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On syzygies of highest weight orbits

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Abstract. We consider the graded space R of syzygies for the coordinate algebra A of projective variety $X = G/P$ embedded into projective space as an orbit of the highest weight vector of an irreducible representation of semisimple complex Lie group G. We show that R is isomorphic to the Lie algebra cohomology $H = H^\bullet(L_{\geqslant 2}, \mathbb{C})$, where $L_{\geqslant 2}$ is graded Lie subalgebra of the graded Lie s-algebra $L = L_1 \oplus L_{\geqslant 2}$ Koszul dual to A. We prove that the isomorphism identifies the natural associative algebra structures on R and H coming from their Koszul and Chevalley DGA resolutions respectively. For subcanonically embedded X a Frobenius algebra structure on the syzygies is constructed. We illustrate the results by several examples including the computation of syzygies for the Plücker embeddings of grassmannians $\mathrm{Gr}(2, N)$.

Introduction

A. Losev brought to our attention the fact that some computations made by N. Berkovits in the framework of string theory[1] contain an intricate description of minimal resolution for the projective coordinate algebra of connected component of complex isotropic grassmannian $\mathrm{Gr}^+_{\mathrm{iso}}(5, 10)$ associated with non degenerate quadratic form on $\mathbb{C}^{10}$. Subsequent papers by M. Movshev, A. Schwarz and others[2] clarify the Berkovits computations as well as the interplay between the syzygies and a graded Lie superalgebra Koszul dual[3] to the coordinate algebra of $\mathrm{Gr}^+_{\mathrm{iso}}(5, 10)$. But the main focus of these papers remains with s-symmetric field theories, and the presentation of underlying mathematics seems to us rather tangled and overdiligent.

The main goal of these notes is to give relatively simple and clear presentation of the mathematics behind the sophisticated computations cited above and to illustrate it by some classical geometric examples. We tried to make the text

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[1]see [**1**], [**2**]

[2]see [**9**], [**32**], [**33**], [**34**], [**35**], etc.

[3]in the sense of [**17**]

self contained and understandable for advanced students. We believe that the subject is fruitful, and hope that this exposition could promote mutual understanding between mathematical and physical communities.

The paper consists of five sections. The first three of them are essentially independent. In §1 we collect necessary algebraic geometric properties of projective varieties

$$X = \mathbb{P}(G \cdot v_{\mathrm{h}w}) \subset \mathbb{P}(V)$$

that appear as the projectivization of the orbit of the highest weight vector in an irreducible representation V of a simply connected semisimple complex Lie group G. Class of these varieties clearly contains the isotropic grassmannians and serves a convenient generalization for the smooth[4] varieties of 'pure spinors' considered in s-symmetric theories cited above. In 1.3 we write down explicitly the quadratic equations generating the homogeneous ideal[5]

$$I(X) = \{f \in S(V^*) \mid f|_X \equiv 0\} \ .$$

In 1.2.2 we show that if X is subcanonical, i.e. $\omega_X = \mathscr{O}_X(-N)$ for some $N \in \mathbb{N}$, then non zero cohomologies $H^q(X, \mathscr{O}_X(m))$ can appear only for $q = 0$ or for $q = \dim X$.

In §2 we consider an arbitrary smooth[6] subcanonical projective variety satisfying the previous vanishing conditions on $H^q(X, \mathscr{O}_X(m))$. By §1, all subcanonical projective highest weight orbits belong to this class. Using Movshev's strategy from [**32**], [**35**], we show that for such X the space R of the syzygies for the graded projective coordinate algebra

$$A = S(V^*)/I(X) = \bigoplus_{m \geqslant 0} H^0\left(X, \mathscr{O}_X(m)\right)$$

inherits a natural structure of the Frobenius algebra. In other words, we construct an s-commutative multiplication on A and a trace form

$$A \xrightarrow{\mathrm{tr}} \mathbb{C}$$

such that the bilinear pairing $(a, b) = \mathrm{tr}\,(a \cdot b)$ is non degenerated. This generalizes and clarifies the duality isomorphisms constructed in [**32**], [**35**] for the smooth varieties of pure spinors.

In §3 we deal with an arbitrary commutative Koszul quadratic algebra A. All the coordinate algebras of projective highest weight orbits are of this sort due to Bezrukavnikov's result [**3**]. It is well known that such an algebra is canonically identified with the cohomology algebra

$$A \simeq H^\bullet(L, \mathbb{C})$$

[4]In fact, besides the grassmannians, Berkovits and others consider also a singular quadratic cone over $\mathrm{Gr}_{\mathrm{iso}}(5, 11)$, which is especially interesting for physicists because it serves some model of gravity. But the mathematical statements here are rather unclear to us and most likely require appropriate A_∞ enchantment of the smooth case. In these notes we restrict ourselves to the smooth homogeneous spaces only.

[5]these equations go back to Kostant and are known to specialists (comp. with [**28**], [**29**]); they have famous infinite dimensional extension developed by V. Kac and D. Peterson (see [**23**], [**24**]); for convenience of readers we sketch a short geometric proof for the finite dimensional case

[6]in fact all constructions and results of §2 hold for locally complete intersection varieties as well

of the graded Lie superalgebra $L = \bigoplus_{m\geqslant 1} L_m$ whose universal enveloping algebra $A^! = U(L)$ is Priddy dual to A. We show that there is an isomorphism of algebras

$$R \simeq H^\bullet(L_{\geqslant 2}, \mathbb{C}) ,$$

where the space of syzygies R is considered with the algebra structure constructed in §2, and $L_{\geqslant 2} = \bigoplus_{m\geqslant 2} L_m \subset L$ is graded Lie subalgebra started with L_2-component of L (the algebra structure on $H^\bullet(L_{\geqslant 2})$ is standard). The proof is based on a kind of differential perturbation lemma (see 3.5.1).

In §4 we illustrate the previous technique by several non trivial examples. Namely, we compute the syzygies of the most singular commutative quadratic algebra $A = \mathbb{T}(V^*)/\mathrm{Skew}(V^* \otimes V^*)$ (see 4.2.1), the syzygies of rational normal curves (see 4.3.1), and the syzygies of grassmannians $\mathrm{Gr}(2, N)$ under the Plücker embeddings (see 4.4.1). These results are also known for experts and can be extracted from [**18**], [**22**], [**37**], [**40**] and references therein. Our approach allows to treat all three examples uniformly: we use the description of the Lie algebra cohomology $H^\bullet(L_{\geqslant 2})$. In the first two examples (actually served by free Lie algebras) the computation is very simple and takes just a few rows. In the grassmannian case the algebra of syzygies is what we call *a hook algebra*. In the last §5 we collect generic properties of the hook algebras, in particular, we prove that each hook algebra is quadratic and koszul.

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1. Projective orbit of the highest weight vector

1.1. Basic notations. Let G be connected and simply connected complex semisimple algebraic group and $V = V_\lambda$ be its complex irreducible linear representation with a highest weight λ. We fix Cartan and Borel subgroups $T \subset B \subset G$ and write $v_{\mathrm{hw}} \in V$ for a highest weight vector and $P \subset G$ for a parabolic subgroup stabilizing 1-dimensional subspace $\mathbb{C}\cdot v_{\mathrm{hw}}$. In this part we consider the projectivization of the highest weight orbit

$$X = G/P \simeq \mathbb{P}(G \cdot v_{\mathrm{hw}}) \xhookrightarrow{\varphi} \mathbb{P}(V) , \tag{1.1}$$

which is a homogeneous G-space with the natural left action of G. We always put $\dim X = d$, $\dim V = n + 1$. Our especial interest is in the case when the canonical class of X is a negative integer multiple of the hyperplane section, i. e.

$$\omega_X = \mathscr{O}_X(-N) \quad \text{for some } N \in \mathbb{N} , \tag{1.2}$$

where $\mathscr{O}_X(1) = \varphi^*\mathscr{O}_{\mathbb{P}(V)}(1)$ is the ample line bundle coming from the projective embedding (1.1). We will call such a variety X *a subcanonical* highest weight orbit (or *a SHW-orbit* for shortness). To clarify this condition, let us fix some standard notations related to Lie algebras and recall some basic facts about vector bundles on homogeneous spaces.

1.1.1. *Lie algebra notations.* We denote by $\mathfrak{g}\supset\mathfrak{p}\supset\mathfrak{b}\supset\mathfrak{h}$ the Lie algebras of $G\supset P\supset B\supset T$ and write $\Lambda^{\mathrm{rt}}\subset\Lambda^{\mathrm{wt}}\subset\mathfrak{h}^*$ for the root and weight lattices of $\mathfrak{g}$ and $\Delta\subset\Lambda^{\mathrm{rt}}$ for the set of all *positive* roots. As usual, we put

$$\varrho=\frac{1}{2}\sum_{\alpha\in\Delta}\alpha$$

to be the half sum of all positive roots. Let $\{\alpha_i\}$ be the basic simple positive roots[7] and $\{\omega_i\}$ be the corresponding fundamental weights, which are dual to α_i w.r.t. the Killing form. Then we have

$$\lambda=\sum n_i\cdot\omega_i\quad\text{with integer } n_i\geqslant 0\,,\tag{1.3}$$

$$\mathfrak{p}=\mathfrak{b}\oplus\bigoplus_{\alpha\in\Delta^{\mathfrak{p}}}\mathfrak{g}_{-\alpha}\,,\quad\text{where }\Delta_{\mathfrak{p}}=\Delta\cap\lambda^{\perp}\tag{1.4}$$

(a simple root $\alpha_i\in\Delta_{\mathfrak{p}}$ iff the corresponding $n_i=0$). We write

$$\Lambda^{\mathrm{wt}}_{\mathfrak{p}}=\Delta^{\perp}_{\mathfrak{p}}\cap\Lambda^{\mathrm{wt}}=\{\,\mu\in\Lambda^{\mathrm{wt}}\mid(\mu,\alpha)=0\ \ \forall\,\alpha\in\Delta_{\mathfrak{p}}\,\}=\bigoplus_{i\,|\,n_i\neq 0}\mathbb{Z}\cdot\omega_i$$

for the set of all weights that produce the characters for $\mathfrak{p}$ and write $\langle\mu\rangle$ for the 1-dimensional P-module coming from such a character $\mu\in\Lambda^{\mathrm{wt}}_{\mathfrak{p}}$. We also put

$$\varrho_{\mathfrak{p}}=\frac{1}{2}\sum_{\alpha\in\Delta_{\mathfrak{p}}}\alpha\,.$$

Besides B, we will sometimes consider its opposite Borel subgroup B'. If $B=T\ltimes U$, where U is the unipotent part of B, then $B'=T\ltimes U'$ and $U'TU$ is a dense open subset in G. Similarly, we will write $\varrho'=-\varrho$ for the half sum of all the negative roots.

