictly hyperbolic, genuinely nonlinear (in the sense of Definition 1.1) systems of type (1.2). Without going into details, we will give only the book [LF] for illustration and then focus only on specific non-standard models.

The first model that we will consider is a model of the so-called pressureless gas dynamics. The corresponding system of conservation laws in the case of a single spatial variable ($m=1$) has the form

$$\begin{cases}\partial\rho/\partial t+\partial(\rho u)/\partial x=0\\ \partial(\rho u)/\partial t+\partial(\rho u^2)/\partial x=0\end{cases},\qquad x\in\mathbb{R},\quad t\in\mathbb{R}_+. \tag{1.4}$$

Here $\rho>0$ is a density, u is a velocity. The system (1.4) can be formally derived from the system of equations of isentropic gas dynamics, assuming the pressure P to be zero. This system has a single eigenvalue $\lambda=u$ and a single right eigenvector $\boldsymbol{r}=(1,u)$. According to Definition 1.2, the system (1.4) is degenerate non-strictly hyperbolic.

The study of the motion of media in which one can neglect its own pressure drop at a given time (briefly: pressureless media) is of both mathematical and applied interest. From the point of view of applications, pressureless media arise when describing various physical phenomena, such as the evolution of multiphase flows, the movement of dispersed media, in particular dust particles or droplets, the phenomenon of cumulation, the interaction of hypersonic flows in some extreme cases, the movement of granular media, etc., see, for example, [Che], [Se], [St]. From a mathematical point of view, the absence of pressure leads to the appearance of non-classical shock waves, the laws of evolution of which are radically different from, for example, gas-dynamic shock waves. The most interesting effects are obtained for a two-dimensional system of equations of pressureless gas dynamics

$$\begin{cases} \partial\rho / \partial t + \nabla \cdot \rho \boldsymbol{U} = 0 \\ \partial(\rho \boldsymbol{U}) / \partial t + \nabla \cdot (\rho \boldsymbol{U} \otimes \boldsymbol{U}) = 0 \end{cases}, \ \boldsymbol{x} \equiv (x, y), (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^2, \nabla = (\partial / \partial x, \partial / \partial y), \quad (1.5)$$

where $\rho > 0$ has the meaning of the density of matter, $\boldsymbol{U}$ is the velocity vector, and $\otimes$ denotes the tensor product. The system (1.5) can also be derived from the equations of traditional gas dynamics, assuming the pressure to be zero. Here we also note that, generally speaking, the complete system of equations of gas dynamics also includes the equation of conservation of total energy, therefore, putting the pressure equal to zero in the traditional law of conservation, the following equation should be added to the system (1.5)

$$\partial E / \partial t + \nabla \cdot E \boldsymbol{U} = 0, \quad (1.6)$$

where $E \equiv \rho\left(e + |\boldsymbol{U}|^2 / 2\right)$, e is specific internal energy. However, equation (1.6), unlike ordinary gas dynamics, turns out to be independent of the system (1.5) in the sense that the evolution of singularities is determined only by (1.5), and (1.6) determines the law of variation of an additional value e – specific internal energy – "along" already known singularities.

In this thesis, it is shown that non-classical shock waves are measures, generally speaking, on manifolds of different dimensions, the laws of evolution of these measures are obtained, which differ significantly from the Rankine-Hugoniot relations for gas dynamics, and the laws of formation of a hierarchy of singularities (a set of singularities on manifolds of different dimensions) in the two-dimensional case (1.5) are also obtained.

The next model is a two-dimensional system of incompressible Navier-Stokes equations, to which a term of the second time derivative with a small parameter is added. This system of the form (1.1) is written as follows, $\boldsymbol{x} = (x, y)$,

$$\varepsilon \frac{\partial^2 \boldsymbol{U}}{\partial t^2} + \frac{\partial \boldsymbol{U}}{\partial t} + \operatorname{div}(\boldsymbol{U} \otimes \boldsymbol{U}) + \nabla P = \Delta \boldsymbol{U} + g, \operatorname{div} \boldsymbol{U} = 0, \quad \boldsymbol{x} \in \mathbb{R}^2, \quad t \in \mathbb{R}_+, \quad (1.7)$$

here $\varepsilon > 0$ is a small parameter, P is a pressure, Δ is the Laplacian, g is an external force. The system (1.7) can be obtained from the kinetic Boltzmann equation by taking into account additional terms after the averaging operation and is defined in [El1], [El2] as a quasi-gas-dynamic system of equations. The same system of equations can be obtained by introducing a relaxation mechanism into the Euler equations, see, for example, [BNP], in fact (1.7) represents the so-called hyperbolization of the Navier-Stokes system of equations. The system (1.7) is used for an alternative description of gas dynamic flows, in particular, flows of rarefied gas, and its modifications arise when describing viscoplastic phenomena [CK]. In the thesis for a system of equations of the form (1.7), it is shown that a strong solution to the initial boundary value problem in a bounded domain exists for sufficiently small ε and bounded initial data energy. For the one-dimensional version of the system (1.7), an example is given that even for small ε the solution explodes in a finite time if the initial energy is large enough.

Another model is a one-dimensional system of equations for compressible two-phase multicomponent filtration, $x \in \mathbb{R}, t \in \mathbb{R}_+$,

$$\frac{\partial}{\partial t}\left(\phi\left(x_{iG}\rho_G s + x_{iL}\rho_L(1-s)\right)\right) + \frac{\partial}{\partial x}\left(x_{iG}\rho_G V_G + x_{iL}\rho_L V_L\right) = 0, \quad i = 1, \ldots, N, \tag{1.8}$$

where N is the number of components in the mixture, $\phi > 0$ is the porosity, $x_{iG} > 0$, $x_{iL} > 0$ are thermodynamic equilibrium constants (i.e. they represent the molar concentrations of the component i) for the gas (G) and liquid (L) phases, respectively, $\rho_G > 0$, $\rho_L > 0$ are the densities of the gas and liquid phases, respectively, $0 < s < 1$ is the saturation of the gas phase. In this case, the filtration rates of the gas V_G and liquid V_L phases have the form

$$V_G = -\frac{Kk_{rG}}{\mu_G}\frac{\partial P}{\partial x}, \quad V_L = -\frac{Kk_{rL}}{\mu_L}\frac{\partial P}{\partial x}, \tag{1.9}$$

where $K > 0$ is the so-called absolute permeability of the rock, $k_{rG} \geq 0$, $k_{rL} \geq 0$, $k_{rG} + k_{rL} > 0$ are relative permeabilities of gas and liquid phases, $\mu_G > 0$, $\mu_L > 0$ represent the viscosities of the gas and liquid phases, respectively, $P > 0$ denotes the pressure. Taking into account the relations of thermodynamics of multicomponent mixtures, the system of equations (1.8), (1.9) can be written as (1.1), but without second-order operators, and can be considered as a system of conservation laws. Mathematical models of filtration are well studied in the case of one component and one phase, when the unknown function is only pressure (or density), see, for example, the book [BER]. In this case, the situation is modeled using degenerate parabolic equations, see, for example, the review [Ka]. In the case of many components, there is an extensive literature devoted to the incompressible case. In this situation, the system of equations of multicomponent filtration (in the case of a single spatial variable and assuming a constant filtration rate) can be transformed into a non-strictly hyperbolic system of conservation laws, see [Or]. It is worth emphasizing that the property of non-strict hyperbolicity significantly complicates the study, see an article [KM] devoted to model systems of equations (but some of which have a physical origin, in particular, comes from the description of filtration processes) and illustrates different sides of the mathematical concept of non-strict hyperbolicity. When taking into account the compressibility property, as in the system (1.8), (1.9), the property of hyperbolicity, generally speaking, is lost, it is also not parabolic, and the question arises from which positions to consider this system of equations. The thesis shows that if we approach this system of equations as a degenerate system of conservation laws, then we can come to a natural concept of characteristics, formulate the concept of a generalized solution and apply methods developed in the theory of conservation laws, for example, consider the Riemann problem. However, all these concepts acquire specific features that are not typical for systems of hyperbolic equations.

Finally, the last model is an equation of the form (1.1) with nonlinear viscosity and a so-called bounded dissipation flow

$$\partial u / \partial t + \partial f(u) / \partial x = \partial Q(\partial u / \partial x) / \partial x, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+, \tag{1.10}$$

here $f \in C^1(\mathbb{R}), Q \in C^2(\mathbb{R})$, $f(0) = Q(0) = 0$, $Q' > 0$ and $Q(\pm\infty) = \text{const}$. Due to these properties, at large $\partial u / \partial x$, equation (1.10), which has the form (1.1), becomes close to the law of conservation of form (1.2) at $m = n = 1$. Equation (1.10) is a modification of the Burgers equation and is a highly degenerate parabolic equation that admits discontinuous solutions of a special kind. In the context of degenerate parabolic equations, as a rule, only continuous solutions have been studied, see, for example, [Ka]. From a physical point of view, the appearance of equations of type (1.10) is described, for example, in [Ro] and is associated with a special regularization of the Chapman-Enskog decomposition, as well as in [BBP] in relation to the theory of special turbulent flows. Equation (1.10) is the simplest model describing the interaction between nonlinear transport and nonlinear dissipative processes. The thesis examines the issues of the formulation of the concept of a generalized solution, the conditions of its existence and uniqueness.

2. List of the main results of the thesis submitted for defense

The main results submitted for defense are contained in the paragraphs below. A complete and mathematically rigorous description of the results, including all the necessary concepts and definitions, will be set out in section 3.

1) A number of new representations of generalized solutions of systems of quasi-linear conservation laws have been found, which we will define by the term *variational representation*, namely:

a) in the case of a single spatial variable $m=1$ and a system of n conservation laws, a functional $\boldsymbol{J}$ associated with a generalized solution $\boldsymbol{U}(t,x)$ and defined on trajectories $\boldsymbol{w} \equiv (w,x) = (\omega(t),\chi(t))$ in space $(t,\boldsymbol{w})$ is found; here w is one of the Riemann invariants, such that the fulfillment of equality $\delta \boldsymbol{J} = 0$ along a certain path $\boldsymbol{w} = (\bar{\omega}(t),\bar{\chi}(t))$ implies the fulfillment of characteristic relations at points of smoothness of $(\bar{\omega}(t),\bar{\chi}(t))$ and Rankine–Hugoniot relations on breaks; in addition, if two trajectories $x=\bar{\chi}_1(t)$, $x=\bar{\chi}_2(t)$ come to a certain curve $x=X(t)$ and $\delta \boldsymbol{J}_{x=\bar{\chi}_1(t)} = \delta \boldsymbol{J}_{x=\bar{\chi}_2(t)} = 0$, $\boldsymbol{J}_{x=\bar{\chi}_1(t)} = \boldsymbol{J}_{x=\bar{\chi}_2(t)}$ then the Rankine-Hugoniot relations also hold on the curve $x=X(t)$; this result is a generalization of the results obtained by E. By Hopf for one equation, see [H];

b) in the case of one and two spatial variables $m=1,2$ and a system of n conservation laws, a functional $\boldsymbol{J}$ is found, which is associated with a generalized solution $\boldsymbol{U}(t,\boldsymbol{x})$, $\boldsymbol{x}=(x,y)$ for $m=2$ and $\boldsymbol{x}=x$ for $m=1$, and defined on trajectories/surfaces in space $(t,\boldsymbol{x})$, such that the fulfillment of the relation $\delta \boldsymbol{J} = 0$ along any trajectory/surface $\boldsymbol{x} = (\bar{\chi}(t,s),\bar{\gamma}(t,s))$, in the case of a surface s is a parameter along it, entails the validity of the system (1.2) in the classical sense if $\boldsymbol{U}(t,\boldsymbol{x})$ is a smooth function and the Rankine-Hugoniot relations, if there is a discontinuity;

c) in the case of one spatial variable $m=1$ and a system of n conservation laws, a functional $\mathcal{L}$ associated with a generalized solution $\boldsymbol{U}(t,x)$ is found such that the generalized solution $\boldsymbol{U}(t,x)$ is expressed in terms of a minimum of $\mathcal{L}$ in some Banach space; the property of minimizing some functional can be used as an alternative definition of the concept of a generalized solution of the system (1.2).