1.1.2. *Vector bundles on G/P.* With any representation E of P is associated a vector bundle

$$\widetilde{E}=G\underset{P}{\times}E$$

over $X=G/P$ with the fiber E. Its total space consists of pairs $(g,e)\in G\times E$ modulo the equivalence $(gp,e)\sim(g,pe)$ for $p\in P$. Global sections $X\xhookrightarrow{s_f}\widetilde{E}$ are naturally identified with functions $G\xrightarrow{f}E$ such that

$$pf(g)=f(gp^{-1})\quad\text{for all } p\in P\,.\tag{1.5}$$

The left G-action on G/P is extended canonically to the left G-action on the whole of $\widetilde{E}$ by the rule

$$g\cdot(g_1,e)=(gg_1,e)$$

and induces a linear representation of G in the space $\Gamma(X,\widetilde{E})$, of the global sections of $\widetilde{E}$. In terms of equivariant functions (1.5), in this representation an element $g\in G$ sends a function f to a function $g\cdot f$ defined by prescription

$$g\cdot f(g_1)=f(g^{-1}g_1)\,.\tag{1.6}$$

If the action of P on E can be extended to an action of G, then we have a vector bundle isomorphism $\widetilde{E}\xrightarrow{\sim}X\times E$, which takes an equivalence class of (g,e) to (g,ge).

[7] recall that $(\varrho,\alpha_i)=1$ for any simple root α_i

In particular, each 1-dimensional P-module $\langle\mu\rangle$, where $\mu \in \Lambda_{\mathfrak{p}}^{\mathrm{wt}}$, leads to the line bundle

$$[\mu] = G \underset{P}{\times} \langle\mu\rangle \ . \tag{1.7}$$

We will write $[\mu]$ for this bundle considered as an element of the Picard group[8] $\mathrm{Pic}(X)$ and will write $\mathscr{O}_X\langle\mu\rangle$ for the corresponding invertible sheaf of its local sections.

For example, 1-dimensional P-module $\langle\lambda\rangle = \mathbb{C}\cdot v_{\mathrm{hw}} \subset V_\lambda$, spanned by the highest weight vector in G-module V_λ, produces the tautological line subbundle

$$\mathscr{O}_X\langle\lambda\rangle = \mathscr{O}_X(-1) \subset \widetilde{V}_\lambda = X \times V_\lambda \ ,$$

which coincides with the restriction of the tautological line bundle $\mathscr{O}_{\mathbb{P}(V_\lambda)}(-1)$ onto X.

It is easy to see that the tangent bundle $T_X = G \underset{P}{\times} Q$ comes from the representation

$$Q = \mathfrak{g}/\mathfrak{p} = \bigoplus_{\alpha\in\Delta\smallsetminus\Delta_{\mathfrak{p}}} \mathfrak{g}_{-\alpha} \ .$$

Hence, the anticanonical line bundle is expressed in $\mathrm{Pic}(X)$ as the sum

$$\omega_X^* = \Lambda^d T_X = \sum_{\alpha\in\Delta\smallsetminus\Delta_{\mathfrak{p}}} [-\alpha] = -2\,(\varrho - \varrho_{\mathfrak{p}}) \ .$$

Thus, a projective orbit (1.1) is subcanonical in $\mathbb{P}(V_\lambda)$, i.e. satisfies (1.2), iff

$$2\,(\varrho - \varrho_{\mathfrak{p}}) = N\,\lambda \quad \text{for some } N \in \mathbb{N} \ . \tag{1.8}$$

This is significant restriction on λ (e.g. we will see in 1.2 that it forces quite strong vanishing condition on the cohomologies of invertible sheaves $\mathscr{O}_X(k)$).

In the next examples we use the standard Bourbaki notations from [**5**].

Example 1.1.3 (the grassmannian $\mathrm{Gr}(2,5)$). Let $G = \mathrm{SL}(5,\mathbb{C})$ with the diagonal torus $T \subset G$,

$$\mathfrak{h} = \{a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_5\varepsilon_5 \mid \textstyle\sum a_i = 0\} \ ,$$

and the simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leqslant i \leqslant 4$. Then the projective embedding

$$\mathrm{Gr}(2,5) \xrightarrow[\sim]{\varphi_\lambda} X = \mathbb{P}(G\cdot v_{\mathrm{hw}}) \subset \mathbb{P}(V_\lambda)$$

by means of the representation V_λ with the highest weight $\lambda = [0,m,0,0] = m\cdot\omega_2$, $m \geqslant 1$, is subcanonical only for $m = 1,\ 5$. Indeed, we have

$$\begin{aligned} \omega_2 &= (3\,\varepsilon_1 + 3\,\varepsilon_2 - 2\,\varepsilon_3 - 2\,\varepsilon_4 - 2\,\varepsilon_5)\,/5 \ , \\ 2\,\varrho &= 4\,\varepsilon_1 + 2\,\varepsilon_2 - 2\,\varepsilon_4 - 4\,\varepsilon_5 \ , \\ 2\,\varrho_{\mathfrak{p}} &= \varepsilon_1 - \varepsilon_2 + 2\,\varepsilon_3 - 2\,\varepsilon_5 \ . \end{aligned}$$

Thus, $\lambda = m\omega_2$ divides $2\,(\varrho - \varrho_{\mathfrak{p}}) = 5\,\omega_2$ only for $m = 1$ and $m = 5$. The former, non tautological, case gives the Plücker embedding $\mathrm{Gr}(2,5) \hookrightarrow \mathbb{P}_9$ and leads to $N = 5$ in (1.2).

[8] in fact, $\mathrm{Pic}(X)$ is spanned by the line bundles (1.7), see [**39**]

Example 1.1.4 (the group $G = \mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$). This group[9] has the diagonal torus $T \subset G$ with $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$, where

$$\mathfrak{h}' = \{a_1\varepsilon_1' + a_2\varepsilon_2' + a_3\varepsilon_3' \mid a_1 + a_2 + a_3 = 0\} ,$$
$$\mathfrak{h}'' = \{b_1\varepsilon_1'' + b_2\varepsilon_2'' \mid b_1 + b_2 = 0\} ,$$
$$2\,\varrho = 2\,\varepsilon_1' - 2\,\varepsilon_3' + \varepsilon_1'' - \varepsilon_2'' .$$

The representation $V_\lambda = V' \otimes V''$ (where V', V'' are the tautological 3- and 2-dimensional $\mathrm{SL}(3)$- and $\mathrm{SL}(2)$-modules) has

$$\lambda = \omega_1' + \omega_1'' = (2\,\varepsilon_1' - \varepsilon_2' - \varepsilon_3')\,/3 + (\varepsilon_1'' - \varepsilon_2'')\,/2 .$$

For $P = \mathrm{Stab}(v_\lambda)$ we have $2\varrho_\mathfrak{p} = \varepsilon_2' - \varepsilon_3'$, which gives

$$2\,(\varrho - \varrho_\mathfrak{p}) = 2\,\varepsilon_1' - \varepsilon_2' - \varepsilon_3' + \varepsilon_1'' - \varepsilon_2'' = 3\,\omega_1' + 2\,\omega_2'' .$$

We conclude that the tautological highest weight embedding of G/P is itself not subcanonical. But the embedding

$$G/P \xrightarrow[\sim]{\varphi_\mu} X = \mathbb{P}(G \cdot v_\mu) \subset \mathbb{P}(S^3V' \otimes S^2V'')$$

(corresponding to $V_\mu = S^3V' \otimes S^2V''$ with $\mu = 3\omega_1' + 2\omega_2''$) is subcanonical and (1.2) holds with $N = 1$.

Example 1.1.5 (even dimensional pure spinors). Let $G = \mathrm{Spin}(10,\mathbb{C})$ be the universal covering for $\mathrm{SO}(10,\mathbb{C})$ and $Y = \mathrm{Gr}^+_{\mathrm{iso}}(5,10)$ be a connected component of the grassmannian of 5-dimensional isotropic subspaces[10] in $\mathbb{C}^{10}$ (this is the case originally considered by Berkovits in [**1**]). Here $\mathfrak{g}$ is the semisimple Lie algebra of type D_5. Using the standard notations of Bourbaki (see [**5**]) as above, we can write

$$2\,\varrho = 8\,\varepsilon_1 + 6\,\varepsilon_2 + 4\,\varepsilon_3 + 2\,\varepsilon_4$$

and $Y = G/P$, where P has $2\,\varrho_\mathfrak{p} = 4\,\varepsilon_1 + 2\,\varepsilon_2 - 2\,\varepsilon_4 - 4\,\varepsilon_5$. This gives

$$2\,(\varrho - \varrho_\mathfrak{p}) = 4\,(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) .$$

The HW-orbit embedding corresponding to $\lambda = \omega_5 = (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_5)\,/2$

$$Y \xrightarrow[\sim]{\varphi_{\omega_5}} \mathbb{P}(G \cdot v_{\omega_5}) \subset \mathbb{P}(V_{\omega_5}) ,$$

is subcanonical with $N = 8$. More generally, for any m the variety of $2m$-dimensional pure spinors $\mathrm{Gr}^+_{\mathrm{iso}}(m, 2m)$ has subcanonical HW-embedding into $\mathbb{P}(V_\lambda)$ with

$$\lambda = (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_m)\,/2 .$$

Indeed,

$$2\,\varrho = 2\,(m-1)\,\varepsilon_1 + 2\,(m-2)\,\varepsilon_2 + \cdots + 2\,\varepsilon_{m-1}$$
$$2\,\varrho_\mathfrak{p} = (m-1)\,\varepsilon_1 + (m-3)\,\varepsilon_2 + \cdots - (m-1)\,\varepsilon_m ,$$

and (1.2) holds with $N = 2\,(m-1)$.

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[9] note that it coincides with the semisimple component for the complexification of the compact Lie group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(1)$

[10] since Y does also parameterize the decomposable elements of the Clifford algebra (see [**10**]), it is often called the variety *of 10-dimensional pure spinors*

1.2. Cohomologies of line bundles. The computation of cohomologies of line bundles on $X = G/P$ is reduced to the computation of cohomologies on the flag variety $Y = G/B$ via G-equivariant projection $Y \xrightarrow{\pi} X$. Namely, for each weight $\mu \in \Lambda_{\mathfrak{p}}^{\mathrm{wt}}$ we can consider the restriction $\langle\mu\rangle_B$, of the 1-dimensional P-module $\langle\mu\rangle$ onto B, and form a line bundle $[\mu]_B = G \underset{B}{\times} \langle\mu\rangle_B$, which is clearly isomorphic to the pull back of $[\mu]$ along π, i.e. $\pi^*\mathscr{O}_X\langle\mu\rangle = \mathscr{O}_Y\langle\mu\rangle_B$. Then, the Leray spectral sequence gives canonical isomorphisms

$$H^q(X, \mathscr{O}_X\langle\mu\rangle) \simeq H^q(Y, \mathscr{O}_Y\langle\mu\rangle_B)$$

for all q. By this reason in the rest of this section we replace P by B, X by Y and write simply $[\mu]$ and $\langle\mu\rangle$ instead of $[\mu]_B$, $\langle\mu\rangle_B$.

It is convenient to describe the representation of G in the space $\Gamma(Y, \mathscr{O}_Y\langle\mu\rangle)$ in terms of its *lowest* vector. Namely, the lowest weight of T on $\Gamma(Y, \mathscr{O}_Y\langle\mu\rangle)$ is μ and a lowest weight vector is unique up to proportionality. Indeed, if we interpret the sections of $\mathscr{O}_Y\langle\mu\rangle$ as the functions

$$G \xrightarrow{f} \langle\mu\rangle \simeq \mathbb{C}$$

via (1.5) and (1.6), then for a lowest weight function f we have $u' \cdot f = f$ for all $u' \in U'$ (see notations on page 4). Hence, for any $t \in T$, $b \in B$

$$t \cdot f(u'b) = f(t^{-1}u'b) = f(u''t^{-1}b) = f(t^{-1}b)\ ,$$

where $u'' = t^{-1}ut \in U'$ fixes f and $t^{-1}b \in B$. On the other hand, all three pairs

$$(\,t^{-1}b\,,\, f(t^{-1}b)\,) \sim (\,e\,,\, \mu(t^{-1}b)(f(t^{-1}b))\,) \sim (\,e\,,\, f(e)\,)$$

represent the same point in the total space of the bundle $\mathscr{O}_Y\langle\mu\rangle = G \underset{B}{\times} \langle\mu\rangle$. Hence,

$$f(t^{-1}b) = \mu(t^{-1}b)^{-1}f(e) = \mu(t)f(b)$$

(were we write $\mu(b)$ for the operator corresponding to $b \in B$ in the representation $\langle\mu\rangle$). This means that $t \cdot f = \mu(t)f$ over the open dense subset $U'B \subset G$ and, moreover, f is uniquely defined there by its value $f(e)$. So, the weight of f is μ and this weight subspace is 1-dimensional.

1.2.1. *The Borel–Weil–Bott theorem,* being formulated in our notations, describes all G-modules $H^q(Y, \mathscr{O}_Y\langle\mu\rangle)$ as irreducible representations presented by their *lowest* weights[11]. Namely, given $\mu \in \Lambda^{\mathrm{wt}}$, we consider its shift $\mu + \varrho'$, by the half sum all the *negative* roots. There are two possibilities:

(1) $\mu + \varrho'$ lies in the interior part of some Weyl chamber C
(2) $\mu + \varrho'$ belongs to a wall separating the Weyl chambers

In the first case $(\alpha, \mu + \varrho') \neq 0\ \ \forall\, \alpha \in \Delta$ and there exist a unique weight μ' in the *lowest* Weyl chamber C_{low} and a unique element w of the Weyl group such that

$$\mu + \varrho' = w(\mu' + \varrho') \tag{1.9}$$

[11]the formulation most commonly used in the representation theory (see, for example, [**21**]) actually describes G-modules $H^q(\mathscr{O}\langle\mu\rangle)$ in terms highest weights but the underlying homogeneous space is always taken to be G/B', that is the lowest weight vector orbit

(indeed, w has to be the symmetry that takes C_{low} to C and then μ' is determined uniquely). In this case $H^q(Y, \mathcal{O}_Y\langle\mu\rangle) \neq 0$ iff q equals the length[12] of w in the Weyl group. This non zero space is an irreducible G-module of the *lowest* weight μ'.

In the second case $\mu + \varrho'$ is orthogonal to some root $\alpha \in \Delta$ and the equation (1.9) is unsolvable in a sense that $\mu + \varrho'$ is not congruent to any weight in the interior part of C_{low} modulo the Weyl group action. In this case $H^q(Y, \mathcal{O}_Y\langle\mu\rangle) = 0$ for all q.

For example, the above description of $\Gamma(Y, \mathcal{O}_Y\langle\mu\rangle)$ fits the Borel–Weil–Bott setup as the case when $\mu' = \mu$, $w = e$, $q = 0$. In particular, $\Gamma(Y, \mathcal{O}_Y\langle\mu\rangle) \neq 0$ iff μ lies in the lowest weights Weyl chamber C_{low}.

Proposition 1.2.2. *If $X = G/P$ is a d-dimensional SHW-orbit with $\omega_X = \mathcal{O}_X(-N)$, then all the non zero cohomologies $H^q(X, \mathcal{O}_X(k))$ are only*

$$H^0(X, \mathcal{O}_X(m)) \qquad \text{and} \qquad H^d(X, \mathcal{O}_X(-N-m)) \ ,$$

where $m \geqslant 0$ in the both cases.

Proof. Since λ is a highest weight, $-m\lambda \in C_{\text{low}}$ lies in the lowest weights chamber for all $m \geqslant 0$. So, for all $\mathcal{O}_X(m) = \mathcal{O}_X\langle -m\lambda\rangle$ we have $H^0(X, \mathcal{O}_X(m)) \neq 0$ and $H^q(X, \mathcal{O}_X(m)) = 0$ when $q > 0$. By the Serre duality, this implies

$$H^d(X, \mathcal{O}_X(-N-m)) = H^0(X, \mathcal{O}_X(m))^* \neq 0$$

and vanishing of all $H^q(X, \mathcal{O}_X(-N-m))$ with $q \neq d$.

To manage the remaining values $m = 1, 2, \dots, (m-1)$, let us note that in the Borel–Weil–Bott setup (see 1.2.1) the triviality of the representation

$$H^d(X, \mathcal{O}_X(-N)) = H^0(X, \mathcal{O}_X)^* = \mathbb{C}$$

means that $N\lambda + \varrho' = w(\varrho')$ for some w from the Weyl group. So, all the weights

$$\lambda + \varrho' \,,\ 2\lambda + \varrho' \,,\ \dots \,,\ (N-1)\lambda + \varrho' \tag{1.10}$$

are internal points of the segment $I = \{\, x\lambda + \varrho' \mid 0 \leqslant x \leqslant N \,\}$ whose endpoints are ϱ' and $N\lambda + \varrho' = w(\varrho')$. The both endpoints have Euclidean length $||\varrho'|| = \sqrt{(\varrho', \varrho')}$, which is the minimal length of the weights lying in the interior part of a Weyl chamber. By the convexity arguments, all the interior points of I lay strictly closer to the origin. So, the weights (1.10) can not be interior points of a chamber. Hence they are not congruent to the interior points of C_{low} modulo the Weyl group action and we deal with the second case of the Borel–Weil–Bott theorem (in the sense of 1.2.1). Therefore, all the cohomologies $H^q(X, \mathcal{O}\langle -m\lambda\rangle)$ do vanish for $1 \leqslant m \leqslant (N-1)$. □

1.3. Quadratic equations for X. In this section we write explicit quadratic equations generating the homogeneous ideal of the projectivization of an arbitrary highest weight vector orbit (1.1) (not necessary subcanonical). Infinite dimensional versions of two propositions below were proved by Kac and Peterson in [**23**], [**24**]. Finite dimensional case goes back to Kostant (comp. with [**28**], [**29**]). For the convenience of readers we sketch here an easy finite dimensional proof.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Consider the Casimir element

$$\Omega = \sum a_i b_i \in U(\mathfrak{g}) \ , \tag{1.11}$$

[12] here the length should be defined w.r.t. the reflections by the walls of the lowest chamber C_{low}

where a_i and b_i form a pair of dual bases of $\mathfrak{g}$ w.r.t. the Killing form. It does not depend on the choice of dual bases, lays in the center of $U(\mathfrak{g})$, and acts on each irreducible $\mathfrak{g}$-module by a scalar operator:

$$\Omega|_{V_\lambda} = c_\lambda \mathrm{Id}_{V_\lambda} .$$

The constant c_λ can be computed by the following formula (see, for example, [**14**])

$$c_\lambda = (\lambda+\varrho\,,\,\lambda+\varrho) - (\varrho\,,\,\varrho) = (\lambda+2\varrho\,,\,\lambda)\ , \tag{1.12}$$

where ϱ is a half sum of all positive roots, and we use the scalar product on $\mathfrak{h}^*$ induced by the Killing form.

With the Casimir element (1.11) it is associated an operator Ω_2 acting on a tensor product $V'\otimes V''$ of any two $\mathfrak{g}$-modules V', V'' by the rule

$$\Omega_2(v'\otimes v'') = \sum a_i(v')\otimes b_i(v'')\ , \tag{1.13}$$

which clearly does not depend on the choice of dual bases a_i, b_i for $\mathfrak{g}$. The actions of Ω and Ω_2 are related by the formula

$$\Omega(v'\otimes v'') = \Omega(v')\otimes v'' + 2\,\Omega_2(v'\otimes v'') + v'\otimes\Omega(v'') \tag{1.14}$$

Applying this formula to $v_{\mathrm{hw}}\otimes v_{\mathrm{hw}}\in V_\lambda\otimes V_\lambda$, which is the highest vector of weight 2λ, we get

$$\begin{aligned} 2\,\Omega_2(v_{\mathrm{hw}}\otimes v_{\mathrm{hw}}) &= (c_{2\lambda}-2c_\lambda)\cdot v_{\mathrm{hw}}\otimes v_{\mathrm{hw}} = \\ &= \big(4(\lambda+\varrho\,,\,\lambda) - 2\,(\lambda+2\varrho\,,\,\lambda)\big)\cdot v_{\mathrm{hw}}\otimes v_{\mathrm{hw}} = 2\,(\lambda,\lambda)\cdot v_{\mathrm{hw}}\otimes v_{\mathrm{hw}}\ . \end{aligned}$$

At the same time the formula (1.14) shows that Ω_2 does commute with $\mathfrak{g}$-action, because the Casimir element Ω does. We conclude that for any $x\in G\cdot v_{\mathrm{hw}}\subset V_\lambda$

$$\Omega(x\otimes x) = c_{2\lambda}\cdot x\otimes x = 4\,(\lambda+\varrho\,,\,\lambda)\cdot x\otimes x\ , \tag{1.15}$$

$$\Omega_2(x\otimes x) = (\lambda\,,\,\lambda)\cdot x\otimes x\ . \tag{1.16}$$

PROPOSITION 1.3.1. *The following four statements about $x\in V_\lambda$ are pairwise equivalent:*

(1) *$x\in G\cdot v_{\mathrm{hw}}$;*
(2) *$x\otimes x$ lays in the irreducible component $V_{2\lambda}\subset V_\lambda\otimes V_\lambda$;*
(3) *$\Omega(x\otimes x) = 4\,(\lambda+\varrho\,,\,\lambda)\cdot x\otimes x$;*
(4) *$\Omega_2(x\otimes x) = (\lambda\,,\,\lambda)\cdot x\otimes x$.*

PROOF. We have seen already that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longleftrightarrow$ (4). The implication (3) $\Longrightarrow$ (1) follows from the next more precise statement. $\square$

PROPOSITION 1.3.2. *The quadratic equations* (1.15) *generate the homogeneous ideal of the projective variety $X = \mathbb{P}(G\cdot v_{\mathrm{hw}})\subset\mathbb{P}(V)$.*

PROOF. As a $\mathfrak{g}$-module, the whole coordinate algebra of $\mathbb{P}(V_\lambda)$ is isomorphic to $S^\bullet(V_\lambda^*)\simeq S^\bullet(V_\mu)$, where $\mu = -w_{\max}(\lambda)$ is the highest weight of V_λ^* and $w_{\max}$ is the maximal length element in the Weyl group. It is easy to check (comp. with [**14**]) that its q-th homogeneous component S^qV_μ splits as

$$S^qV_\mu = V_{q\mu}\oplus(\Omega - c_{q\mu}\mathrm{Id})\cdot S^qV_\mu\ , \tag{1.17}$$

because the highest weight $q\mu$ appears in S^qV_μ with the multiplicity one and the eigenvalues of Ω on the irreducible submodules $V_\nu \subset S^qV_\mu$ with $\nu < q\mu$ are strictly less than $c_{q\mu}$. Let us write

$$J_q = (\Omega - c_{q\mu}\mathrm{Id}) \cdot S^qV_\mu$$

for the right summand in (1.17), which collects all irreducible submodules V_ν with $\nu < q\mu$. Since all the weights of $S^pV_\mu \cdot J_q \subset S^{p+q}V_\mu$ are strictly less then $(p+q)\mu$, it is clear that

$$J = \bigoplus_q J_q$$

is a homogeneous ideal in $S^\bullet V_\mu$. We would like to check that J is generated by J_2.