2) For a system of equations of pressureless gas dynamics:

a) in the case of a single spatial variable for the system (1.4), as well as for its generalization, including an external force, a definition of the concept of a generalized solution in the Radon measures space is proposed, the theorem of the existence of a generalized solution for either a continuous or completely discrete distribution of matter is proved; a variational representation of the generalized solution, which looks like a characteristic of continuity, is also found. Namely, in the case of system (1.4) we have: a point (t,x), $x = a + tu_0(a)$ is a point of absolute continuity of the measure of mass of matter $P_t(dx)$ and continuity of velocity $u(t,x)$ if and only if the following relation is fulfilled for any $a^+, a^-, a^- < a < a^+$

$$\frac{\int\limits_{[a^-,a)} (s + tu_0(s)) P_0(ds)}{\int\limits_{[a^-,a)} P_0(ds)} < \frac{\int\limits_{[a,a^+)} (s + tu_0(s)) P_0(ds)}{\int\limits_{[a^-,a)} P_0(ds)},$$

where $P_0(da), u_0(a)$ are the initial distribution of matter and the initial velocity;

b) in the case of two spatial variables, a system of equations is found describing the evolution of strong singularities along surfaces $\Gamma \equiv (t, \chi(t,l), \gamma(t,l))$, which differs in type from the traditional Rankine-Hugoniot relations

$$\begin{cases} \dfrac{\partial P}{\partial t} = \dfrac{\partial \chi}{\partial l}\{V[\rho] - [\rho v]\} - \dfrac{\partial \gamma}{\partial l}\{U[\rho] - [\rho u]\} \\ \dfrac{\partial \boldsymbol{I}}{\partial t} = \dfrac{\partial \chi}{\partial l}\{V[\rho \boldsymbol{u}] - [\rho v \boldsymbol{u}]\} - \dfrac{\partial \gamma}{\partial l}\{U[\rho \boldsymbol{u}] - [\rho u \boldsymbol{u}]\}, \\ \dfrac{\partial(\chi,\gamma)}{\partial t} = \boldsymbol{U} \equiv (U,V) \end{cases}$$

where $\boldsymbol{I} = P\boldsymbol{U}$ and for any value f it is indicated: $f^{\pm}$ – the value of f on both sides of Γ and $[f] \equiv f^+ - f^-$;

the emergence of a hierarchy of strong singularities, i.e. a system of strong singularities arising on manifolds of different dimensions, is proved, including the Rankine-Hugoniot relations for strong singularities along curves;

an approximate dynamics of adhesion in the two-dimensional case is constructed and estimates of the degree of deviation from the weak solution for a discrete particle system are obtained;

a variational description of generalized solutions is obtained, which differs significantly from the one-dimensional version and is associated with the use of vector functionals on domains G

$$\boldsymbol{F}(t,\boldsymbol{x};G) \equiv \iint_G \left[\boldsymbol{u}_0(\boldsymbol{a}) - \frac{\boldsymbol{x}-\boldsymbol{a}}{t} \right] \rho_0(\boldsymbol{a})\, d\boldsymbol{a}.$$

3) For the hyperbolization of Navier-Stokes equations (1.7) in the case of two spatial variables, a theorem for the existence of a strong solution with a small hyperbolization parameter and bounded initial energy is obtained in the space

$$\mathrm{E}_\varepsilon^1 \equiv \left\{ \{\boldsymbol{U},\boldsymbol{V}\} \in \left[H^2(\Omega) \cap H_0^1(\Omega) \right]^2 \times \left[H_0^1(\Omega) \right]^2, \nabla \cdot \boldsymbol{U} = 0, \nabla \cdot \boldsymbol{V} = 0 \right\}$$

with the norm

$$\|\mathfrak{A}\|_{\mathrm{E}_\varepsilon^1}^2 \equiv \varepsilon \left\| \partial \boldsymbol{U}(t) / \partial t \right\|_{H^1}^2 + \left\| \partial \boldsymbol{U}(t) / \partial t \right\|_{L^2}^2 + \left\| \boldsymbol{U}(t) \right\|_{H^2}^2,$$

where $\mathfrak{A}(t) = \{ \boldsymbol{U}(t), \partial \boldsymbol{U}(t) / \partial t \}$; an example of the destruction of a strong solution in a finite time is given when these conditions are violated , even in the one-dimensional case.

4) For a one-dimensional system of equations of compressible two-phase multicomponent filtration (1.8), (1.9), such a reformulation of the basic equations is given, which allows the use of methods of the theory of systems of conservation laws. The resulting system of equations is defined as almost hyperbolic, having the properties of both hyperbolic and parabolic systems of equations. For this system of equations, the Rankine-Hugoniot relation is derived and the theorem on the structure of the set of shocks is proved. In the case of two components, the structure of the solution to the Riemann problem is obtained, the properties of which differ significantly from traditional systems of conservation laws. So, for example, it turns out that the solution to the Riemann problem is always discontinuous and, in addition, there is an infinite velocity of propagation of perturbations. In the incompressible case, an expression is found for all entropy-flux pairs, which turns out to be significantly richer than for a single scalar conservation law.

5) For an equation with a bounded dissipation flux (1.10), the concept of a generalized solution is formulated, taking into account the possibility of discontinuities in, generally speaking, a parabolic equation. For initial functions from space $W^{2,1}(\mathbb{R})$ that are piecewise smooth on a compact set with a finite set of discontinuity points, the existence theorem of a generalized solution is proved. The uniqueness theorem of a generalized solution satisfying the Oleinik E condition and the condition of continuity of the dissipation flux is proved for a rather

narrow, although reflecting all the features of the problem, class of functions that are piecewise $C^2\left(\left(0,T\right)\times\mathbb{R}\right)$ with a finite number of discontinuity lines belonging to $C^1\left(0,T\right)$.

3. Detailed presentation of the results obtained

In this section, the results announced in section 2 will be presented at the mathematical level of rigor, including all definitions and formulations. Additional literature will also be provided characterizing the contribution to the problem that the thesis has made.

3.1 Variational representation for generalized solutions of systems of quasi-linear conservation laws

Let us first introduce the necessary concepts and definitions. Consider the Cauchy problem for a system of quasi-linear equations of general form, see also (1.2),

$$\frac{\partial}{\partial t}\boldsymbol{U}\left(t,\boldsymbol{x}\right)+\sum_{j=1}^{m}\frac{\partial}{\partial x_j}\left(\boldsymbol{F}_j\left(\boldsymbol{U}\left(t,\boldsymbol{x}\right)\right)\right)=0 \quad , \quad \boldsymbol{U}\left(0,\boldsymbol{x}\right)=\boldsymbol{U}_0\left(\boldsymbol{x}\right), \qquad (3.1.1)$$

where $\left(t,\boldsymbol{x}\right)\in\Pi_T\equiv\left\{\left(t,\boldsymbol{x}\right):\left(t,\boldsymbol{x}\right)\in\left[0,T\right]\times\mathbb{R}^m\right\}$, $\boldsymbol{U}\left(t,x\right)=\left(u_1\left(t,\boldsymbol{x}\right),...,u_n\left(t,\boldsymbol{x}\right)\right)$, $\left(t,\boldsymbol{x}\right)\equiv\left(t,x_1,...,x_m\right)$, $\boldsymbol{U}\left(t,\boldsymbol{x}\right):\mathbb{R}^+\times\mathbb{R}^m\rightarrow\mathbb{R}^n$, a $\boldsymbol{F}_j=\left(f_{1j},...,f_{nj}\right)$ are sufficiently smooth (at least $\boldsymbol{F}_j\in C^1\left(\mathbb{R}^n\right)$) vector functions of variables $\left(u_1,...,u_n\right)$. Multidimensional integrals over spaces of many variables and over surfaces in these spaces will be written as single or double integrals to shorten the record, if this does not lead to ambiguity. For example, writing $\iint\limits_{\Pi_T}\left[...\right]dxdt$ would mean the integration with respect to the Lebesgue measure $dxdt$ in the space $\mathbb{R}^+\times\mathbb{R}^m$. Individual exceptions will be specifically specified.

The solutions of the system (3.1.1), which take the given initial values, are understood in a generalized sense in accordance with the following Definition 3.1.1.

Definition 3.1.1. Let $\boldsymbol{U}_0\left(\boldsymbol{x}\right)\in\mathbb{R}^n$ be a locally bounded measurable function. Then we call a locally bounded measurable in Π_T function $\boldsymbol{U}\left(t,\boldsymbol{x}\right)$ *a generalized solution* of the problem (3.1.1) if for any test function $\varphi\in C^\infty\left(\left[0,T\right)\times\mathbb{R}^m\right)$, $\varphi\left(t,\cdot\right)\in C_0^\infty\left(\mathbb{R}^m\right)$ for a fixed $t\in\left[0,T\right]$, $\varphi\equiv 0$ at $T_1\le t\le T$, $T_1<T$, the following integral identity is fulfilled

$$\iint_{\Pi_T}\left[\boldsymbol{U}\frac{\partial\varphi}{\partial t}+\sum_{j=1}^{m}\boldsymbol{F}_j\left(\boldsymbol{U}\right)\frac{\partial\varphi}{\partial x_j}\right]d\boldsymbol{x}dt+\int_{\mathbb{R}^m}\boldsymbol{U}_0\varphi\left(0,\boldsymbol{x}\right)d\boldsymbol{x}=0. \qquad (3.1.2)$$

If $\boldsymbol{U}\left(t,\boldsymbol{x}\right)$ is a continuously differentiable function, then it is easy to see the equivalence of formulations (3.1.1) and (3.1.2). Let $\boldsymbol{U}\left(t,\boldsymbol{x}\right)$ be a continuously differentiable function outside some hypersurface $\Omega\subset\mathbb{R}_+\times\mathbb{R}^m$ of codimension 1 having a continuous normal vector $\left(n_0,n_1,...,n_m\right)$. Let there be a discontinuity of $\boldsymbol{U}\left(t,\boldsymbol{x}\right)$ on this hypersurface, and the values $\boldsymbol{U}^{\pm}=\boldsymbol{U}\left(t,\boldsymbol{x}\pm 0\right)$ are determined along Ω. Then (3.1.2) will be fulfilled if the equation (3.1.1) is valid in the smoothness regions of $\boldsymbol{U}\left(t,\boldsymbol{x}\right)$, and along Ω the so-called Rankine-Hugoniot relation is true

$$\left(\boldsymbol{U}^{-}-\boldsymbol{U}^{+}\right)n_0+\sum_{j=1}^{m}\left(\boldsymbol{F}_j\left(\boldsymbol{U}^{-}\right)-\boldsymbol{F}_j\left(\boldsymbol{U}^{+}\right)\right)n_j=0. \qquad (3.1.3)$$

On the other hand, it is well known that within the definition 3.1.1, the generalized solution of the problem (3.1.1) is not unique. Therefore, for the system (3.1.1), additional concepts of entropy and entropy solution are introduced as follows.

Definition 3.1.2. Let's call a convex positive function $\eta\left(\boldsymbol{U}\right)\in C^1\left(\mathbb{R}^n\right)$ *entropy* for the system (3.1.1) if an additional conservation law holds for classical solutions to (3.1.1)

$$\frac{\partial}{\partial t}\eta\left(\boldsymbol{U}\left(t,\boldsymbol{x}\right)\right)+\sum_{j=1}^{m}\frac{\partial}{\partial x_j}\left(q_j\left(\boldsymbol{U}\left(t,\boldsymbol{x}\right)\right)\right)=0 \qquad (3.1.4)$$

with some sufficiently smooth flux functions $q_j\left(u_1,...,u_n\right)$.

Definition 3.1.3. A function $\boldsymbol{U}\left(t,x\right)$ that is a generalized solution to the problem (3.1.1) in the sense of definition 3.1.1, is called *an entropy solution* of the system (3.1.1) if for each entropy $\eta\left(\boldsymbol{U}\right)$ from definition 3.1.2 and the test function $\varphi\left(t,\boldsymbol{x}\right)\geq 0$ from definition 3.1.1 the following inequality is fulfilled

$$\iint_{\Pi_T}\left[\eta\left(\boldsymbol{U}\right)\frac{\partial\varphi}{\partial t}+\sum_{j=1}^{m}q_j\left(\boldsymbol{U}\right)\frac{\partial\varphi}{\partial x_j}\right]d\boldsymbol{x}dt+\int_{\mathbb{R}^m}\eta\left(\boldsymbol{U}_0\right)\varphi\left(0,\boldsymbol{x}\right)d\boldsymbol{x}\geq 0. \qquad (3.1.5)$$

In the case of a piecewise smooth function $\boldsymbol{U}\left(t,x\right)$, which is an entropy solution to (3.1.1), an additional to (3.1.3) relation will be true on the discontinuity hypersurfaces Ω (with appropriate determination of the positive and negative sides of Ω)

$$\left(\eta\left(\boldsymbol{U}^{-}\right)-\eta\left(\boldsymbol{U}^{+}\right)\right)n_0+\sum_{j=1}^{m}\left(q_j\left(\boldsymbol{U}^{-}\right)-q_j\left(\boldsymbol{U}^{+}\right)\right)n_j\geq 0. \qquad (3.1.6)$$

The question of the existence and uniqueness of generalized entropy solutions for (3.1.1) is currently solved only for special cases, despite great efforts, see section 1. Within the framework of section 3.1, the thesis presents a new view on the structure of generalized solutions of quasi-linear systems of conservation laws based on a special representation for discontinuous generalized solutions.