The key point is that for any $v \in V_\mu$ we have

$$[\Omega - c_{q\mu}\mathrm{Id}]\,(v^q) = \frac{q(q-1)}{2}\,[\Omega - c_{2\mu}\mathrm{Id}]\,(v^2)\cdot v^{q-2}\,. \tag{1.18}$$

Indeed, using the Leibnitz rule and the relation (1.14), we get

$$\begin{aligned}\Omega(v^q) &= q\cdot\Omega(v)\cdot v^{q-1} + q(q-1)\cdot\Omega_2(v^2)\cdot v^{q-2} = \\ &= q\cdot c_\mu\cdot v^q + \frac{q(q-1)}{2}\cdot\big(\Omega(v^2) - 2\cdot\Omega(v)\cdot v\big)\cdot v^{q-2} = \\ &= \big(-q^2+2q\big)\cdot c_\mu\cdot v^q + \frac{q(q-1)}{2}\cdot\Omega(v^2)\cdot v^{q-2}\,.\end{aligned}$$

This reduces (1.18) to the purely numerical identity

$$\big(q^2-2q\big)\,c_\mu + c_{q\mu} = \frac{q(q-1)}{2}c_{2\mu}\,,$$

which is verified by straightforward computation using (1.12).

Since the powers v^q span S^qV_μ as a linear space, the identity (1.18) says actually that the ideal J is generated by its quadratic component

$$J_2 = (\Omega - c_{2\mu}\mathrm{Id})\cdot S^2V_\mu\,.$$

Taking into account the equality $c_{2\lambda} = c_{2\mu}$, we see that these quadratic equations coincide with (1.15) as well as with ones mentioned in the condition (3) of the previous proposition.

Now, rewriting the decomposition (1.17) as $S^qV_\mu = V_{q\mu} \oplus J_q$, we see immediately that any $\mathfrak{g}$-invariant ideal $I \supsetneq J$ should contain the irreducible submodule $V_{q\mu}$ for all $q \gg 0$, i.e. should be of finite codimension in $S^\bullet(V_\mu)$ as a vector space. This means that $J = \sqrt{J}$ coincides with the homogeneous ideal of the projective variety $Y \subset \mathbb{P}(V_\lambda)$ defined by the quadratic equations (1.15). Moreover, this means that Y does not contain proper $\mathfrak{g}$-invariant closed algebraic subsets. Since any G-orbit of minimal dimension inside $Y \smallsetminus X$ would be such a subset, we conclude that $Y = X$. □

2. Syzygies of the projective coordinate algebra

The content of this section was influenced by our discussions with M. Movshev. We streamline and clarify arguments used in [**35**], [**32**] and construct the Frobenius algebra structure on the syzygies of an arbitrary projective variety $X \subset \mathbb{P}(V)$ satisfying the following three properties

(1) X is smooth[13];
(2) for $m \in \mathbb{Z}$ the cohomologies $H^i(X, \mathscr{O}_X(m)) = 0$, if $i \neq 0$, $d = \dim X$;
(3) X is subcanonical, i.e. $\omega_X = \mathscr{O}_X(-N)$ for some $N \in \mathbb{N}$.

As we have seen in §1, these conditions hold for the SHW-orbits (1.1). So, the syzygy space of any SHW-orbit always carries a Frobenius algebra structure.

2.1. Coordinate algebra of a projective variety. By the definition, the coordinate algebra of a projective variety $X \subset \mathbb{P}_n = \mathbb{P}(V)$ is the graded algebra

$$A = \bigoplus_{m \geqslant 0} H^0(X, \mathscr{O}_X(m)) = S/J \ , \tag{2.1}$$

where $S = \bigoplus_{m \geqslant 0} S^m V^*$ is the symmetric algebra of V^* (the homogeneous coordinate algebra of $\mathbb{P}(V)$) and $J = \{f \in S \mid f|_X \equiv 0\}$ is the homogeneous ideal of X.

In terms of generators and relations, such an algebra A is described by its *minimal free resolution*

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0 \ , \tag{2.2}$$

which is an exact sequence of graded free S-modules of the form

$$F_p = \bigoplus_{q \geqslant m_p} R_{p,q} \underset{\mathbb{C}}{\otimes} S[-q] \, , \tag{2.3}$$

where $R_{p,q}$ are finite dimensional vector spaces of *p-th order syzygies of degree q* for A and we write m_p for the minimal degree appearing among the order p syzygies. We will call p the *homological* degree and q — the *internal* degree.

The *minimality* of the resolution (2.2) means that all homogenous components of all matrix elements of each its differential are polynomials of strictly positive degree, i.e. for any p the differential $F_p \longrightarrow F_{p-1}$ takes each syzygy submodule $R_{p,q} \otimes S[-q]$ into $\bigoplus_{\nu \leqslant q-1} R_{p-1,\nu} \otimes S[-\nu]$. Thus, the tensor multiplication by the trivial S-module $\mathbb{C}$ annihilates all the differentials in a minimal free resolution (2.2) and we get for each p an isomorphism of graded vector spaces

$$R_p \overset{\text{def}}{=} \bigoplus_{q \geqslant m_p} R_{p,q} = \operatorname{Tor}_p^S(A, \mathbb{C}) \ . \tag{2.4}$$

In particular, the dimensions $\dim R_{p,q}$ do non depend on the choice of a minimal resolution.

Example 2.1.1 (the HW-orbits). If $X = G/P$ is embedded into $\mathbb{P}(V)$ as the highest vector orbit (1.1), then its ideal $J = (Q)$ is generated by the quadratic equations (1.15), which form a linear subspace

$$Q \subset S^2 V^* \subset S \ .$$

Thus, the resolution (2.2) starts with $F_0 = S$, i.e. $m_0 = 0$, $R_0 = R_{0,0} = \mathbb{C}$. Then, $F_1 = Q \otimes S[-2]$, i.e. $m_1 = 2$ and $R_1 = R_{1,2} = Q$. Further, it follows from the minimality that $m_p \geqslant p+1$ for all $p \geqslant 1$.

Note that the group G acts naturally on the coordinate algebra A and on the Tor-spaces, thus on the syzygies. During the proof of the proposition 1.3.2 we have seen that $Q \subset S^2V^*$ collects all the irreducible direct summands of S^2V^* except for $V_{2\mu}$, where μ is the highest weight of V^*. This gives an effective way to compute at least the starting term of minimal resolution.

[13] in fact, all results of this section (and their proofs) hold for any locally complete intersection varieties (see [**19**], [**12**] for details); all we need is well defined invertible dualizing sheaf ω_X

Say, for the Plücker embedding $\mathrm{Gr}(2,5) \simeq \mathbb{P}(G \cdot v_{\omega_2}) \subset \mathbb{P}(V_{\omega_2}) = \mathbb{P}_9$, which corresponds to $\lambda = \omega_2 = [0,1,0,0]$ in the notations of 1.1.3, we have $\mu = \omega_3 = [0,0,1,0]$, thus $2\mu = [0,0,2,0]$, and $S^2V^* = V_{[0,0,2,0]} \oplus V_{[0,0,0,1]}$ (this is immediate from $\dim S^2\mathbb{C}^{10} = 55$). Hence, $Q = V_{[0,0,0,1]}$ and $\dim Q = 5$.

Similarly, for the variety of 10-dimensional pure spinors $\mathrm{Gr}^+_{\mathrm{iso}}(5,10)$ (see example 1.1.5) we have $\mu = \lambda = \omega_5$, $\dim V_\lambda = 16$, $\dim S^2V^* = 136$, but $\dim V_{2\mu} = 126$. This implies $Q = V_{\omega_1}$, $\dim V_{\omega_1} = 10$. Note that in the both cases, $\mathrm{Gr}(2,5) \subset \mathbb{P}_9$ and $\mathrm{Gr}^+_{\mathrm{iso}}(5,10) \subset \mathbb{P}_{15}$, the quadratic equations Q form an irreducible G-module.

2.2. DGA resolution for the algebra of syzygies. The syzygies $R_{p,q}$ can be computed using the standard Koszul resolution for the trivial S-module $\mathbb{C}$

$$\cdots \xrightarrow{d_{\mathrm{K}}} K_2 \xrightarrow{d_{\mathrm{K}}} K_1 \xrightarrow{d_{\mathrm{K}}} K_0 \xrightarrow{d_{\mathrm{K}}} \mathbb{C} \longrightarrow 0 \tag{2.5}$$

that has $K_p = \Lambda^pV^* \underset{\mathbb{C}}{\otimes} S[-p]$ and the differential $d_{\mathrm{K}} = \sum \frac{\partial}{\partial\vartheta_i} \otimes x_i$, which takes

$$\omega \otimes f \longmapsto \sum \frac{\partial \omega}{\partial \vartheta_i} \otimes x_i \cdot f\,, \tag{2.6}$$

where $\vartheta_0, \vartheta_1, \dots, \vartheta_n$ is some basis in V^* considered inside the exterior algebra ΛV^* and $x_0, x_1, \dots, x_n$ is the same basis of V^* but considered inside the symmetric algebra SV^*. The derivation $\partial/\partial\vartheta_i$ takes[14]

$$\vartheta_I = \vartheta_{i_1} \wedge \vartheta_{i_2} \wedge \cdots \wedge \vartheta_{i_k} \longmapsto (-1)^{\nu-1}\vartheta_{I \smallsetminus i}\,, \quad \text{when } i = i_\nu \in I\,,$$

and annihilates all ϑ_J with $J \not\ni i$.

Tensoring the Koszul resolution (2.5) by A over S, we get a complex

$$\begin{aligned} 0 \longrightarrow \Lambda^{n+1}V^* \underset{\mathbb{C}}{\otimes} A[-n-1] \xrightarrow{\partial} \cdots \\ \cdots \xrightarrow{\partial} \Lambda^2V^* \underset{\mathbb{C}}{\otimes} A[-2] \xrightarrow{\partial} V^* \underset{\mathbb{C}}{\otimes} A[-1] \xrightarrow{\partial} A \longrightarrow 0\,, \end{aligned} \tag{2.7}$$

whose differential is given by the same formula (2.6) considered modulo the quadratic relations $(Q) \subset S$. The p-th homology group of (2.7) coincides with

$$\mathrm{Tor}^S_p(A, \mathbb{C}) = R_p$$

from (2.4). The Koszul complex (2.7) can be considered from two different viewpoints. First of all, taking the direct sum of its elements, we get a DG-algebra

$$\mathscr{A} = \underset{p\geqslant 0}{\oplus} \mathscr{A}_p\,, \quad \mathscr{A}_p = \Lambda^pV^* \underset{\mathbb{C}}{\otimes} A[-p]\,, \tag{2.8}$$

whose multiplication is induced by exterior and symmetric multiplication of the tensor factors:

$$(\omega \otimes f) \cdot (\eta \otimes g) = (-1)^{|f||\eta|}(\omega \wedge \eta) \otimes (fg)\,,$$

where we write $|x| \in \mathbb{Z}/2\mathbb{Z}$ for the parity of x induced by the internal degree of x. It satisfies $a \cdot b = (-1)^{|a|\cdot|b|}\, b \cdot a$ for homogeneous[15] $a \in \mathscr{A}_{|a|}$, $b \in \mathscr{A}_{|b|}$ and agrees with the differential (2.6):

$$\partial(a \cdot b) = (\partial a) \cdot b + (-1)^{|a|}\, a \cdot (\partial b)\,.$$

---

[14] note that grassmann partial derivatives skew commute $\frac{\partial}{\partial\vartheta_i}\frac{\partial}{\partial\vartheta_j} = -\frac{\partial}{\partial\vartheta_j}\frac{\partial}{\partial\vartheta_i}$ and satisfy the graded Leibniz rule $\frac{\partial}{\partial\vartheta_i}(\omega_1 \wedge \omega_2) = \left(\frac{\partial}{\partial\vartheta_i}\omega_1\right) \wedge \omega_2 + (-1)^{|\omega_1|}\omega_1 \wedge \left(\frac{\partial}{\partial\vartheta_i}\omega_2\right)$

[15] here and further we write $|x| = p$ for the degree of a homogeneous element $x \in \mathscr{A}_p$

This implies that the syzygies $R = \bigoplus_p R_p = H(\mathscr{A})$ also inherit an associative algebra structure.