3.1.1 On the generalization of E. Hopf's results to systems of quasi-linear conservation laws in the case of a single spatial variable

The variational representation for generalized solutions in the one-dimensional case is known for a single equation. In the paper [H] for the Cauchy problem

$$u_t+\left(u^2/2\right)_x=0,\ (t,x)\in\mathbb{R}_+\times\mathbb{R},\ u(0,x)=u_0(x), \qquad (3.1.7)$$

where $u_0(x)$ was assumed to be integrable at each finite interval with some natural restriction of growth at infinity, the following representation was obtained for a generalized solution

$$u(t,x)=\left(x-y(t,x)\right)/t;\ y(t,x)=\min_y F(t,x,y)\equiv\min_y\left[\int_0^y u_0(s)ds+\frac{(x-y)^2}{2t}\right], \qquad (3.1.8)$$

where $\min_y$ implies a global minimum. If there is one such minimum, then (t,x) is a point of continuity of $u(t,x)$, and if there are several such points, then (t,x) is a discontinuity point. It is important that the resulting set of discontinuity points automatically satisfies the Rankine-Hugoniot relations (3.1.3) for equation (3.1.7). Further, formulas of the type (3.1.8) were generalized to the case of a more general flux $\varphi(t,x,u)$ instead $u^2/2$, see [L3], [O]. In papers [1], see also short version in [ERS], the formulas (3.1.8) were generalized to a degenerate non-strict, see definition 1.2, hyperbolic system (1.3). An analogue of the representation (3.1.8) for general systems was not proposed, this gap was filled in the thesis, the results were published in [2].

Consider the system of conservation laws (3.1.1). Let $m=1$ and the index j of the corresponding variables and functions be omitted. Let this system be strictly hyperbolic, see Definition 1.1.

Definition 3.1.4. *The Riemann invariant* for the system (3.1.1) in the case $m=1$ is called a function $w(u_1,...,u_n)$ such that $\nabla w=\boldsymbol{l}$, where $\boldsymbol{l}$ is any left eigenvector of the system (3.1.1).

The system (3.1.1), $m=1$, can have no more than n Riemann invariants. Let $IR\in\{1,...,n\}, IR\neq\varnothing$ be such a set of indices that there are Riemann invariants $w_k(u_1,...,u_n)$ for (3.1.1), $m=1$, i.e. $\nabla w_k=\boldsymbol{l}_k$, $k\in IR$. For the rest of the set of indices $O\in\{1,...,n\}$ we define additional functions $p_k(u_1,...,u_n)$ such that $\nabla p_k\times\boldsymbol{l}_k\neq 0$, $k\in O$, and the combination w_k , $k\in IR$ and p_k , $k\in O$ represents a non-degenerate change of variables in the phase space. For convenience, we introduce the notation $\omega_k=w_k, k\in IR$ and $\omega_k=p_k, k\in O$.

Now let's consider special classes for functions $\boldsymbol{U}(t,x)$ and trajectories $\chi(t)$ that are close to the corresponding classes in [O].

Definition 3.1.5. Let $x=\chi(t)\subset\Pi_T$, $0<t<T$. Let's call the path $x=\chi(t)$ *belonging to the class* Γ, if the following conditions are met. Let there be a finite set of points $\{t_i\}\in(0,T), i=1,...,N$ (the set of points for each path is its own, but the maximum number of points for all considered paths is the same) such that $\chi(t)\in C^1((t_{i-1},t_i)); i=1,...,N+1; t_0=0, t_{N+1}=T$ and $\chi(t)\in C([0,T])$. In addition, assume that under small transformations of $\chi(t)$ the points $\{(t_i,\chi(t_i))\}\in(0,T)\times\mathbb{R}, i=1,...,N$ form piecewise continuously differentiable curves $x=s_i(t)$ (which, in fact, are discontinuity lines of $\boldsymbol{U}(t,x)$). Now let's consider piecewise continuously differentiable functions $\boldsymbol{U}(t,x)$, we will say that $\boldsymbol{U}(t,x)$ *belongs to the class* K if for any path $x=\chi(t)$ from the class Γ the following is true. There is a function $\boldsymbol{U}(t)\equiv\boldsymbol{U}(t,\chi(t))$, $\boldsymbol{U}(t)\in C^1((t_{i-1},t_i)); i=1,...,N+1$, and there exist onesided limits $\boldsymbol{U}(t_i\pm 0), i=1,...,N$.

Further, variables can be changed by $(u_1,...,u_n)\to(\omega_1,...,\omega_n)$. Let's choose some $k_0\in IR$, in the next theorem, when considering variation $\delta\boldsymbol{U}$, only ω_{k_0} and x will be considered as varied, the remaining variables $\omega_k, k\neq k_0$ will be considered as functions of t,x

that need to be defined along with the trajectories $\omega_{k_0}(t), \chi(t)$. The corresponding variation is denoted as δ_{k_0}.

We fix some $0 < t < T$. Let's define a vector functional $\boldsymbol{J}$, defined on a set of trajectories $\chi(\tau), \boldsymbol{U}(\tau)$,

$$\boldsymbol{J} \equiv \int_0^y \boldsymbol{U}_0(s)ds + \int_0^t \boldsymbol{L}(\dot{\chi}, \boldsymbol{U})d\tau\,; \boldsymbol{L}(\dot{\chi}, \boldsymbol{U}) \equiv \boldsymbol{U}\dot{\chi} - \boldsymbol{F}(\boldsymbol{U});\, \chi(0) = y, \chi(t) = x. \qquad (3.1.9)$$

Theorem 3.1.1. Fix an arbitrary $k_0 \in IR$. Let the functions $\omega_k, k \in O$ satisfy the Liouville equation $\boldsymbol{r}_{k_0} \cdot \nabla \omega_k = 0$. Then for functions $\boldsymbol{U}(t,x)$ from the class K the fulfillment of equality $\delta_{k_0}\boldsymbol{J} = 0$ along a certain path $x = \chi(\tau) \in \Gamma$ entails the fulfillment of the characteristic relations (1.3) at the points of smoothness of $\chi(\tau)$, where the curve $x = \chi(\tau)$ turns out to be a characteristic, and the Rankine-Hugoniot relations (3.1.3) on the breaks.

We will now assume that $\omega_k, k \neq k_0$ are fixed and characteristic relations (1.3) are fulfilled. Let's construct a trajectory connecting the points $(0, y)$ and (t, x) with the aid of relations $\dot{\omega}_{k_0} = 0$, $\dot{\chi} = \lambda_{k_0}(\boldsymbol{U})$. At this the trajectory can have breaks at points $(\tau_i, \chi(\tau_i)), i = 1,...,N$ that we will consider lying on fixed discontinuity curves $\chi = s_i(\tau)$. Then $\boldsymbol{J}$ from (3.1.9) turns from a functional into a function of variables $\{y, \tau_i\}, i = 1,...,N$. We will write $\boldsymbol{J} = \boldsymbol{J}(y, \tau_1,...,\tau_N; t, x) \equiv \boldsymbol{J}(y, \tau_i; t, x)$.

Theorem 3.1.2. For functions $\boldsymbol{U}(t,x)$ belonging to the class K, the vector function $\boldsymbol{J}$ constructed above satisfies the equalities $\partial \boldsymbol{J} / \partial \tau_i = 0, i \neq 0$ and $\partial \boldsymbol{J} / \partial y = \boldsymbol{U}_0(y) - \boldsymbol{U}(0, y)$. In addition, if for each point of a curve $x = X(t)$ for some k_0 there are two trajectories connecting the points $(0, y_1)$, $(0, y_2)$ and $(t, X(t))$, and, besides that, $\boldsymbol{J}\left(y_1, \tau_i^{(1)}; t, X(t)\right) = \boldsymbol{J}\left(y_2, \tau_i^{(2)}; t, X(t)\right)$, then the curve $x = X(t)$ satisfies the Rankine-Hugoniot relations (3.1.3).

3.1.2 Variational representation for generalized solutions of systems of quasi-linear conservation laws in the case of one and two spatial variables

As far as the author knows, the results presented below are of an original nature. Unlike second-order equations, the variational approach for first-order equations has not been formulated. Some thoughts in this direction were expressed by E. Tadmor, [Ta]. The results obtained are published in [2], see also additional work [R1].

Consider successively the one-dimensional, $m=1$, and two-dimensional, $m=2$, case.

Let $G=[0,T]\times[-X,X]\subset\Pi_T$, where X is some positive number. Let us consider a class $\bar{K}$ of functions $\boldsymbol{U}(t,x)$ in G that are piecewise doubly continuously differentiable with a finite number of piecewise continuously differentiable discontinuity lines, see also [O]. Let's also consider the trajectory space $\chi(t)\in C_X^1\equiv C^1([0,T],[-X,X])$.

Definition 3.1.6. Let's call a certain set of trajectories $\Gamma\subset C_X^1$ *acceptable* for $\boldsymbol{U}(t,x)\in\bar{K}$ if for any point $(t,x)\in[0,T]\times[-X,X]$ there exists the single trajectory $\chi(\tau)\in\Gamma$ such that $\chi(t)=x$, as well as for any $\chi(\tau)\in\Gamma$ and every discontinuity line $s(\tau)$ of the function $\boldsymbol{U}(t,x)$ there are only a finite number of points of their intersection, and in case $\chi(\tau_0)=s(\tau_0)$ the relation $\dot{\chi}(\tau_0)\neq\dot{s}(\tau_0)$ is also true.

Consider a vector functional $\bar{\boldsymbol{J}}$ similar to (3.1.9), and such that $\bar{\boldsymbol{J}}:\chi(\tau)\in C_X^1\to\mathbb{R}^n$,

$$\bar{\boldsymbol{J}}\equiv\int_0^t \boldsymbol{L}(\dot{\chi},\boldsymbol{U})\,d\tau\,;\;\boldsymbol{L}(\dot{\chi},\boldsymbol{U})\equiv\boldsymbol{U}\dot{\chi}-\boldsymbol{F}(\boldsymbol{U})\,;\;\chi(0)=y,\chi(t)=x. \qquad (3.1.10)$$

Theorem 3.1.3. Let $\boldsymbol{U}(t,x)\in\bar{K}$ and let for some acceptable trajectory $x=\bar{\chi}(\tau)$ the equality $\delta\bar{\boldsymbol{J}}=0$ is fulfilled. Then, at those points $\bar{\chi}$ where $\boldsymbol{U}(t,x)$ is smooth, equation (3.1.1) is fulfilled in the classical sense, and at the points of intersection of $\bar{\chi}$ and discontinuity line of the function $\boldsymbol{U}(t,x)$ the Rankine-Hugoniot relations (3.1.3) are true. In addition, the expression for $\delta^2\bar{\boldsymbol{J}}$ on the trajectory $x=\bar{\chi}(\tau)$ where $\delta\bar{\boldsymbol{J}}=0$, contains only the terms depending on $(\delta\chi)^2$.

Let's now $m=2$, denote $\boldsymbol{F}_1(\boldsymbol{U})\equiv\boldsymbol{F}(\boldsymbol{U}),\boldsymbol{F}_2(\boldsymbol{U})\equiv\boldsymbol{G}(\boldsymbol{U})$. Let $G=[0,T]\times[-X,X]\times[-Y,Y]\subset\Pi_T$, where X,Y are some positive numbers. Let us consider a class $\bar{K}$ of functions $\boldsymbol{U}(t,x,y)$ in G that are piecewise doubly continuously differentiable, with one (for simplicity) continuously differentiable discontinuity surface Ω. Consider a set of surfaces $\boldsymbol{S} \equiv \{\tau, \chi(\tau,s), \gamma(\tau,s)\}$, s is the internal parameter of the surface, τ is the time parameter, $\chi(\tau,s), \gamma(\tau,s) \in C^1_{X,Y} \equiv C^1([0,T]\times[0,1],[-X,X]\times[-Y,Y])$. Let the discontinuity surface Ω of the function $\boldsymbol{U}(t,x,y)$ be given by the equations $t=\tau, x=\varphi(\tau,s), y=\psi(\tau,s)$.

Definition 3.1.7. Let's call a certain set of surfaces $\Gamma \subset C^1_{X,Y}$ *acceptable* for $\boldsymbol{U}(t,x,y) \in \bar{K}$ if for any point $(t,x,y) \in [0,T]\times[-X,X]\times[-Y,Y]$ there exist a single surface $S \in \Gamma$ and a value s_0 such that $\chi(t,s_0)=x, \gamma(t,s_0)=y$, and the set $S \cap \Omega$ consists of a finite number of sufficiently smooth lines, moreover on this set the following relation is true $\psi_s(\varphi_\tau - \chi_\tau) - \varphi_s(\psi_\tau - \gamma_\tau) \neq 0$.

Consider an analogue $\bar{\boldsymbol{J}}$ of the vector functional (3.1.10) such that $\bar{\boldsymbol{J}} : S(\tau,s) \in C^1_{X,Y} \to \mathbb{R}^n$,

$$\bar{\boldsymbol{J}} \equiv \iint_S \boldsymbol{U} dx \wedge dy + \boldsymbol{F}(\boldsymbol{U}) dy \wedge dt + \boldsymbol{G}(\boldsymbol{U}) dt \wedge dx, \tag{3.1.11}$$

here the sign $\wedge$ denotes an exterior product.

Theorem 3.1.4. Let $\boldsymbol{U}(t,x,y) \in \bar{K}$ and let the equality $\delta\bar{\boldsymbol{J}} = 0$ is fulfilled on some acceptable surface $\bar{S}$. Then at those points of $\bar{S}$ where $\boldsymbol{U}(t,x,y)$ is smooth, the equation (3.1.1) is fulfilled in the classical sense, and at the points of intersection of $\bar{S}$ and discontinuity surface of the function $\boldsymbol{U}(t,x,y)$ the Rankine-Hugoniot relations (3.1.3) are true.

Expression (3.1.11) is essentially a differential form, therefore, generally speaking, the generalization of this entry to the multidimensional case is not difficult.

3.1.3 Representation of a generalized solution of system of quasi-linear conservation laws in the case of one spatial variable as a minimum of some functional

The content of this paragraph is a development of the previous one. Again, as far as the author knows, the results presented below are of an original nature. They are published in [3], see also the additional work [R1].

In the case of one spatial variable, $m=1$, it is possible to obtain another form of representation of generalized solutions using the functional minimization procedure. Within the framework of this paragraph, we will use the concepts and notations introduced in section 3.1.2. We will denote $\boldsymbol{V}(t,x) \equiv \int \boldsymbol{U}(t,p)dp$. Let's introduce a vector functional of the form (3.1.10)

$$\mathbf{J}_{\alpha} \equiv \int_0^T \left[\mathbf{U}(\tau, \chi + \alpha\Delta\chi)(\dot{\chi} + \alpha\Delta\dot{\chi}) - \mathbf{F} \circ \mathbf{U}(\tau, \chi + \alpha\Delta\chi) \right] d\tau \,, \tag{3.1.12}$$

where $\Delta\chi(\tau)$ belongs to the same class as $\chi(\tau)$.

Theorem 3.1.5. Let $\boldsymbol{U}(t,x) \in \bar{K}$ and satisfy the system (3.1.1) in a weak sense, i.e. in the sense of Definition 3.1.1. Then the following relation is true

$$\frac{d}{d\alpha}\mathbf{J}_{\alpha} = \frac{d}{d\alpha}\int_0^T \frac{d}{d\tau}\mathbf{V}(\tau, \chi + \alpha\Delta\chi)d\tau \text{, and} \tag{3.1.13}$$

the function

$$\mathbf{M}_V(t,x) \equiv \frac{\partial \mathbf{V}}{\partial t} + \mathbf{F}\left(\frac{\partial \mathbf{V}}{\partial x}\right) \tag{3.1.14}$$

is continuous and does not depend on x.

The equalities (3.1.12), (3.1.13) are related to the transformations used in [R1] and to the functionals introduced there. Based on Theorem 3.1.5, the following definition can be proposed for the generalized solution of the system (3.1.1), which is not based on the fulfillment of the integral identity (3.1.2).

Definition 3.1.8. Consider the function $\mathbf{V}(t,x) \in W^{1,\infty}(\Pi_T) \equiv B$ and the initial condition $\mathbf{V}_0(x) \in W^{1,\infty}(\mathbb{R}), \mathbf{V}_0'(x) = \mathbf{U}_0(x)$. Denote by $\mathcal{V}$ a subset B such that $\mathbf{V}(+0,x) = \mathbf{V}_0(x)$. Also, in the space B consider the functional

$$\mathcal{L}(\mathbf{V}) \equiv \underset{t}{esssup} \sum_{i=1}^{n} \underset{x}{Var}\left(\frac{\partial v_i}{\partial t} + f_i\left(\frac{\partial \mathbf{V}}{\partial x}\right)\right), \tag{3.1.15}$$

where $\mathbf{V} = (v_1, \ldots, v_n)$, and $\underset{x}{Var}$ denotes a variation of the expression with respect to variable x. Then we will call the function $\mathbf{U} = \partial \mathbf{V} / \partial x$ назовем *a generalized solution* to the problem (3.1.1), $m=1$, if the function $\mathbf{V}$ implements the minimum of the functional $\mathcal{L}$ on the set $\mathcal{V}$.

The connection of a generalized solution in the sense of Definition 3.1.8 with a weak solution, i.e. a solution in the sense of Definition 3.1.1, illustrates the following statement.

Theorem 3.1.6. Let $\mathcal{W} \subset B$ denotes a set of such $\mathbf{V}$ that the functions $\mathbf{M}_V(t,x)$ from (3.1.14) have a bounded variation with respect to x for almost all t. Then the functional $\mathcal{L}$

from (3.1.15) is semi-continuous from below on the set $\mathcal{W}$. Let the minimum of $\mathcal{L}$ be achieved. Denote $\bar{m} \equiv \min_{\mathbf{V}\in\mathcal{V}} \mathcal{L}(\mathbf{V})$, then if $\bar{m} = 0$, then $\mathbf{U} = \partial \mathbf{V} / \partial x$ satisfies the system (3.1.1) in the sense of Definition 3.1.1.

The meaning of the introduction of Definition 3.1.8 is that, firstly, it indicates a natural way to construct a generalized solution and, secondly, it can serve as a generalization of the concept of a weak solution if it is necessary to consider as solutions of a nonlinear equation the delta functions, for example, a Keifitz-Kranzer type system, see [KK]. Definition 3.1.8 avoids direct consideration of delta functions in quasi-linear systems of conservation laws.

3.2 Singular solutions of the system of equations of pressureless gas dynamics in the case of one and two spatial variables

3.2.1 The case of a single spatial variable

The study of the laws of concentration of matter, based on physical considerations, was already contained in the article [Z1] and the book [Z2]. In [Kra], it was also shown at the physical level of rigor that singularities arise in solving equations for pressureless media, but these equations make sense even after the singularities arise. In this case, a new type of discontinuous solutions arises, in which strong density singularities are formed on hypersurfaces of different codimension. In particular, in [Kra], also at the physical level of rigor, the laws of evolution of such hypersurfaces were obtained. In the three-dimensional case, the author called such singularities as "sheets" and "filaments" to distinguish them from gas-dynamic shock waves. The appearance of the [Bou] article stimulated the study of pressureless media from a mathematical point of view. For example, in [Ov], [Chu], classical solutions of a multidimensional system of equations of pressureless gas dynamics were studied up to the moment of occurrence of singularities based on the technique of group-theoretic analysis. In the works of the thesis author with co-authors [1], [ERS] for the first time, in parallel with the article [Gr], the existence of generalized solutions for the system (1.4) in the measure space is proved at the mathematical level of rigor and a variational principle is proposed that allows finding generalized solutions using the minimization procedure of some function constructed from initial data. Later, this approach was developed in the works of other authors in the case of a single spatial variable, see, for example, [HW], [LiW], [Hy1], [Hy2]. Additionally, we note that, despite the apparent simplicity of the system (1.4), it allows for a fairly rich set of solutions, in particular, it can describe the processes of not only concentration, but also the decomposition of matter [KlR].

Along with the system (1.4), we will also consider its more general version with an external force

$$\begin{cases} \partial\rho / \partial t + \partial\left(\rho u\right) / \partial x = 0 \\ \partial\left(\rho u\right) / \partial t + \partial\left(\rho u^2\right) / \partial x = -\rho g\ , \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}_+ . \\ \partial g / \partial x = 2\rho \end{cases} \tag{3.2.1}$$

We will define the concept of a generalized solution of the Cauchy problem only for (3.2.1), since the system (1.4) is obtained from (3.2.1), assuming $g = 0$ and excluding the third equation. Due to the degeneracy of the systems (1.4), (3.2.1), it is natural to assume that they set the dynamics in the space of pairs of measures $\left(P_t, I_t\right)$, where $P_t \geq 0$ is responsible for the distribution of mass, and, generally speaking, alternating I_t – for the distribution of momentum.

Definition 3.2.1. Let $\left(P_t, I_t\right)$ be families of Radon measures defined on Borel subsets of $\mathbb{R}$, weakly continuous in t, and, moreover, $P_t \geq 0$, and the measure I_t is absolutely continuous with respect to P_t for almost all positive t. Let's define the function $u\left(t,x\right)$ as Radon-Nikodym derivative $u\left(t,x\right) = dI_t / dP_t$. Then we call the pair $\wp_t \equiv \left(P_t, I_t\right)$ a *generalized solution* to the Cauchy problem for (3.2.1) if

1) for any vector function $\boldsymbol{\varphi}: \mathbb{R} \to \mathbb{R}^2, \boldsymbol{\varphi} = \left(\varphi_1, \varphi_2\right) \in C_0^1\left(\mathbb{R}^2\right)$ and every $0 < t_1 < t_2 < +\infty$ the following integral identity is true

$$\int\limits_{\mathbb{R}} \boldsymbol{\varphi}\left(x\right) \odot \wp_{t_2}\left(dx\right) - \int\limits_{\mathbb{R}} \boldsymbol{\varphi}\left(x\right) \odot \wp_{t_1}\left(dx\right) = \int\limits_{t_1}^{t_2} d\tau \int\limits_{\mathbb{R}} \frac{\partial \boldsymbol{\varphi}}{\partial x} u \odot \wp_\tau\left(dx\right) - \\ (0,1) \cdot \int\limits_{t_1}^{t_2} d\tau \int\limits_{\mathbb{R}} \varphi_2\left(x\right)\left(P_\tau\left(-\infty, x\right) - P_\tau\left(x, +\infty\right)\right) P_\tau\left(dx\right) , \tag{3.2.2}$$

where $\odot$ denotes a component-by-component product (Hadamard's product);

2) in a weak sense as $t \to +0 \quad P_t \to P_0, \quad I_t \to I_0$.

Theorem 3.2.1. Let the initial measures $\wp_0 \equiv \left(P_0 \geq 0, I_0\right)$ be Radon measures on $\mathbb{R}$ and let the following conditions be fulfilled: 1) P_0 is discrete measure or absolutely continuous measure with respect to Lebesgue measure, in the case of absolute continuity its density $\rho_0\left(x\right) > 0, x \in \operatorname{supp}\left(P_0\right)$, and with unbounded support the following is true

$\int_0^x s dP_0(s) \to +\infty$ as $|x| \to +\infty$; 2) the measure I_0 is absolutely continuous with respect to P_0, $u_0 \equiv dI_0 / dP_0$ and in the case of absolute continuity of P_0 the function u_0 is also continuous; 3) for any $s > 0$ the following relations are fulfilled $\sup_{|x|\le s} |u_0(x)| \le b_0(s)$, $\lim_{s\to\infty} b_0(s)/s = 0$; 4) $P_0(\mathbb{R}) < \infty$. Then a generalized solution of the system (3.2.1) exists. In the case of system (1.4), a generalized solution exists without fulfilling condition 4).

Now we describe the variational principle for generalized solutions of the system (3.2.1), for the system (1.4) the expressions are similar.

Theorem 3.2.2. The point $(t,x),\ x = a + tu_0(a) - g_0(a)t^2/2$, where $g_0(a) = \int_{-\infty}^{a} dP_0 - \int_a^{+\infty} dP_0$, is the point of absolute continuity of the measure $P_t(dx)$ and of continuity of the function $u(t,x)$ if and only if for any $a^+, a^-, a^- < a < a^+$ the following relation is true

$$\frac{\int_{[a^-,a)} (s + tu_0(s)) P_0(ds)}{\int_{[a^-,a)} P_0(ds)} - \frac{t^2}{2}\left(P_0(-\infty, a^-) - P_0(a, +\infty)\right) < \frac{\int_{[a,a^+)} (s + tu_0(s)) P_0(ds)}{\int_{[a^-,a)} P_0(ds)} - \frac{t^2}{2}\left(P_0(-\infty, a) - P_0(a^+, +\infty)\right). \quad (3.2.3)$$

This theorem allows us to determine from the initial data all the points of continuity of the generalized solution, and, consequently, the places of origination of the singularities. And thus build a generalized solution.