We are going to show that this structure is *graded Frobenius*, i.e. there is a non degenerated scalar product

$$R \underset{\mathbb{C}}{\otimes} R \xrightarrow{a\otimes b\mapsto (a,b)} \mathbb{C}$$

such that $(a, b\cdot c) = (a\cdot b, c)$ and $(a,b) = \pm(b,a)$, where the sign rule depends on the parity of $\operatorname{codim} X$ (see page 16). This scalar product can be written as $(a,b) = \operatorname{tr}(a\cdot b)$, where

$$\operatorname{tr}\, : R \xrightarrow{a\mapsto(e,a)=(a,e)} \mathbb{C} \tag{2.9}$$

is some natural *trace form* that comes from well known geometric construction, which will be described in the next section using another viewpoint on the Koszul complex (2.7).

Namely, let us fix some isomorphism[16] $\Lambda^{n+1}V \xrightarrow[\sim]{\delta} \mathbb{C}$, which performs to identify $(\Lambda^p V)^*$ with $\Lambda^{n+1-p}V$ via non degenerated pairing

$$\Lambda^p V \otimes \Lambda^{n+1-p} V \xrightarrow{\omega\otimes\eta\mapsto\omega\wedge\eta} \Lambda^{n+1}V \xrightarrow[\sim]{\delta} \mathbb{C}\,. \tag{2.10}$$

Further, we identify $\Lambda^p(V^*)$ with $(\Lambda^p V)^*$ via non degenerated pairing induced by lifting $\Lambda^p(V^*)$ into $(V^*)^{\otimes p}$, $\Lambda^p V$ into $V^{\otimes p}$, and taking the complete contraction[17]. It is easy to see that under this identification the Koszul complex (2.7) turns to the complex

$$\begin{aligned} 0 \longrightarrow A[-n-1] \xrightarrow{\partial} V \underset{\mathbb{C}}{\otimes} A[-n] \xrightarrow{\partial} \Lambda^2 V \underset{\mathbb{C}}{\otimes} A[-n+1] \xrightarrow{\partial} \cdots \\ \cdots \xrightarrow{\partial} \Lambda^{n-1}V \underset{\mathbb{C}}{\otimes} A[-2] \xrightarrow{\partial} \Lambda^n V \underset{\mathbb{C}}{\otimes} A[-1] \xrightarrow{\partial} \Lambda^{n+1} V \underset{\mathbb{C}}{\otimes} A \longrightarrow 0\,, \end{aligned} \tag{2.11}$$

whose components also form an associative graded superalgebra[18]

$$\mathscr{A}' = \bigoplus_p \Lambda^p V \underset{\mathbb{C}}{\otimes} A[p-n-1]$$

w.r.t. the multiplication induced by exterior multiplication in ΛV and the usual one in A. The differential ∂ sends a homogeneous element $a \in \mathscr{A}_p$ to $p\,\mathrm{Id}_V \cdot a$, where $\mathrm{Id}_V \in V \otimes V^*$ is considered as an element of $V \otimes A[-n] \subset \mathscr{A}'$ and the multiplication is taken inside $\mathscr{A}'$. Note that ∂ is no longer compatible with the multiplication. However the interpretation (2.11) reveals projective geometrical meaning of ∂.

2.3. Euler-Dolbeault bicomplex. There is canonical Euler exact triple of coherent sheaves on $\mathbb{P}_n = \mathbb{P}(V)$

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}_n} \xrightarrow{1\mapsto\sum \vartheta_i^*\otimes x^i} V \otimes \mathscr{O}_{\mathbb{P}_n}(1) \longrightarrow T \longrightarrow 0\,,$$

which describes the tangent sheaf $T = T_{\mathbb{P}_n}$ over any point $p \in V$ as the factor space $V/\mathbb{C}\cdot p$. The exterior powers of this triple

$$0 \longrightarrow \Lambda^{m-1}T \longrightarrow \Lambda^m V \otimes \mathscr{O}_{\mathbb{P}_n}(m) \longrightarrow \Lambda^m T \longrightarrow 0$$

---

[16] recall that $\dim V = n+1$ and $\mathbb{P}_n = \mathbb{P}(V)$

[17] in terms of dual bases $\vartheta_i^* \in V$, $\vartheta_i \in V^*$, the full contraction between ϑ_I^* and ϑ_J equals $\delta_{IJ}\cdot\frac{1}{p!}$, where p is the degree of the monomials

[18] in a physical cant the exchange $\mathscr{A} \leftrightsquigarrow \mathscr{A}'$ is known as 'odd Fourier transform'

are naturally organized into a long exact sequence of locally free $\mathscr{O}_{\mathbb{P}_n}$-modules

$$(2.12)\quad 0 \longrightarrow \mathscr{O}_{\mathbb{P}_n} \longrightarrow V\otimes\mathscr{O}_{\mathbb{P}_n}(1) \longrightarrow \Lambda^2V\otimes\mathscr{O}_{\mathbb{P}_n}(2) \longrightarrow \cdots$$
$$\cdots \longrightarrow \Lambda^{n-1}V\otimes\mathscr{O}_{\mathbb{P}_n}(n-1) \longrightarrow \Lambda^nV\otimes\mathscr{O}_{\mathbb{P}_n}(n) \longrightarrow \mathscr{O}_{\mathbb{P}_n}(n+1) \longrightarrow 0$$

whose maps are given by the left multiplication by $\mathrm{Id}_V=\sum\vartheta_i^*\otimes x_i$, where $\vartheta_i^*\in V$ form the dual base to $\vartheta_i\in V^*$ but $x_i\in V^*$ are considered now as global sections of $\mathscr{O}(1)$, and 'the multiplication' means exterior multiplication in the first factor and tensor multiplication in the second one. Hence, twisting by $\mathscr{O}_{\mathbb{P}_n}(k-n-1)$ and using the identification $\Lambda^pV\simeq\Lambda^{n+1-p}V^*$ described above, we can rewrite (2.12) as

$$(2.13)\quad 0 \longrightarrow \mathscr{O}_{\mathbb{P}_n}(k-n-1) \xrightarrow{d_{\mathrm{K}}} \Lambda^nV^*\otimes\mathscr{O}_{\mathbb{P}_n}(k-n) \xrightarrow{d_{\mathrm{K}}} \cdots$$
$$\cdots \xrightarrow{d_{\mathrm{K}}} \Lambda^2V^*\otimes\mathscr{O}_{\mathbb{P}_n}(k-2) \xrightarrow{d_{\mathrm{K}}} V^*\otimes\mathscr{O}_{\mathbb{P}_n}(k-1) \xrightarrow{d_{\mathrm{K}}} \mathscr{O}_{\mathbb{P}_n}(k) \longrightarrow 0\,,$$

where the differential $\Lambda^pV^*\otimes\mathscr{O}_{\mathbb{P}_n}(k-p) \xrightarrow{d_{\mathrm{K}}} \Lambda^{p-1}V^*\otimes\mathscr{O}_{\mathbb{P}_n}(k-p+1)$ is given by the same formula (2.6).

Since X is smooth, restricting (2.13) onto X we get the following exact sequence of locally free coherent sheaves on X:

$$(2.14)\quad 0 \longrightarrow \mathscr{O}_X(k-n-1) \xrightarrow{\partial} \Lambda^nV^*\otimes\mathscr{O}_X(k-n) \xrightarrow{\partial} \cdots$$
$$\cdots \xrightarrow{\partial} \Lambda^2V^*\otimes\mathscr{O}_X(k-2) \xrightarrow{\partial} V^*\otimes\mathscr{O}_X(k-1) \xrightarrow{\partial} \mathscr{O}_X(k) \longrightarrow 0\,.$$

Note that the Koszul complex (2.7) is a direct sum of complexes obtained from (2.14) by applying the global sections functor $\Gamma(X,*)$.

Now consider the flabby Dolbeault $\overline{\partial}$-resolutions[19] for all coherent sheaves in (2.14). They are organized in the exact bicomplex of flabby sheaves of abelian groups $\mathscr{E}_p^q=\Lambda^pV^*\otimes\Omega_X^{0,q}(k-p)$ on X:

$$(2.15)\quad \begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle\overline{\partial}} & & \uparrow{\scriptstyle\overline{\partial}} & & \\
\cdots & \xrightarrow{\partial} & \Lambda^pV^*\otimes\Omega_X^{0,q+1}(k-p) & \xrightarrow{\partial} & \Lambda^{p-1}V^*\otimes\Omega_X^{0,q+1}(k-p+1) & \xrightarrow{\partial} & \cdots \\
 & & \uparrow{\scriptstyle\overline{\partial}} & & \uparrow{\scriptstyle\overline{\partial}} & & \\
\cdots & \xrightarrow{\partial} & \Lambda^pV^*\otimes\Omega_X^{0,q}(k-p) & \xrightarrow{\partial} & \Lambda^{p-1}V^*\otimes\Omega_X^{0,q}(k-p+1) & \xrightarrow{\partial} & \cdots \\
 & & \uparrow{\scriptstyle\overline{\partial}} & & \uparrow{\scriptstyle\overline{\partial}} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

Writing $\mathscr{E}_p^q$ in the $(-p,q)$-cell of the second coordinate quadrant of (p,q)-plane, we get a diagram bounded by inequalities $-n-1\leqslant -p\leqslant 0$, $0\leqslant q\leqslant d$, where $d=\dim X$. It has the following obvious properties:

[19]We use the Dolbeault complex just by tradition. In fact, any natural flabby right resolution $\mathscr{E}_p^q$ of complex (2.14) would be OK for the forthcoming computation. Say, the usage of the canonical Godemant resolution could make the computation even more transparent and works for non smooth locally complete intersection varieties as well.

(1) All the rows of (2.15) are exact and acyclic w.r.t. the functor $\Gamma(X, *)$. In particular, taking the global sections in (2.15), we get a bicomplex $\Gamma(X, \mathscr{E}_p^q)$, with exact rows, whose associated total complex is exact.
(2) For each p the p-th column of (2.15) gives a flabby resolution for the coherent sheaf $\Lambda^p V^* \otimes \mathscr{O}(k-p)$. So, applying $\Gamma(X, *)$ to p-th column of (2.15), we get a complex whose q-th cohomology group equals $\Lambda^p V^* \otimes H^q(X, \mathscr{O}_X(k-p))$.