3.2.2 The case of two spatial variables

The case of many, and even two, spatial variables is much less studied. For the case of two spatial variables, in parallel in the book [LZY] and in the work of the thesis's author [R2] (for a more detailed description of this paper, see [4]), Rankine-Hugoniot type relations in differential form were obtained for the first time at the mathematical level of rigor for the case of strong singularities on surfaces in space (t,x,y). Solutions of the two-dimensional Riemann problem, which contain such singularities, are also studied in [LZY], this is done as part of the study of solutions to the two-dimensional Riemann problem for a system of equations of traditional gas dynamics. The obtained Rankine-Hugoniot relations in the case of pressureless media represent an evolutionary system of partial differential equations, which includes, in addition to the dynamics of the singularities surface, the dynamics of the density of the concentration of matter. Therefore, generally speaking, they are more complex than the Rankine-Hugoniot relations for ordinary gas dynamics and represent a new type of shock waves. In the paper [4] by the author of thesis, the Rankine-Hugoniot relations were obtained in an original way based on the theory of new generalized functions [Co], in addition, a description of these relations in integral form was also obtained there. The integral description of the Rankine-Hugoniot relations actually suggests the possibility of the emergence of evolving singularities on manifolds of different dimensions and the emergence of a hierarchy of shock waves. This fact is new, not noted by other authors. It was concretized in the related works [R3], [AR1] by thesis's author and co-authors and justified in [5], [6]. It turns out that the form of the Rankine-Hugoniot relations varies depending on the codimension of the surface of the strong singularity. In the foreign literature, the results of [LZY] were generalized towards considering the complete system of equations (1.5), (1.6), see, for example, [Pa]. When considering discrete solutions of the system (1.5) in the two-dimensional case, in contrast to the one-dimensional setting, a situation of crossing particle trajectories arises. It turns out that this will be the case for most trajectories for at least a complete set of initial particle distributions, see [BD]. If there is an infinite initial set of particles in a bounded set on the plane, then situations of non-existence and non-uniqueness of the generalized solution are possible, see [BrN]. Nevertheless, the thesis's author with co-authors in [7] were the first to propose a numerical method for finding generalized solutions to a system of equations of two-dimensional pressureless gas dynamics based on adhesion principle, and estimates for the corresponding system of interacting particles were obtained. A numerical study of the phenomenon of the emergence of a hierarchy of shock waves based on this methodology is contained in [KRy]. Due to the difficulties of constructing solutions using a system of interacting particles, the thesis proposed an approach for variational description of generalized solutions in the two-dimensional case [5] based on the integral representation of works [4] and [R3], which qualitatively differs from the one-dimensional version (3.2.3). Additionally, we note that the remarks made in [4] about the non-hyperbolic nature of the Rankine-Hugoniot system of equations in the two-dimensional case were continued in recent work [BCH].

The concentration of matter in the multidimensional case on surfaces of codimension one was studied, for example, in [LY], [Sh1], [ARS]. In addition, the subtle issues of the propagation of the velocity vector field to points (t,x,y) located inside singularities of different dimensions are considered in [KS1], [KS2] in the case when the system (1.5) can be represented as the Hamilton-Jacobi equation; generalization to an arbitrary convex Hamiltonian is also acceptable. In this case, a variational interpretation of the "viscous solutions" of this equation is used. The results obtained are useful, including in astrophysical applications, see, for example, [GSS].

Let's consider the Cauchy problem for the system (1.5). Introduce the notations $\boldsymbol{x} \equiv (x,y), (t,\boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^2, \nabla = (\partial/\partial x, \partial/\partial y)$, $d\boldsymbol{x} \equiv (dx,dy), \boldsymbol{u}(t,\boldsymbol{x}) \equiv (u(t,\boldsymbol{x}), v(t,\boldsymbol{x}))$, $\boldsymbol{I}_t(d\boldsymbol{x}) \equiv (I_t(dx,dy), J_t(dx,dy))$. Similarly to the case of a single spatial variable, the generalized solution of the Cauchy problem for (1.5) is understood in the sense of a family of measures $P_t(d\boldsymbol{x}) \geq 0, \boldsymbol{I}_t(d\boldsymbol{x})$, where the index t means a time variable. The initial data $P_0(d\boldsymbol{x}) \geq 0, \boldsymbol{I}_0(d\boldsymbol{x})$ are also, generally speaking, measures, in the case of their absolute continuity relative to the standard Lebesgue measure, the corresponding densities are denoted as $\rho_0(\boldsymbol{x}) \geq 0, \rho_0(\boldsymbol{x})\boldsymbol{u}_0(\boldsymbol{x})$.

Definition 3.2.2. Let $\wp_t(d\boldsymbol{x}) \equiv (P_t(d\boldsymbol{x}), \boldsymbol{I}_t(d\boldsymbol{x}))$ be a family of Radon measures defined on Borel subsets of $\mathbb{R}^2$, weakly continuous in t and, moreover, $P_t \geq 0$ and the measure $\boldsymbol{I}_t$ is absolutely continuous with respect to P_t for almost all positive t. Let's define the vector function $\boldsymbol{u}(t,\boldsymbol{x})$ as a Radon-Nikodym derivative $\boldsymbol{u}(t,\boldsymbol{x}) = d\boldsymbol{I}_t / dP_t$. Then we call $\wp_t$ the *generalized solution* of the Cauchy problem for the system (1.5) if

1) for any vector function $\boldsymbol{\varphi}: \mathbb{R}^2 \to \mathbb{R}^3, \boldsymbol{\varphi} \in C_0^1(\mathbb{R}^2)$ and any $0 < t_1 < t_2 < +\infty$ the following integral identity is fulfilled

$$\iint_{\mathbb{R}^2} \boldsymbol{\varphi}(\boldsymbol{x}) \odot \wp_{t_2}(d\boldsymbol{x}) - \iint_{\mathbb{R}^2} \boldsymbol{\varphi}(\boldsymbol{x}) \odot \wp_{t_1}(d\boldsymbol{x}) = \int_{t_1}^{t_2} d\tau \iint_{\mathbb{R}^2} (\nabla \otimes \boldsymbol{\varphi})^T \cdot \boldsymbol{u} \odot \wp_\tau(d\boldsymbol{x}), \tag{3.2.4}$$

where $\otimes$ denotes the tensor product, $\odot$ denotes the component-by-component product (Hadamard product), and the index T is the transposition operation;

2) in a weak sense as $t \to +0$ $\quad P_t \to P_0, \quad \boldsymbol{I}_t \to \boldsymbol{I}_0$.

For the subsequent formulation of the theorems, consider the following geometric construction. Let within this paragraph the index i takes a finite set of integer values $i = 1,\dots,Q.$

Consider in space $(t,\boldsymbol{x})$ in a time interval $t\in[t_1,t_2]$ Q surfaces $\Gamma_i(t,\boldsymbol{x})=0$, $\Gamma_i\in C^1([t_1,t_2]\times\mathbb{R}^2)$. Suppose that all surfaces intersect along some curve L, at this Γ_i and L are set parametrically as $\boldsymbol{x}=\boldsymbol{X}_i(t,l)\in C^1([t_1,t_2]\times\mathbb{R}),\ \boldsymbol{X}\equiv(\chi,\gamma)$ and $\boldsymbol{x}=\boldsymbol{S}(t)\in C^1([t_1,t_2]),\ \boldsymbol{S}\equiv(s^x,s^y)$. Let's also assume that equality $\boldsymbol{X}_i(t,l_i(t))=\boldsymbol{S}(t),\ t\in[t_1,t_2]$ holds for some continuously differentiable $l_i(t)$. We will consider the surfaces Γ_i not for all values l, namely, for Γ_i the parameter $l\le l_i(t)$. For the resulting parts of the surfaces we will keep the designation Γ_i. We orient Γ_i in accordance with the orientation $(t,\boldsymbol{x})$ and in the direction of the positive normal we define positive «+» and negative «-» sides of Γ_i, which are set using inequalities $\Gamma_i(t,\boldsymbol{x})>0$, $\Gamma_i(t,\boldsymbol{x})<0$. Accordingly, the values related to the two sides of the surfaces will be provided with the same indices. For each value t we define the areas $G_i(t)\subset\mathbb{R}^2$ between the introduced surfaces: $G_i(t)=\{\boldsymbol{x}:\Gamma_i(t,\boldsymbol{x})>0\,,\Gamma_{i+1}(t,\boldsymbol{x})<0\}$, $i=1,...,Q$. At this $\Gamma_{Q+1}\equiv\Gamma_1$. We will also assume that the characteristics of the system (1.5), namely the straight lines

$$\boldsymbol{x}=\boldsymbol{a}+t\boldsymbol{u}_0(\boldsymbol{a}), \tag{3.2.5}$$

lying on both the positive and negative sides of Γ_i cross it at some point in time. Here $\boldsymbol{a}\equiv(a,b)$ is the point on the initial plane from which the characteristic originates.

Let's define a family of measures

$$\begin{aligned} P_t &= P_1^- + \sum_{i=1}^{Q}\left(P_i^+ - P_i^-\right)H(\Gamma_i) + \sum_{i=1}^{Q}P_i(t,l)\delta(\Gamma_i) + M(t)\delta\left(\boldsymbol{x}-\boldsymbol{S}(t)\right) \\ \boldsymbol{I}_t &= \boldsymbol{I}_1^- + \sum_{i=1}^{Q}\left(\boldsymbol{I}_i^+ - \boldsymbol{I}_i^-\right)H(\Gamma_i) + \sum_{i=1}^{Q}\boldsymbol{I}_i(t,l)\delta(\Gamma_i) + \boldsymbol{\Pi}(t)\delta\left(\boldsymbol{x}-\boldsymbol{S}(t)\right) \end{aligned}, \tag{3.2.6}$$

where H is the Heaviside function, δ is the Dirac measure on curves and at a point, $P_i^\pm,\boldsymbol{I}_i^\pm$ are measures that are absolutely continuous with respect to the Lebesgue measure with densities $\rho_i^\pm,\rho_i^\pm\boldsymbol{u}_i^\pm$ respectively. In this case, the measures $P_i^+,\boldsymbol{I}_i^+$ are defined only in domains G_i, and for any value f it is assumed that $f_i^+=f_{i+1}^-$ and the number $Q+1$ is replaced by 1. Functions $\rho_i^\pm,\rho_i^\pm\boldsymbol{u}_i^\pm$, $P_i,\boldsymbol{I}_i,M,\boldsymbol{\Pi}$ are assumed to be piecewise continuously differentiable on their domains of definition.

Theorem 3.2.3. Let the family of measures defined by expression (3.2.6) be a generalized solution of system (1.5) in the sense of Definition 3.2.2. Then the following system of equations holds for any surface Γ_i (index i is omitted for brevity)

$$\begin{cases} \dfrac{\partial P}{\partial t} = \dfrac{\partial \chi}{\partial l}\left\{V\left[\rho\right]-\left[\rho v\right]\right\} - \dfrac{\partial \gamma}{\partial l}\left\{U\left[\rho\right]-\left[\rho u\right]\right\} \\ \dfrac{\partial \boldsymbol{I}}{\partial t} = \dfrac{\partial \chi}{\partial l}\left\{V\left[\rho \boldsymbol{u}\right]-\left[\rho v \boldsymbol{u}\right]\right\} - \dfrac{\partial \gamma}{\partial l}\left\{U\left[\rho \boldsymbol{u}\right]-\left[\rho u \boldsymbol{u}\right]\right\}, \\ \dfrac{\partial\left(\chi,\gamma\right)}{\partial t} = \boldsymbol{U} \equiv \left(U,V\right) \end{cases} \qquad (3.2.7)$$

where for any value f it is denoted $\left[f\right] \equiv f^{+} - f^{-}$.

The relations (3.2.7) are Rankine-Hugoniot relations for (1.5) in the case when the support of the singularity is concentrated on the surface, and differ significantly from the traditional Rankine-Hugoniot relations in gas dynamics. The original inference of these relations is presented in [4] based on the work of the author of the thesis [R2].

Theorem 3.2.4. Let the family of measures defined by expression (3.2.6) be a generalized solution of the system (1.5) in the sense of Definition 3.2.2. Then 1) there is a Lagrangian mapping $\mathfrak{A}_t : \boldsymbol{a} \to \boldsymbol{x}$ that shows which points on the initial plane $\boldsymbol{a}$ will come to a point $\boldsymbol{x}$ at a time t; the functions $\rho_i^{\pm}, \boldsymbol{u}_i^{\pm}, i = 1,\ldots,Q$ satisfy (1.5) in the classical sense; 3) along the surfaces Γ_i for each i the relations (3.2.7) are fulfilled; 4) along the curve L the following equations hold

$$M\left(t\right)\frac{d\boldsymbol{S}}{dt} = \boldsymbol{\Pi}\left(t\right), \qquad (3.2.8)$$

where $\left(M\left(t\right),\boldsymbol{\Pi}\left(t\right)\right) = \wp_0\left(\mathfrak{A}^{-1}\left(\boldsymbol{S}\left(t\right)\right)\right)$.

Until the moment of occurrence of the singularities, the Lagrangian mapping $\mathfrak{A}_t$ is given by the formulas (3.2.5). The relations (3.2.8) are Rankine-Hugoniot relations for (1.5) in the case when the support of the singularity is concentrated on the curve, these relations have no gas-dynamic analogues. The formulas (3.2.8) were first obtained by the thesis's author in [R3] and further co-authored in [AR2], the final proof was published in [6]. Combination of formulas (3.2.7), (3.2.8) describes the situation of the formation of a hierarchy of shock waves, which was discovered by the author of the thesis and was not noted by other authors.

The real case of a hierarchy of singularities is shown for solutions to the Riemann problem for (1.5).

Definition 3.2.3. *The Riemann problem* for system (1.5) is called the Cauchy problem, in which the initial data is constant in each quadrant of $\mathbb{R}^2$, i.e.

$$\rho_0(\boldsymbol{x}) = \rho_k = \text{const}, \; \boldsymbol{u}_0(\boldsymbol{x}) = \boldsymbol{u}_k = \text{const}, \tag{3.2.9}$$

where $k = 1, 2, 3, 4$ is the natural numbering of quadrants in $\mathbb{R}^2$.

Let's consider the specific data (3.2.9). Namely, let $u > 0, v > 0, 0 < \rho < R$ and let

$$\begin{array}{l} u_1 = u_4 = -u, \quad u_2 = u_3 = u, \quad v_1 = v_2 = -v, \quad v_3 = v_4 = v, \\ \rho_k = \rho, \quad k \neq 4, \quad \rho_4 = R \end{array}, \tag{3.2.10}$$

Theorem 3.2.5. For the generalized solution of the problem (1.5), (3.2.10) there is such a straight line $\boldsymbol{x} = t \cdot (X^*, Y^*)$ that for any t the measure P_t has at point $t \cdot (X^*, Y^*)$ a delta singularity. In this case, the point (X^*, Y^*) is the intersection point of shock waves described by the self-similar version of the system (3.2.7)

$$\begin{cases} X' = (X - U)/\bar{l} \\ Y' = (Y - V)/\bar{l} \\ m' = m/\bar{l} - (\rho^+ d^+ - \rho^- d^-)/\bar{l}^2 \\ U' = \left(\rho^+ (U - u^+) d^+ - \rho^- (U - u^-) d^-\right)/(m\bar{l}^2) \\ V' = \left(\rho^+ (V - v^+) d^+ - \rho^- (V - v^-) d^-\right)/(m\bar{l}^2) \end{cases}, \tag{3.2.11}$$

where $\bar{l} \equiv l/t$, «stroke» denotes the differentiation by $\bar{l}$, $(\chi, \gamma) = t \cdot (X, Y)(\bar{l})$, (U, V) are considered as dependent only on $\bar{l}$, $P = t \cdot m(\bar{l})$, $\boldsymbol{I} = P \cdot (U, V)$,

$$d^{\pm} \equiv \begin{vmatrix} X - U & U - u^{\pm} \\ Y - V & V - v^{\pm} \end{vmatrix}.$$

The proof of this theorem is contained in [5], [6]. The main difficulty lies in the fact that waves of the type (3.2.11) do not necessarily intersect, this difficulty can be overcome using symmetry considerations.

Now let the initial measure $\wp_0(d\boldsymbol{x})$ be discrete, that is, at the initial moments of time in $\mathbb{R}^2$ there are N particles with masses m_i and velocities (u_i, v_i), $i = 1, \ldots, N$, and these particles are placed in a uniform auxiliary grid $[x_{j-1/2}, x_{j+1/2}] \times [y_{k-1/2}, y_{k+1/2}]$,

$x_j \equiv j\Delta x, y_k \equiv k\Delta y$, $\varepsilon \equiv \sqrt{(\Delta x)^2 + (\Delta y)^2}$ is sufficiently small. Let us also assume that as $N \to \infty$ the system of particles under consideration converges in a weak sense to a measure that is absolutely continuous with respect to the Lebesgue measure. Next, consider the following approximate adhesion dynamics. Each particle with a certain number i_1 moves in a straight line with the speed prescribed for it until it approaches another particle with a number i_2 at a distance $k \cdot \varepsilon$, where $k < 1$ is some constant. Let's call the fact of such an approaching a type E event. When an event E occurs, the particles i_1 and i_2 stick together into one, the mass and motion of which are determined by the laws of conservation of mass and momentum, the new particle is located in the center of mass of the particles i_1 and i_2. A similar procedure is applied if not two, but a larger number of particles stick together. It is easy to see that the resulting particle system will, in a weak sense, satisfy (1.5) not exactly, but with some error R.

Theorem 3.2.6. For the adhesion dynamics described above, there is an estimate

$$|R| \le C\varepsilon \sum_E \left(\frac{\sum_{i<l} m_i m_l \left(|u_i - u_l| + |v_i - v_l| \right)}{\sum_l m_l} + \varepsilon \sum_i m_i \left(1 + |u_i| + |v_i| \right) \right),$$

where C is some positive constant, the outer summation goes for all events of the type E, and the rest of the summation goes for all particles involved in this event E.

The presented estimate was obtained in [7], algorithms for numerical calculation of generalized solutions (1.5) were obtained there, additionally see [KRy]. Generally speaking, it turned out that in order to find out the convergence of the constructed particle system to any evolving measure, additional assumptions about the structure of the generalized solution are necessary. For example, suppose that the generalized solution contains a finite set of evolving curves with the corresponding delta mass function. Then the masses of the particles making up this curve $\sim \varepsilon$, while the mass of non-interacting particles $\sim \varepsilon^2$, but the number the particles that undergo interaction $\sim 1/\varepsilon$. Further $\sum_l m_l \sim \varepsilon$, $\sum_{i<l} m_i m_l \sim \varepsilon^3$, the velocities are bounded, that is $|R| \sim \varepsilon^3 \cdot N_E$, where N_E is the number of type E events. However $N_E < C/\varepsilon^2$, since the initial mass is finite in any bounded region, hence $|R| \sim \varepsilon$.

Thus, the insufficiency of the estimates given, as well as the results of [BrN], [BD], force us to look for other approaches to describing generalized solutions of the system (1.5). An example of such approach is the search for variational descriptions of generalized solutions.

In the case of a single spatial variable, the variational principle (3.2.3) describes from a single point of view both the smooth and non-smooth parts of the generalized solution. In the two-dimensional case, the expression of type (3.2.3) cannot be used due to the fact that the trajectories of the particles in the general position, generally speaking, are crossing. However, it is possible to formulate a variational description. To do this, we will need additional constructions.

Let's fix $t > 0$. Let A_1 be a finite set of mutually disjoint domains in $\mathbb{R}^2$, and A_2 is a set of domains obtained as follows. Let's also take the finite set of mutually disjoint domains $G_\alpha \subset \mathbb{R}^2 \setminus A_1$ of the form $G_\alpha = \left\{ \boldsymbol{a}_\alpha(s,l), s \in \left[s_\alpha^1, s_\alpha^2\right], l \in \left[l_\alpha^1, l_\alpha^2\right] \right\}$, the functions $\boldsymbol{a}_\alpha$ are assumed to be continuously differentiable on their domain of definition. Then some domain G will belong to A_2 if and only if there exists such α and l_G^1, l_G^2, that $G = \left\{ \boldsymbol{a}_\alpha(s,l), s \in \left[s_\alpha^1, s_\alpha^2\right], l \in \left[l_G^1, l_G^2\right] \right\}$, $0 < \left|l_G^2 - l_G^1\right| < \left|l_\alpha^2 - l_\alpha^1\right|$. Let now A_3 represent the set of all domains lying in $\mathbb{R}^2 \setminus \left(A_1 \cup A_2\right)$, and $B = A_1 \cup A_2 \cup A_3$. Denote

$$\boldsymbol{F}(t, \boldsymbol{x}; G) \equiv \iint_G \left[\boldsymbol{u}_0(\boldsymbol{a}) - \frac{\boldsymbol{x} - \boldsymbol{a}}{t} \right] \rho_0(\boldsymbol{a}) d\boldsymbol{a}\,,\ G \in B. \tag{3.2.12}$$

Definition 3.2.4. Let the entry $|G| \downarrow \min$ means that the area G tends to a minimum under the condition $G \in B$. Let's call *the derivative* of $\boldsymbol{F}$ *with respect to the area* containing the point $\bar{\boldsymbol{a}}$, the value

$$\frac{\delta \boldsymbol{F}}{\delta \bar{\boldsymbol{a}}} \equiv \lim_{|G| \downarrow \min} \frac{\boldsymbol{F}(t, \boldsymbol{x}; G)}{|G|}, \quad \bar{\boldsymbol{a}} \in G \in B. \tag{3.2.13}$$

Theorem 3.2.7. Let $\wp_0(d\boldsymbol{x})$ be absolutely continuous with respect to the Lebesgue measure. Let there be a generalized solution of system (1.5) under the conditions of Theorems 3.2.3, 3.2.4. Then for almost all $t > 0$ and almost all $\bar{\boldsymbol{a}} \in \mathbb{R}^2$ there exist such $\boldsymbol{x}$ that $\delta \boldsymbol{F} / \delta \bar{\boldsymbol{a}} = 0$.

This theorem was obtained in [5], [6], see also [R3].

Theorem 3.2.8. Let $\wp_0(d\boldsymbol{x})$ be absolutely continuous with respect to the Lebesgue measure and let there be a generalized solution of the system (1.5) in the sense of Definition 3.2.2. Let there also be such a parametrization $\boldsymbol{a}(s,l)$ and a value l^* that the vector function

$$\boldsymbol{\Phi}\left(\tau,l^*\right)\equiv\int_0^\tau\left[\boldsymbol{u}_0\left(\boldsymbol{a}\left(\tau,l^*\right)\right)-\frac{\boldsymbol{x}-\boldsymbol{a}\left(\tau,l^*\right)}{t}\right]\rho_0\left(\boldsymbol{a}\left(\tau,l^*\right)\right)\left.\frac{\partial\boldsymbol{a}\left(\tau,l\right)}{\partial\left(\tau,l\right)}\right|_{l=l^*}d\tau \quad (3.2.14)$$

has component wise more than one joint global minimum with respect to τ, then at the point $\left(t,\boldsymbol{x}\right)$ the generalized solution of the system (1.5) has a singularity in the form of a concentration of a mass measure.

The function (3.2.14) is derived from the function (3.2.12) by parameterizing the domain G, and the behavior of the function (3.2.14) actually means that the derivative (3.2.13) vanishes. This theorem is obtained in [AR3], see also [AR1]. As a further illustration, [KRy] shows that the condition for the existence of two global minima (3.2.14) leads to formulas for the evolution of the surface of singularities, without relying on the system (3.2.7).

3.3 Existence of strong solutions of a quasi-gas-dynamic system of equations in the case of two spatial variables

In this section, we consider the initial boundary value problem for the system (1.7), rewritten in a slightly different form,

$$\varepsilon\frac{\partial^2\boldsymbol{U}}{\partial t^2}+\frac{\partial\boldsymbol{U}}{\partial t}+\left(\boldsymbol{U}\cdot\nabla\right)\boldsymbol{U}+\nabla P=\Delta\boldsymbol{U}+g,\,\mathrm{div}\boldsymbol{U}=0,\quad t\in\mathbb{R}_+ \quad (3.3.1)$$

in a bounded smooth domain $\boldsymbol{x}=\left(x,y\right)\in\Omega\subset\mathbb{R}^2$

$$\left.\boldsymbol{U}\right|_{t=0}=\boldsymbol{U}_0\left(\boldsymbol{x}\right),\left.\frac{\partial\boldsymbol{U}}{\partial t}\right|_{t=0}=\boldsymbol{U}_1\left(\boldsymbol{x}\right),\left.\boldsymbol{U}\right|_{\partial\Omega}=0. \quad (3.3.2)$$

Recall that here $\varepsilon>0$ is a small parameter, P is a pressure, Δ is Laplace operator, g is external force.