Hence, there is a spectral sequence that converges to the zero cohomologies of the total complex associated with $\Gamma(X, \mathscr{E}_p^q)$ and it has

$$E_1^{-p,q} = H^q\big(\Gamma(X, \mathscr{E}_p^q)\,,\,\overline{\partial}\,\big) = \Lambda^p V^* \otimes H^q(X, \mathscr{O}_X(k-p))\;. \tag{2.16}$$

Now, if X satisfies the condition (2) formulated on page 10, then

$$H^q(X, \mathscr{O}_X(k-p)) = 0 \quad \text{for } q \neq 0, d\,.$$

Therefore all non zero terms of the spectral sequence (2.16) will be situated only in two horizontal rows: $q = 0$ and $q = d = \dim X$.

Proposition 2.3.1. *Under the assumptions* (1−3) *on page 10 for each* $k \in \mathbb{Z}$ *and any* $0 \leqslant p \leqslant n+1$ *there is an isomorphism*

$$\tau_p \;:\; R^*_{n-d-p,n-N+1-k} \xrightarrow{\;\sim\;} R_{p,k} \tag{2.17}$$

provided by the differential in E_d*-term of spectral sequence* (2.16).

Proof. The bottom row $q = 0$, of (2.16), coincides with the internal degree k homogeneous slice of the Koszul resolution (2.7):

$$0 \longrightarrow \Lambda^{n+1}V^* \underset{\mathbb{C}}{\otimes} A_{k-n-1} \xrightarrow{\;\partial\;} \cdots$$
$$\cdots \xrightarrow{\;\partial\;} \Lambda^2 V^* \underset{\mathbb{C}}{\otimes} A_{k-2} \xrightarrow{\;\partial\;} V^* \underset{\mathbb{C}}{\otimes} A_{k-1} \xrightarrow{\;\partial\;} A_k \longrightarrow 0\,.$$

Hence, the bottom row in the E_2-term consists of the following syzygies:

$$E_2^{-p,0} = \operatorname{Tor}^p_S(A, \mathbb{C})_k = R_{p,k}\;.$$

For the upper row $q = d$ in (2.16) we claim that it is dual to the complex

$$0 \longleftarrow A_{n-N+1-k} \xleftarrow{\;\partial\;} V^* \underset{\mathbb{C}}{\otimes} A_{n-N-k} \xleftarrow{\;\partial\;} \Lambda^2 V^* \underset{\mathbb{C}}{\otimes} A_{n-N-k-1} \xleftarrow{\;\partial\;} \cdots$$
$$\cdots \xleftarrow{\;\partial\;} \Lambda^{n+1} V^* \underset{\mathbb{C}}{\otimes} A_{-N-k} \longleftarrow 0\,,$$

because of $H^d(X, \mathscr{O}_X(k-p)) \simeq H^0(X, \mathscr{O}_X(p-k-N))^*$ by the Serre duality[20] and $\Lambda^p V^* \simeq \big(\Lambda^{n+1-p}V^*\big)^*$ via the dual version of pairing (2.10). So, the top row in E_2 is filled by the spaces dual to the following syzygies:

$$E_2^{-p,d} = \operatorname{Tor}^{n+1-p}_S(A, \mathbb{C})^*_{n-N+1-k} = R^*_{n+1-p,n-N+1-k}\;.$$

The differentials in the consequent terms of this spectral sequence will be non trivial only in E_d. Therefore, to get the zero limit, the E_d-differential should map $(-p-d-1, d)$-cell isomorphically onto $(-p, 0)$-cell providing isomorphism (2.17). □

[20] here we use the condition (3) on page 10 : $\omega_X = \mathscr{O}_X(-N)$

COROLLARY 2.3.2. *If the ideal of X is generated by some linear space of quadrics $Q \subset S^2(V^*)$, then non zero syzygies $R_{p,k}$ can appear only for the following values p, k:*

- $p = 0$, $k = 0$ *or* $p = n-d$, $k = n-N+1$, *where* $R_{0,0} = \mathbb{C} \simeq R^*_{n-d,n-N+1}$;
- $p = 1$, $k = 2$ *or* $p = n-d-1$, $k = n-N-1$, *where*

$$R_{1,2} = Q \simeq R^*_{n-d-1,n-N-1} \ ;$$

- $2 \leqslant p \leqslant n-d-2$ *and* $p+1 \leqslant k \leqslant p+d-N$, *where* $R_{p,k} \simeq R^*_{n-d-p,n-N+1-k}$.

PROOF. Indeed the syzygies $R_{p,k} = 0$ automatically vanish in the following four cases:

$$\begin{array}{ll} 1)\quad p<0 \quad \text{or} \quad p>n+1 & 2)\quad p=0 \quad \text{and} \quad k\neq 0 \\ 3)\quad p=1 \quad \text{and} \quad k\neq 2 & 4)\quad 2\leqslant p\leqslant n+1 \quad \text{and} \quad k\leqslant p \end{array}$$

Applying this to the right hand side of (2.17) we see that $R_{p,k} = 0$ also for $p > n-d$ or $p < d-1$, for $p = n-d$ and $k \neq n-N+1$, for $p = n-d-1$ and $k \neq n-N-1$, and finally, for $2 \leqslant p \leqslant n-d-2$ and $k \geqslant p+d-N-1$. $\square$

EXAMPLE 2.3.3 (continuation of 1.1.3, 1.1.5, and 2.1.1). For the grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}_9$ we have $n = 9$, $d = 6$, $N = 5$. Thus, by the above corollary,

$$R^*_{3,5} \simeq R_{0,0} = \mathbb{C} \ , \qquad R^*_{2,3} \simeq R_{1,2} = Q \simeq \mathbb{C}^5$$

and all the other syzygies vanish. For 10-dimensional pure spinors $\mathrm{Gr}^+_{\mathrm{iso}}(5,10) \subset \mathbb{P}_{15}$ we have $n = 15$, $d = 10$, $N = 8$. The previous corollary implies that

$$R^*_{5,8} \simeq R_{0,0} = \mathbb{C} \ , \qquad R^*_{4,6} \simeq R_{1,2} = Q \simeq \mathbb{C}^{10}$$

and all other syzygies of orders 0, 1, 4, 5 vanish. The precise computation of $R_2 \simeq R^*_3$ requires more sophisticated computational analysis of the $\mathrm{Spin}(10,\mathbb{C})$-module R. It was made, e. g. in [**9**], [**11**], [**35**].

2.4. Scalar product and trace on *R*. Let us define the trace functional (2.9) as a linear form on R that annihilates all $R_{p,k}$ except for $R_{n-d,n-N+1}$ and sends $R_{n-d,n-N+1}$ to $R^*_{0,0} = \mathbb{C}$ via isomorphisms τ_0^{-1} inverse to τ_0 defined in 2.3.1. This form provides the syzygy algebra R with a scalar product

$$(a,b) \stackrel{\text{def}}{=} \mathrm{tr}\,(a \cdot b) \tag{2.18}$$

which satisfies the property $(a\cdot b, c) = (a, b\cdot c) = \mathrm{tr}\,(a\cdot b\cdot c)$, because R is associative.

Since for $a \in R_{p,k}$, $b \in R_{n-d-p,n-N+1-k}$ we have $a \cdot b = (-1)^{p(n-d-p)} b \cdot a$, the scalar product (2.18) is purely symmetric, if $\mathrm{codim}_{\mathbb{P}_n} X = (n-d)$ is odd. If $\mathrm{codim}_{\mathbb{P}_n} X$ is even, then (2.18) is *even supersymmetric*, that is symmetric, when the both arguments are even, and skew-symmetric, when the both arguments are odd.

PROPOSITION 2.4.1. *The scalar product* (2.18) *is non degenerate, and* $\forall\, a \in R$

$$\tau_0^{-1}(a) = (a, *)$$

as the linear forms on R.

Proof. There is a natural scalar product on E_1-term of the spectral sequence (2.16) induced by the multiplication and the trace form provided by the Serre duality. It takes

$$\begin{array}{c} \Lambda^{p_1}V^*\otimes H^0\left(\mathscr{O}_X(m_1)\right)\times\Lambda^{p_2}V^*\otimes H^d\left(\mathscr{O}_X(m_2)\right) \\ \big\downarrow \\ \Lambda^{n+1}V^*\otimes H^d\left(\mathscr{O}_X(-N)\right)\simeq\mathbb{C} \end{array} \tag{2.19}$$

for $p_1+p_2=(n+1)$ and $m_1+m_2=-N$ (all the other components of E_1 are mutually orthogonal). Clearly, it is non degenerated. We extend the isomorphism from the bottom row of (2.19) to a linear map $E_1\xrightarrow{\mathrm{Tr}}\mathbb{C}$, which annihilates all the components of E_1 except $\Lambda^{n+1}V^*\otimes H^d\left(\mathscr{O}_X(-N)\right)$, and define the scalar product on E_1 by prescription

$$(a,b)_1\overset{\mathrm{def}}{=}\mathrm{Tr}\,(ab)\,. \tag{2.20}$$

Since d_1 satisfies $d_1(ab)=(d_1a)\,b+(-1)^{|a|}\,a\,(d_1b)$ and $\operatorname{im}d_1$ is annihilated by Tr, this scalar product interacts with d_1 by the rule

$$\left(d_1a\,,\,b\right)_1=\mathrm{Tr}\,\left((d_1a)\,b\right)=(-1)^{|a|+1}\mathrm{Tr}\,\left(a\,(d_1b)\right)=(-1)^{|a|+1}\left(a\,,\,d_1b\right)_1\,. \tag{2.21}$$

Let us show that this scalar product induces well defined non degenerate pairing on the cohomologies $H(E_1,d_1)=E_2=R$ and that this pairing coincides with (2.18) and produces the duality maps (2.17). We can fix some vector space decomposition (compatible with the p-grading on E_1): $E_1=Z\oplus W=d_1W\oplus C\oplus W$ such that d_1 takes W isomorphically onto $d_1W=\operatorname{im}(d_1)$, $Z=d_1W\oplus C=\ker(d_1)$, and $C\simeq E_2$ consists of representatives for the cohomology classes. It follows from (2.21) that $Z\subset(d_1W)^\perp$. Hence, the scalar product of cohomology classes is well defined, and for each p the pairing

$$\frac{W_p}{W_p\cap(d_1W)^\perp}\times d_1W_{n-p}\longrightarrow\mathbb{C}$$

induced by (2.20) is non degenerate[21]. Let

$$\begin{aligned} w_p&=\dim_{\mathbb{C}}W_p=\dim_{\mathbb{C}}d_1W_p\,,\\ w'_p&=\dim_{\mathbb{C}}\Big(W_p/\big(W_p\cap(d_1W)^\perp\big)\Big)\,. \end{aligned}$$

Then $w'_p=w_{n-p}$ for each p and evident inequalities $w'_p\leqslant w_p=w'_{n-p}\leqslant w_{n-p}=w'_p$ imply that $w'_p=w_p$ for all p. So, $W\cap(d_1W)^\perp=0$, that is $(d_1W)^\perp=Z$ and $Z^\perp=(d_1W)$. This implies $C^\perp\cap C=0$, which means that (2.20) gives a non degenerate pairing on C.

Now write $\omega\in\Lambda^{n+1}V^*\otimes H^d\left(\mathscr{O}_X(-N)\right)$ for the basic element being sent to 1 by the isomorphism from the bottom row of (2.19). For any $a\in E_2^{-p,0}$ there exists some $b\in E_2^{-p-d-1,d}$ such that $\mathrm{Tr}\,(ab)=1$. Then $ab=\omega$ in $R_{n-d,n-N+1}$. Since in E_d-term $d_{E_d}(a)=0$, we have $1=d_{E_d}(\omega)=d_{E_d}(ab)=(-1)^p\,a\,d_{E_d}(b)$. Therefore $\mathrm{tr}\,(a\,d_{E_d}(b))=1$, which means that for any $a\in R$ the linear form $\mathrm{tr}\,(a\cdot *)$ is non zero and coincides with $\tau_0^{-1}(a)$. $\square$

[21] note that $d_1W_{n-p}\subset(d_1W)_{n+1-p}$

3. Cohomology of the dual graded Lie superalgebra

We are going to compare an algebra of syzygies for an arbitrary commutative graded quadratic Koszul algebra A and an algebra of cohomologies of a graded Lie superalgebra L Koszul dual to A in the sense of Ginzburg and Kapranov [**17**]. Namely, in 3.6 we identify the syzygies of A with the cohomologies of Lie subalgebra $L_{\geqslant 2} \subset L$ and give alternative description for the algebra structure on the syzygies.

By quite deep theorem of R. Bezrukavnikov (see [**3**]) the projective coordinate algebra of any highest weight orbit $X = G/P$ (not necessary subcanonical) is Koszul. Thus our results can be applied to the syzygies of the highest weight vector orbits.

Certainly, the coincidence of two algebra structures on the space of syzygies (one constructed in §2 and another we will construct in this section) could be extracted from the general bar-cobar equivalence staff[22]. But in our situation this will be clearly apparent, fortunately, and we will check it 'by hands'. We begin with recallment of some standard resolutions and the experts could jump directly to 3.4.1.

3.1. Dual quadratic algebra. Recall that a *quadratic algebra*[23] generated by a vector space V^* is an associative algebra A of the form

$$A = \mathbb{T}(V^*)/(I) ,$$

where $\mathbb{T}(V^*)$ is the tensor algebra of a vector space V^* and $(I) \subset \mathbb{T}(V^*)$ is a double side ideal spanned by a vector subspace $I \subset V^* \otimes V^*$, of homogeneous quadratic relations. Well known Priddy's construction (see [**16**, p. 108], [**38**]) attaches to any such an algebra A the dual quadratic algebra $A^!$ generated by the dual space V with the orthogonal relation ideal

$$A^! = \mathbb{T}(V)/(I^\perp) ,$$

where $I^\perp \subset V \otimes V$ is the annihilator of I. Clearly, $A^{!!} = A$.

A projective coordinate algebra $A = \underset{m\geqslant 0}{\oplus} H^0(X, \mathscr{O}(m))$ of any variety $X \subset \mathbb{P}(V)$ whose ideal is generated by quadratic equations $\{q^\nu\} \subset S^2V^*$ fits into this framework as

$$A = S^\bullet(V^*)/(Q) = \mathbb{T}(V^*)/(C + Q) ,$$

where $C = \mathrm{Skew}(V^* \otimes V^*) \simeq \Lambda^2 V^*$ consists of commutativity relations and $Q \subset \mathrm{Sym}(V^* \otimes V^*)$ is the linear span of symmetric bilinear forms $\widetilde{q^\nu}$, polarizing the quadratic equations for X. In this case the Priddy dual algebra

$$A^! = \mathbb{T}(V)/(Q^\perp \cap \mathrm{Sym}(V \otimes V))$$

can be tautologically treated as universal enveloping algebra for graded Lie superalgebra[24]

$$L = \underset{m\geqslant 1}{\oplus} L_m = \mathscr{L}ie(V)/(\mathrm{Ann}\,(Q)) , \tag{3.1}$$

[22] see [**25**], [**26**, sec. 4] for the most comprehensive approach

[23] recall that in this paper we restrict ourself by $\mathbb{C}$-algebras only; but in this section the reader can everywhere replace $\mathbb{C}$ by an arbitrary field of zero characteristic

[24] this is the simplest motivating example for much more wide Koszul duality between graded Com and Lie operads, see [**17**]

which is a factor of free graded Lie s-algebra generated by V (taken with *odd* parity) through graded Lie ideal generated by $\mathrm{Ann}\,(Q) \subset \mathrm{Sym}(V \otimes V)$, i.e. by

$$\{\, v_1 \otimes v_2 + v_2 \otimes v_1 \mid \widetilde{q}(v_1, v_2) = 0 \;\; \forall\, q \in Q \,\} \; .$$

Thus, $L_1 = V$, $\quad L_2 = S^2V/\mathrm{Ann}\,(Q) \simeq Q^*$, etc.

In terms of coordinates, if we fix some dual bases $\{v_i\}$, $\{x^i\}$ for V, V^* and $\{q^\nu\}$, $\{z_\nu\}$ for Q, Q^*, then we can describe L as graded Lie s-algebra generated by L_1 with elements v_i of parity 1 as a basis for L_1, elements z_ν as a basis for L_2, and Lie s-brackets given by

$$[v_i, v_j] = \sum_\nu \widetilde{q^\nu}(v_i, v_j) \cdot z_\nu = \sum_\nu a^\nu_{ij} z_\nu \; , \tag{3.2}$$

where a^ν_{ij} is the matrix of q^ν in the basis v_i, i.e. $q^\nu = \sum a^\nu_{ij} x^i x^j$.

3.2. Bar construction. Recall that for any graded associative $\mathbb{C}$-algebra B with unity and augumentation $B \xrightarrow{\varepsilon} \mathbb{C}$, there is the bar-complex of free graded left B-modules

$$\cdots \xrightarrow{\overline{\partial}} B \otimes [B^{\otimes 3}] \xrightarrow{\overline{\partial}} B \otimes [B^{\otimes 2}] \xrightarrow{\overline{\partial}} B \otimes [B] \xrightarrow{\overline{\partial}} B \xrightarrow{\varepsilon} \mathbb{C} \longrightarrow 0 \, , \tag{3.3}$$

where the tensor products are taken over $\mathbb{C}$ and the B-linear differential is defined on the free generators by prescription

$$\begin{aligned} \overline{\partial}(1 \otimes [b_1 \otimes b_2 \otimes \cdots \otimes b_m]) = b_1 \otimes [b_2 \otimes \cdots \otimes b_m] + \\ + 1 \otimes \sum_{i=1}^{m-1} (-1)^i [b_1 \otimes \cdots \otimes (b_i b_{i+1}) \otimes \cdots \otimes b_m] \, . \end{aligned} \tag{3.4}$$

It is clearly contracted onto $\mathbb{C}$ by the homotopy taking

$$b_0 \otimes [b_1 \otimes \cdots \otimes b_m] \mapsto 1 \otimes [b_0 \otimes b_1 \otimes \cdots \otimes b_m] \tag{3.5}$$

and gives the standard free resolution for the trivial B-module $\mathbb{C}$ in the category of graded left B-modules. The space of free generators $\mathbb{T}^c(B)$ carries the natural coalgebra structure dual to the tensor multiplication

$$\mathbb{T}^c(B) \xrightarrow{\Delta} \mathbb{T}^c(B) \otimes \mathbb{T}^c(B)$$

$$[b_1 \otimes \cdots \otimes b_m] \mapsto \sum_{i=0}^{m} [b_1 \otimes \cdots \otimes b_i] \, \otimes \, [b_{i+1} \otimes \cdots \otimes b_m]$$

(where $[] \stackrel{\mathrm{def}}{=} 1$) and the bar differential is a coderivation w.r.t. this coproduct, i.e. satisfies[25]

$$(1 \otimes \overline{\partial} + \overline{\partial} \otimes 1) \circ \Delta = \Delta \circ \overline{\partial} \, .$$

Thus, $\mathrm{Ext}^\bullet_B(\mathbb{C}, \mathbb{C})$ can be described as the cohomology algebra of the DG algebra

$$\mathrm{Hom}_B(B \otimes \mathbb{T}^c(B), \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{T}^c(B), \mathbb{C}) = \mathbb{T}(B^*) \tag{3.6}$$

[25] applying a homogeneous operator monomial $f_1 \otimes f_2 \otimes \cdots \otimes f_m$ to a homogeneous vector monomial $v_1 \otimes v_2 \otimes \cdots \otimes v_m$ we always assume the Koszul sign agreements: $f_1 \otimes f_2 \otimes \cdots \otimes f_m(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = (-1)^\varepsilon \cdot f_1(v_1) \otimes f_2(v_2) \otimes \cdots \otimes f_m(v_m)$, where $\varepsilon = |f_m| \cdot (|v_1| + \cdots + |v_{m-1}|) + |f_{m-1}| \cdot (|v_1| + \cdots + |v_{m-2}|) + \cdots + |f_2| \cdot |v_1|$

whose multiplication is the standard tensor multiplication[26] and the differential is dual to (3.4), i. e. takes 1 to zero, acts on degree 1 generators $\beta \in B^*$ as

$$\overline{\partial}^* \beta(b_1, b_2) = \beta(b_1 b_2) \ ,$$

and is extended onto the whole of $\mathbb{T}(B^*)$ by the Leibnitz rule. We call (3.6) the *cobar complex* of B. It is naturally bigraded. In what follows we always call the degree w. r. t. the natural grading in the tensor algebra as *the (co) homological degree* in a contrast with *the internal degree*, which equals the total sum of degrees of all tensor factors w. r. t. the internal grading of B.

3.3. Koszulity. Let $B = A^! = \mathbb{T}(V)/(I^\perp)$ be the dual quadratic algebra for $A = \mathbb{T}(V^*)/(I)$. There is the Koszul complex of graded left B-modules

$$K_B = (B \otimes A^*, d_{\mathrm{K}}) \tag{3.7}$$

whose differential d_{K} comes from the right $B \otimes A$-module[27] structure on $B \otimes A^*$ given by the right multiplication in B and dual to the left multiplication in A. Namely, it is easy to see that the Casimir element

$$\mathrm{Id}_V \in \mathrm{End}_{\mathbb{C}}(V) = V \otimes V^* = B_1 \otimes A_1 \subset B \otimes A$$

has the zero square in $B \otimes A$. By the definition, d_{K} is given by the right action of Id_V on $B \otimes A^*$. In 'low level' notations, if v_i, x_i are dual bases for V and V^*, then

$$d_{\mathrm{K}}(b \otimes \alpha) = \sum (b \cdot v_i) \otimes (\alpha \circ x_i) \ ,$$

where $\alpha \circ x_i = \Big(A \xrightarrow{a \mapsto \alpha(x_i \cdot a)} \mathbb{C} \Big) \in A^*$. For example, if

$$B = \Lambda(V) = \mathbb{T}(V)/(\mathrm{Sym}(V \otimes V)) \ ,$$
$$A = S(V^*) = \mathbb{T}(V^*)/(\mathrm{Skew}(V^* \otimes V^*))$$

are the ordinary exterior and symmetric algebras, then

$$K_{S(V)} = \left(S(V) \otimes \Lambda(V) \, , \ \sum v_i \otimes \frac{\partial}{\partial v_i} \right)$$

is the Koszul complex (2.5) but with V instead of V^*.