The problem of the form (3.3.1), (3.3.2) was studied, for example, in [BNP], [PaR], see also [RS 1], [RC2], however, either for the case of periodic boundary conditions, or in the entire space $\mathbb{R}^2$ (note that the case $\mathbb{R}^3$ was also considered). Unlike other works, the results presented in the thesis, see [8], are obtained for a bounded domain. In this setting, the technique of the above-mentioned works is not applicable, and it is necessary to use only energy-type estimates. Accordingly, the results are obtained only for small $\varepsilon>0$, and in the case of one spatial variable, an example of a situation is given that demonstrates the "explosion" of a solution in finite time for finite $\varepsilon>0$. Note also that for the linear version of the quasi-gas dynamic equations, additional results were obtained in [IR].

We will study the strong solutions of the problem (3.3.1), (3.3.2) $\mathfrak{A}(t)=\{\boldsymbol{U}(t),\partial\boldsymbol{U}(t)/\partial t\}$ in the phase space E_ε^1 with the corresponding norm $\|\mathfrak{A}\|_{\mathrm{E}_\varepsilon^1}$:

$$\begin{gathered}\mathrm{E}_\varepsilon^1 \equiv \left\{\{\boldsymbol{U},\boldsymbol{V}\}\in\left[H^2(\Omega)\cap H_0^1(\Omega)\right]^2\times\left[H_0^1(\Omega)\right]^2, \nabla\cdot\boldsymbol{U}=0, \nabla\cdot\boldsymbol{V}=0\right\} \\ \|\mathfrak{A}\|_{\mathrm{E}_\varepsilon^1}^2 \equiv \varepsilon\|\partial\boldsymbol{U}(t)/\partial t\|_{H^1}^2+\|\partial\boldsymbol{U}(t)/\partial t\|_{L^2}^2+\|\boldsymbol{U}(t)\|_{H^2}^2\end{gathered}. \quad (3.3.3)$$

Let $\boldsymbol{\mathcal{V}}\equiv\left\{\boldsymbol{U}\in\left[H_0^1(\Omega)\right]^2, \nabla\cdot\boldsymbol{U}=0\right\}$, $\boldsymbol{\mathcal{H}}\equiv\left\{\boldsymbol{U}\in\left[L^2(\Omega)\right]^2, \nabla\cdot\boldsymbol{U}=0, \boldsymbol{U}\cdot\boldsymbol{n}|_{\partial\Omega}=0\right\}$, $\boldsymbol{\mathcal{D}}\equiv\left[H^2(\Omega)\cap H_0^1(\Omega)\right]^2\cap\{\nabla\cdot\boldsymbol{U}=0\}$.

Definition 3.3.1. Let's call the function $\boldsymbol{U}(t,\boldsymbol{x})$ a *strong solution* to the problem (3.3.1), (3.3.2) on the interval $[0,T]$ if: 1) $\boldsymbol{U}(t,\boldsymbol{x})\in C([0,T],\boldsymbol{\mathcal{D}})$, $\partial\boldsymbol{U}/\partial t\in C([0,T],\boldsymbol{\mathcal{V}})$, $\varepsilon\partial^2\boldsymbol{U}/\partial t^2\in C([0,T],\boldsymbol{\mathcal{H}})$; 2) $\boldsymbol{U}$ satisfies (3.3.1) in $\boldsymbol{\mathcal{H}}$ after applying the orthogonal Leray-Helmholtz projection to the solenoidal vector fields; 3) $\boldsymbol{U}$ satisfies the initial data (3.3.2).

Theorem 3.3.1. Let $g\in\boldsymbol{\mathcal{H}}$. Then for any $R>0$ there exists such $\varepsilon_0\equiv\varepsilon_0(R)$ that for any $\varepsilon\le\varepsilon_0$ and any initial data $\mathfrak{A}(0), \|\mathfrak{A}(0)\|_{\mathrm{E}_\varepsilon^1}\le R$ the problem (3.3.1), (3.3.2) has a unique global strong solution and the following estimate is correct for $t\ge 0$

$$\|\mathfrak{A}(t)\|_{\mathrm{E}_\varepsilon^1}\le Q\left(\|\mathfrak{A}(0)\|_{\mathrm{E}_\varepsilon^1}\right)e^{-\alpha t}+Q\left(\|g\|_{L^2}\right), \quad (3.3.4)$$

where the positive constant α and the monotone function Q do not depend on $R,\varepsilon,t\ge 0$ and on initial data $\mathfrak{A}(0)$.

Let us now consider a one-dimensional version of equation (3.3.1) for the function $u(t,x)$

$$\varepsilon\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}=\frac{\partial^2 u}{\partial x^2}+g,\ x\in[0,L],\ t\in\mathbb{R}_+, \quad (3.3.5)$$

with Dirichlet boundary conditions and initial data $(u_0(x),u_1(x))$.

Theorem 3.3.2. Let $g=0$, and $\varepsilon=1$. Then there are smooth initial data with a compact support such that the corresponding solutions (3.3.5) cease to exist in a finite time.

Theorem 3.3.2 shows that, with a finite ε the requirement of boundedness of the initial data with respect to the energy norm in Theorem 3.3.1 is essential.

3.4 Degenerate parabolic systems of equations describing the processes of compressible two-phase multicomponent filtration from the point of view of the theory of conservation laws

In the system of equations (1.8), (1.9) the variables $x_{1G},...,x_{NG};x_{1L},...,x_{NL}$, taking into account thermodynamic relations (the specific type of which is not important in this context), are subjected to the normalization condition $\sum_i x_{iG}=\sum_i x_{iL}=1$ and are connected by N functional relations expressing the equality of chemical potentials of phases. We will assume that these nonlinear dependencies are expressed using doubly continuously differentiable functions. Thus, of the above variables, there are exactly $N-2$ independent ones, which will be denoted by $y_k, k=1,...,N-2$. Then, by introducing the notations, $i=1,...,N$,

$$X_i\left(x,s,y_1,...,y_{N-2},P\right)\equiv\phi(x)\left(x_{iG}\rho_G s+x_{iL}\rho_L\left(1-s\right)\right)$$

$$F_i\left(x,s,y_1,...,y_{N-2},P\right)\equiv K\left(x\right)\left(x_{iG}\rho_G\frac{k_G\left(s\right)}{\mu_G}+x_{iL}\rho_L\frac{k_L\left(1-s\right)}{\mu_L}\right),$$

$$Q\equiv-\frac{\partial P}{\partial x}$$

the system (1.8), (1.9) can be written in the form of a degenerate system of conservation laws with the right-hand side

$$\frac{\partial}{\partial t}X_i+\frac{\partial}{\partial x}\left(F_iQ\right)=0 \quad , \quad \frac{\partial P}{\partial x}=-Q \quad , \quad i=1,...,N. \tag{3.4.1}$$

System (3.4.1) is a system of $N+1$ partial differential equations of the first order from a vector of unknown functions $\boldsymbol{U}=\left(s,y_1,...,y_{N-2},P,Q\right)$.

The mathematical theory of filtration processes is most developed in the case of a single-component medium, where the main model is a single degenerate parabolic equation for pressure, see, for example, [Vaz] and also [Ka]. The case of two immiscible phases has also been considered for a long time (the corresponding phase densities are constant), see the fundamental work [AKM]. From a mathematical point of view, this case boils down to solving a set of a degenerate parabolic equation for saturation and a uniformly elliptical equation for pressure. In this case, the parabolicity of the saturation equation arises from the use of the capillary pressure concept. The presence of parabolic/elliptic terms allows us to obtain various a priori estimates, which is still the basis for the mathematical study of filtration phenomena. So we can give the following examples of work in this direction. The non-isothermal case, in which an equation for temperature is added, is considered in [Amz]. In the case of the dependence of porosity on pressure [DEH], two degenerate parabolic equations of saturation and pressure arise. The same situation is in the case of compressible immiscible phases [AJK]. The case of many components that can mix, in particular to describe the processes of mutual solubility, is contained in [ASh], where a degenerate parabolic system of equations for concentrations and pressures arises. The introduction to the consideration of phase transitions leads in the incompressible case to the consideration of a hyperbolic system of equations, see, for example, [Orr]. Accordingly, in mathematical research, difficulties arise here related to the development of the theory of quasi-linear systems of conservation laws. The compressible case was considered mainly from a physical and computational point of view, for example, [Bed], [AbP], since the nature of even the simplest system of equations (1.8), (1.9), which describes the process of phase transitions, remains unclear. The results obtained in the thesis, [9], fill this gap. The system of equations (3.4.1) is defined as *almost hyperbolic*, having the properties of both hyperbolic and parabolic systems of equations.

The system of equations (3.4.1) can already be written as a system of first-order equations in a non-divergent form and eigenvalues and eigenvectors can be calculated in a standard way. It turns out that there can be no more than $N-1$ eigenvalues λ_j for which left $\boldsymbol{l}_j$ and right $\boldsymbol{r}_j$ eigenvectors can be searched. Next, along with some vector $\boldsymbol{q}=\left(q^1,...,q^{N+1}\right)$, we will consider its projection $\boldsymbol{q}'=\left(q^1,...,q^{N-1}\right)$, consisting of the first $N-1$ coordinates of the vector $\boldsymbol{q}$.

Definition 3.4.1. Let's call a certain set of vectors $\boldsymbol{q}_j$ *linearly independent by projection*, if the vectors $\boldsymbol{q}'_j$ are linearly independent.

Definition 3.4.2. A system of equations of type (3.4.1) is called *almost hyperbolic* if it has exactly $N-1$ real and different eigenvalues, and there are corresponding sets of left and right eigenvectors, each of which contains exactly $N-1$ linearly independent by projection vectors.

For almost hyperbolic systems (3.4.1) consider the strip

$$\Pi_T \equiv \left\{(t,x):(t,x)\in[0,T]\times\mathbb{R}\right\}$$

and we will set the initial data

$$s(0,x)=s^0(x),\ y_1(0,x)=y_1^0(x),...,\ y_{N-2}(0,x)=y_{N-2}^0(x),\ P(0,x)=P^0(x). \quad (3.4.2)$$

Definition 3.4.3. Let the initial functions (3.4.2) be bounded measurable functions on $\mathbb{R}$. Then we call bounded measurable in Π_T functions $s(t,x),y_1(t,x),...,y_{N-2}(t,x)$ and continuous in Π_T function $P(t,x)$ *a generalized solution* of the problem (3.4.1), (3.4.2), if for any function $\varphi \in C^{\infty}\left(\left[0,T\right)\times\mathbb{R}\right)$, $\varphi(t,\cdot)\in C_0^{\infty}(\mathbb{R})$ at a fixed $t\in\left[0,T\right]$, $\varphi\equiv 0$ as $T_1 \le t \le T$, $T_1 < T$, the following integral identities are fulfilled

$$\begin{aligned} &\iint\limits_{\Pi_T}\left(X_i \frac{\partial\varphi}{\partial t} + F_i Q \frac{\partial\varphi}{\partial x}\right) dxdt + \int\limits_{\mathbb{R}} X_i^0(x)\varphi(0,x)\,dx = 0 \quad , \quad i=1,\dots,N \\ &\iint\limits_{\Pi_T}\left(P\frac{\partial\varphi}{\partial x} - Q\varphi\right) dxdt = 0 \end{aligned} \quad , \qquad (3.4.3)$$

where denoted $X_i^0(x)\equiv X_i\left(s^0(x), y_1^0(x),\dots, y_{N-2}^0(x), P^0(x)\right)$.

Definition 3.4.4. We will say that the generalized solution of the problem (3.4.1), (3.4.2) has a *strong singularity* if the functions $s(t,x), y_1(t,x),\dots, y_{N-2}(t,x)$ undergo a discontinuity along some curve in the strip Π_T. We will say that the generalized solution of the problem (3.4.1), (3.4.2) has a *weak singularity* if the functions $s(t,x), y_1(t,x),\dots, y_{N-2}(t,x)$ are continuous, while their partial derivatives in t and x suffer a discontinuity along some curve in the strip Π_T, and the function $P(t,x)$ is continuously differentiable in the vicinity of this curve.

Proposition 3.4.1. The propagation velocity of weak singularities coincides with one of the eigenvalues of the system (3.4.1).

For strong singularities propagating along a certain curve $x = z(t)$ the Rankine-Hugoniot relations now follow from (3.4.3) in a standard way

$$\frac{dz}{dt}\left(X_i^+ - X_i^-\right) = Q^+F_i^+ - Q^-F_i^-, \quad i=1,\dots,N, \quad P^+ = P^-, \qquad (3.4.4)$$

where the entry of the form $y^{\pm}$ denotes the values of some function y to the right and left of the discontinuity, respectively, i.e. $y^{\pm}\equiv y\left(t, z(t)\pm 0\right)$. Let us further agree within this section that the index s or Q of the quantities denotes the partial derivative with respect to the corresponding variables. Let's introduce the notation

$$A \equiv \left(\frac{F_N\left(X_j\right)_s - F_j\left(X_N\right)_s}{F_N} ; \frac{F_N\left(X_j\right)_{y_k} - F_j\left(X_N\right)_{y_k}}{F_N} \right),$$

where A is a matrix of size $(N-1)\times(N-1)$, $j=1,\dots,N-1$, $k=1,\dots,N-2$, the entry of the form $\left(X_j\right)_{y_k}$ denotes the Jacobi matrix for functions X_j depending on variables y_k. That is, the first element of the entry for A is a column of length $N-1$, and the second element is a matrix of dimension $(N-2)\times(N-1)$.

Theorem 3.4.1. Let the conditions $|F_N|>0, |\det A|>0$ be fulfilled. Then in the phase space $\boldsymbol{U}=(s,y_1,...,y_{N-2},P,Q)$ the set of values $\boldsymbol{U}^+$, determined by the relations (3.4.4), is locally a set consisting of $N-1$ curves $\boldsymbol{U}_j(\varepsilon), \boldsymbol{U}_j(0)=\boldsymbol{U}^-$, $j=1,...,N-1$, $|\varepsilon|\le\varepsilon_0$ is some small parameter. At the same time, there are $N-1$ discontinuity speeds $\sigma_j(\varepsilon)$ corresponding to the curves $\boldsymbol{U}_j(\varepsilon)$. In addition, $d\boldsymbol{U}_j(0)/d\varepsilon \,||\, \boldsymbol{r}_j$, and a local discontinuity speed satisfies the relation $\sigma_j(0)=\lambda_j(0)$.

Now consider the case of two components, i.e. $N=2$. Then there are no variables y_k, the values x_{iG}, x_{iL} depend only on pressure P, thus the system of equations (3.4.1) consists of three equations with unknown functions s,P,Q. Let also the quantities μ_G,μ_L be constant, and $\phi=K\equiv 1$. Let's study the properties of solutions to the Riemann problem in the case $N=2$ taking into account the simplifications just mentioned, i.e. consider the following initial data for (3.4.1)

$$s^0(x)=\begin{cases} s_0^-, & x<0 \\ s_0^+, & x>0 \end{cases} \qquad P^0(x)=\begin{cases} P_0^-, & x<0 \\ P_0^+, & x>0 \end{cases}. \tag{3.4.5}$$

Theorem 3.4.2. If $P_0^+\ne P_0^-$ then the self-similar solutions of the problem (3.4.1) for $N=2,$ (3.4.5) have an infinite velocity of propagation of perturbations.

This property is typical for parabolic equations and is absent in hyperbolic equations with smooth nonlinearities.

Theorem 3.4.3. In the self-similar solution of the problem (3.4.1) for $N=2$, (3.4.5) there is always a saturation s singularity: either a shock, or at least a point with an infinite derivative with respect to self-similar variable.

For hyperbolic systems of equations, there are continuous solutions to the Riemann problem consisting of rarefaction waves. As Theorem 3.4.3 shows, this is not the case in the case of filtration, and the saturation in solving the Riemann problem is discontinuous in the case of a general position.

In the theory of systems of conservation laws, the concept of entropy plays an important role in solving issues related to the uniqueness of a generalized solution. Therefore, we will discuss this concept for the system (3.4.1). We define vectors of length $N+1$:

$$\boldsymbol{X} = \begin{pmatrix} X_i \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{F} = \begin{pmatrix} QF_i \\ P \end{pmatrix} \quad , \quad i = 1,\dots,N .$$

Definition 3.4.5. Let there be such a vector function $\boldsymbol{E}(\boldsymbol{U})$ and such functions $\boldsymbol{\mathcal{E}}(\boldsymbol{U}), \boldsymbol{\mathcal{F}}(\boldsymbol{U})$ that

$$\boldsymbol{E} \cdot D\boldsymbol{X} = D\boldsymbol{\mathcal{E}} , \quad \boldsymbol{E} \cdot D\boldsymbol{F} = D\boldsymbol{\mathcal{F}} , \quad E^{N+1}Q \geq 0,$$

where the sign D denotes the differential of the corresponding functions, and E^{N+1} is the last coordinate of the vector $\boldsymbol{E}$. Then we will call the function $\boldsymbol{\mathcal{E}}$ *entropy* for the system (3.4.1).

Theorem 3.4.4. Let in the system (3.4.1) for $N = 2$ in addition to the above limitations the functions X_i and F_i, $i = 1, 2$, are independent of the pressure variable P. Let $\rho_G \rho_L \left(k_G / \mu_G + k_L / \mu_L\right)\left(x_{1G} x_{2L} - x_{2G} x_{1L}\right) > 0$ and $\chi(s) \equiv F_2\left(X_1\right)_s - F_1\left(X_2\right)_s$, while ω be some doubly continuously differentiable function of two variables satisfying the condition $Q \cdot \omega\left(Q\chi(s), P\right) \geq 0$. Let also $e_m(s,Q), m = 1, 2$, be some continuously differentiable functions satisfying the following relations:

$$\begin{gathered} \left(X_2\right)_s \left(e_2\right)_Q + \left(X_1\right)_s \left(e_1\right)_Q = 0 ; \left(X_2\right)_s \left[F_2\left(e_2\right)_s + F_1\left(e_1\right)_s\right] = -Q\chi'(s)\left(e_1\right)_Q ; \\ \left(X_2\right)_s \left(e_2\right)_s + \left(X_1\right)_s \left(e_1\right)_s > 0 \end{gathered} .$$

Then the convex entropy $\boldsymbol{\mathcal{E}}$ of such a system depends only on saturation s and is expressed using a formula

$$\boldsymbol{\mathcal{E}}_s = e_1\left(X_1\right)_s + e_2\left(X_2\right)_s ,$$

and the corresponding flux $\boldsymbol{\mathcal{F}}$ is defined ambiguously according to the following formula

$$\boldsymbol{\mathcal{F}} = \int \omega\left(Q\chi(s), P\right) dP + A(s,Q),$$

where the sign $\int$ denotes the operation of indefinite integration and the function $A(s,Q)$ satisfies the relation

$$A_s = Q\left[e_1\left(F_1\right)_s + e_2\left(F_2\right)_s\right] \quad , \quad A_Q = e_1 F_1 + e_2 F_2 .$$

Theorem 3.4.4 implies, in particular, that the set of entropy-flux pairs includes the corresponding pairs by S.N. Kruzhkov for one scalar conservation law, but the set of fluxes turns out to be much richer than in the scalar case.

3.5 Equation with a saturated (bounded) dissipation flux

Equations of type (1.10) have been studied quite a lot in the context of the theory of parabolic equations. As a rule, equations of the form

$$\partial u / \partial t = \partial\left(\varphi(u)\psi\left(\partial u / \partial x\right)\right) / \partial x, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+ \tag{3.5.1}$$

were considered, where the function ψ tends to be constant when gradient modulus of u goes to infinity. If ψ becomes constant, then (3.5.1) passes into a quasi-linear hyperbolic equation, which causes the occurrence of discontinuities. Apparently, this effect was studied for the first time in [BdP], [dP], but only with growing initial data. In the future, the need to select fairly narrow classes of initial data remained, see, for example, [LTa]. In contrast to (3.5.1), equation (1.10) is more related to an extension of the theory of conservation laws, and in the literature is often called a Burgers-type equation with saturated dissipation. This type of equations was studied mainly from the point of view of particular properties of solutions, for example, [KuR], [GKR], and nonmonotonic functions Q, [KLR] were also considered. In addition, the computational aspect was also studied, for example, [CKR]. In the work of thesis's author [10], steps were taken to build a theory of the existence and uniqueness of solutions. Similar to the works of [BdP], [dP], the corresponding results are obtained under sufficiently strong restrictions on the class of initial data and the set of generalized solutions under consideration.

For equation (1.10) consider the initial data $u(0,x) = u_0(x)$, satisfying the following conditions: 1) $u_0 \in W^{2,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$; 2) there is a finite set of points $\{x_i\} \in K \subset \mathbb{R}$, where K is a compact, such that $u_0 \in C^2(K \setminus \{x_i\})$, and the points $\{x_i\}$ are discontinuity points of the first kind for u_0; 3) $u_0{}'(x_i \pm 0) = 0$.

Definition 3.5.1. We will say that a bounded measurable function $u(t,x)$ is a *generalized solution to the Cauchy problem* for (1.10) with an initial function u_0 if the following conditions are met:

a) there exist such a set $\mathcal{E} \subset [0,T]$, $\operatorname{mes} \mathcal{E} = 0$, that for $t \in [0,T] \setminus \mathcal{E}$ a continuous function $Q_{\lim}(t,x)$ is defined

$$Q_{\lim}(t,x) = \lim_{\substack{h\to 0 \\ h\in[0,\delta]\setminus\chi}} Q\left(\frac{u(t,x+h) - u(t,x-h)}{2h}\right),$$

where the value δ and the set χ, $\operatorname{mes} \chi = 0$, generally speaking, depend on x;

б) for an arbitrary function $\varphi \in C_0^\infty(\Pi_T)$, $\Pi_T = [0,T]\times\mathbb{R}$ the integral identity is fulfilled

$$\iint_{\Pi_T}\left\{u(t,x)\frac{\partial\varphi}{\partial t} + f\left(u(t,x)\right)\frac{\partial\varphi}{\partial x} - Q_{\lim}(t,x)\frac{\partial\varphi}{\partial x}\right\}dtdx = 0;$$

в) for any segment $[a,b]\subset\mathbb{R}$ it is completed

$$\lim_{\substack{t\to 0\\ t\in[0,T]\setminus\mathcal{E}}}\int_a^b\left|u(t,x)-u_0(x)\right|dx = 0.$$

Theorem 3.5.1. For initial data satisfying conditions 1) – 3), there is a generalized solution $u(t,x)$ of the Cauchy problem for (1.10), moreover, $u(t,x)\in BV_{\text{loc}}(\mathbb{R})$, $Q_{\lim}(t,x)\in BV_{\text{loc}}(\mathbb{R})$ for almost all $t\in[0,T]$.

Definition 3.5.2. Let

$$l(u)\equiv f\left(u^-\right)+\left(u-u^-\right)\frac{f\left(u^+\right)-f\left(u^-\right)}{u^+-u^-}.$$

We will say that a generalized solution $u(t,x)$ in the sense of Definition 3.5.1 *satisfies the condition* E (terminology of O.A. Oleinik), if for each point of discontinuity $u(t,x)$ there exist $u^\pm = u(t,x\pm 0)$ and $l(u)\ge f(u)$ for $u\in\left[u^+,u^-\right]$, $u^->u^+$; $l(u)\le f(u)$ for $u\in\left[u^-,u^+\right]$, $u^+>u^-$.

The uniqueness theorem given below is largely conditional in nature, since it is formulated for a rather narrow class of functions, although reflecting all the features of the problem.

Definition 3.5.3. We will say that a function $u(t,x)$ belongs to a class $\mathcal{K}$ if the following conditions are met:

а) $u(t,x)\in C^2(\Pi_T)$ for all $(t,x)\in\Pi_T$ except a finite set of curves $x_i(t)\in C^1(0,T)$; in addition, $\sup\limits_{[0,T]}\left|u(t,R)\right|\to 0$ при $|R|\to\infty$;

б) for any point of discontinuity, except for a finite number of them, there are unilateral limits $u\left(t,x_i(t)\pm 0\right)=u^\pm, u^+\ne u^-$;

в) $\sup\limits_{[0,T]}\left|Q_{\lim}(t,R)\right|\to 0$ при $|R|\to\infty$.

Theorem 3.5.2. For initial data satisfying conditions 1) – 3), the generalized solution $u(t,x)$ of the Cauchy problem for (1.10), which lies in the class $\mathcal{K}$ and satisfies the condition E, is unique.

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