There is canonical morphism of the differential graded B-modules

$$K_B = B \otimes A^* \xrightarrow{1 \otimes \tau} B \otimes \mathbb{T}^c(B) \tag{3.8}$$

induced by the coalgebra morphism $A^* \xrightarrow{\tau} \mathbb{T}^c(V) = \mathbb{T}^c(B_1) \subset \mathbb{T}^c(B)$ dual to the structure morphism of algebras $\mathbb{T}(V^*) \twoheadrightarrow A = \mathbb{T}(V^*)/(I)$. It is well known and

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[26] it is instructive to see how does it agree with the classic Yoneda product

$$\mathrm{Ext}^k_B(\mathbb{C}, \mathbb{C}) \otimes \mathrm{Ext}^m_B(\mathbb{C}, \mathbb{C}) \xrightarrow{\varphi \otimes \psi \mapsto \varphi \circ \psi} \mathrm{Ext}^{k+m}_B(\mathbb{C}, \mathbb{C})$$

defined by obvious extending of $\psi \in B^{\otimes m\, *}$, $\varphi \in B^{\otimes m\, *}$ to the B-linear homomorphisms

$$\widetilde{\varphi}, \widetilde{\psi} \, : \, B \otimes \mathbb{T}^c(B) \longrightarrow \mathbb{C} \ ,$$

then lifting $\widetilde{\psi}$ to some degree m homomorphism of free resolutions $B \otimes \mathbb{T}^c(B) \xrightarrow{\widetilde{\psi}_\bullet} B \otimes \mathbb{T}^c(B)$, and taking the composition $1 \otimes B^{\otimes(m+k)} \xrightarrow{\widetilde{\psi}_{m+k}} B \otimes B^{\otimes k} \xrightarrow{\varepsilon \otimes 1} 1 \otimes B^{\otimes k} \xrightarrow{\varphi} \mathbb{C}$; clearly, $\widetilde{\psi}_\bullet = (1 \otimes \psi) \circ \Delta$ gives precisely the required lifting of ψ to the morphism of bar resolutions and leads to the Yoneda product $\psi \circ \varphi = (\psi \otimes \varphi) \circ \Delta$, which coincides with the tensor product of multilinear forms

[27] the algebra structure on $B \otimes A$ is given by $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{|a_2||b_1|} (a_1 a_2) \otimes (b_1 b_2)$, where $|x|$ means the internal degree of x modulo 2

not difficult to check (see [**36**], [**38**]) that the following conditions on B are pairwise equivalent:

(1) the Koszul complex K_B gives a free graded left B-module resolution for $\mathbb{C}$, i. e. the mapping (3.8) is a quasiisomorphism;
(2) $A \simeq \mathrm{Ext}^\bullet_B(\mathbb{C},\mathbb{C})$;
(3) $\mathrm{Ext}^{i,j}_B(\mathbb{C},\mathbb{C}) = 0$ for $i \neq j$, where $\mathrm{Ext}^{i,j}$ means the internal degree j graded component of i-th derived functor Ext^i;
(4) for each $m \geqslant 3$ the subspaces $W_\nu = V^{\otimes \nu} \otimes I^\perp \otimes V^{\otimes(m-\nu-2)} \subset V^{\otimes m}$ (where $0 \leqslant \nu \leqslant (m-2)$) form a distributive lattice[28] in $V^{\otimes m}$.

Quadratic algebras satisfying this conditions are called *Koszul algebras*. Of course, B is Koszul iff $A = B^!$ is Koszul, and one can exchange B and A in the above properties. Since $\mathrm{Hom}_B(\,*\,,\mathbb{C})$ kills the Koszul differential, applying this functor to (3.8) we get DGA homomorphism from A, considered as DG algebra with the zero differential, to the cobar complex (3.6)

$$A \xrightarrow{\tau^*} (\mathbb{T}(B^*), \overline{\partial}^*)\,, \tag{3.9}$$

which is a quasiisomorphism as soon as A, B are Koszul. Thus, each Koszul algebra A is *canonically* identified with the algebra $\mathrm{Ext}_B(\mathbb{C},\mathbb{C})$ via (3.9).

3.4. Chevalley complex. For a commutative Koszul algebra $A = S(V^*)/(Q)$ the quaisiisomorphism (3.9) means the natural identification of A with the Lie algebra cohomology

$$A \simeq \mathrm{Ext}^\bullet_{U(L)}(\mathbb{C},\mathbb{C}) \overset{\text{def}}{=} H^\bullet(L,\mathbb{C})\,,$$

which can be computed using another reduction of the bar complex for $B = U(L)$ known as Chevalley's complex. Let us write $\Lambda^c(L)$ for the graded s-exterior coalgebra, which is dual to the tensor algebra of L^* factorized through the relations $u \otimes v + (-1)^{|u||v|} v \otimes u$. It can be considered as the sub-coalgebra of $\mathbb{T}^c(B)$ via the s-alternation embedding

$$e_1 \wedge e_2 \wedge \cdots \wedge e_m \mapsto \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_n} \text{s-sgn}(\sigma)\, e_{\sigma(1)} \wedge e_{\sigma(2)} \wedge \cdots \wedge e_{\sigma(m)} \tag{3.10}$$

where s-sign takes proper account of the internal degree of the permuted elements. One can check (see [**4**, §3 ex.21], [**8**, ch.XIII, ex.14]) that the bar differential on $\mathbb{T}^c(B)$ takes the subcoalgebra $\Lambda^c(L)$ to itself and the resulting subcomplex[29]

$$\cdots \xrightarrow{d_{\mathrm{C}}} B \otimes \Lambda^3 L \xrightarrow{d_{\mathrm{C}}} B \otimes \Lambda^2 L \xrightarrow{d_{\mathrm{C}}} B \otimes L \xrightarrow{d_{\mathrm{C}}} B \xrightarrow{\varepsilon} \mathbb{C} \longrightarrow 0 \tag{3.11}$$

gives the free graded left B module resolution for $\mathbb{C}$ as well. We call it *the Chevalley resolution* and denote by $\mathfrak{C}$ or $\mathfrak{C}(L)$ when the precise reference to L is important.

Practical handling of (3.11) becomes more demonstrative with an alternative Lie theoretic interpretation of $\mathfrak{C}$. Namely, let us write $\bar{L}$ for another copy of the vector superspace L but with the inverse parity and the trivial abelian s-Lie structure. Then we can write $\mathfrak{C}_m = \Lambda^m(L) = S^m(\overline{L})$ (because of the parity change) and treat

$$\mathfrak{C} = B \otimes \Lambda^m(L) = B \otimes S^m(\overline{L}) = U(L \oplus \overline{L})$$

[28] recall that this means the coincidences $W_\alpha \cap (W_\beta + W_\gamma) = W_\alpha \cap W_\beta + W_\alpha \cap W_\gamma$ and $W_\alpha + (W_\beta \cap W_\gamma) = (W_\alpha + W_\beta) \cap (W_\alpha + W_\gamma)$ for all α, β, γ or, equivalently, the existence of a basis $E = \{e_i\} \subset V^{\otimes m}$ for $V^{\otimes m}$ such that $\forall \alpha$ $W_\alpha \cap E$ is a basis for W_α

[29] whose differential d_{C} is the restricted bar differential

as the universal enveloping algebra of an abelian extension $L \oplus \overline{L}$ that contains L as a Lie subalgebra, $\overline{L}$ as an abelian ideal, and has Lie brackets defined by prescriptions[30]

$$\begin{aligned} &[x,y]=[x,y]_L\,, && [\overline{x},\overline{y}]=0\,, \\ &[x,\overline{y}]=\overline{[x,y]}_L\,, && [\overline{y},x]=(-1)^{|x|}\overline{[y,x]}_L \end{aligned} \tag{3.12}$$

for all $x,y\in L$. Thus, in new s-symmetric notations, $\mathfrak{C}_m$ consists of degree m s-symmetric monomials

$$b\cdot\overline{e}_1\overline{e}_2\,\cdots\,\overline{e}_m\,,\quad \text{where } b\in B=U(L)=\mathfrak{C}_0\,,\quad e_i\in L\,. \tag{3.13}$$

They also can be embedded into $\mathbb{T}^c(L)$ via s-symmetrization identical with (3.10).

To get an alternative description for the Chevalley differential (3.11), let us define for a moment a new differential[31] $\mathfrak{C}\xrightarrow{d}\mathfrak{C}$ as the odd right s-algebra derivation[32] whose action on the generating vector space $L\oplus\overline{L}$ is given by by the same rule as the bar differential

$$d(x)=0\,,\qquad d(\overline{x})=x\qquad \forall\, x\in L\ . \tag{3.14}$$

It is clear that d preserves the enveloping algebra relations and automatically satisfies the right Leibnitz rule w.r.t. supercommutators:

$$d([a,b])=[a,d(b)]+(-1)^{|b|}[d(a),b]\,.$$

Its action on the generators (3.13) looks like

$$d\,(\overline{e}_1\overline{e}_2\,\cdots\,\overline{e}_m)= \sum_{1\leqslant j\leqslant m}\pm e_j\cdot\overline{e}_1\,\cdots\,\underset{j}{\widehat{\ }}\,\cdots\,\overline{e}_m+\sum_{1\leqslant i<j\leqslant m}\pm\overline{[e_i,e_j]}\,\overline{e}_1\,\cdots\,\underset{i}{\widehat{\ }}\,\cdots\,\underset{j}{\widehat{\ }}\,\cdots\,\overline{e}_m\,, \tag{3.15}$$

where the precise sign calculation is quite cumbersome, but it is not so important for our purposes[33]. Note that the coalgebra structure on the bar complex agrees with the standard coalgebra structure on the universal enveloping algebra, which is given on generators $\ell\in L\oplus\overline{L}$ by the usual rule

$$\Delta(\ell)=1\otimes\ell+\ell\otimes 1\,, \tag{3.16}$$

and is extended onto the whole of $\mathfrak{C}$ as a homomorphism of graded algebras

$$\mathfrak{C}\xrightarrow{\Delta}\mathfrak{C}\otimes\mathfrak{C},$$

where the algebra structure $\mathfrak{C}\underset{\mathbb{C}}{\otimes}\mathfrak{C}$ is given by $(a\otimes b)\cdot(c\otimes d)=(-1)^{|b||c|}(a\cdot c)\otimes(b\cdot d)$. The differential d agrees with the coalgebra structure (3.16), i.e. satisfies[34]

$$\Delta\circ d=(1\otimes d+d\otimes 1)\circ\Delta\,. \tag{3.17}$$

[30] of course, the last two formulas, describing the action of L on $\overline{L}$, are equivalent, because of $[\overline{y},x]=-(-1)^{|x|(1+|y|)}[x,\overline{y}]=-(-1)^{|x|(1+|y|)}\overline{[x,y]}=(-1)^{|x|(1+|y|)}(-1)^{|x||y|}\overline{[y,x]}$

[31] we will see soon that it coincides with $d_{\mathfrak{C}}$

[32] i.e. satisfying the right Leibnitz rule $d(ab)=ad(b)+(-1)^{|b|}d(a)b$

[33] for example, to see that $d^2=0$, it is enough to mention that d, being an odd right derivation, forces the commutator $[d,d]=2\,d^2$ to be the right derivation of $\mathfrak{C}$ as well; now $d^2=0$ follows from (3.14)

[34] indeed, since Δ is an algebra homomorphism and both d, $(1\otimes d+d\otimes 1)$ are the right derivations, the both sides of (3.17) are right derivations of $\mathfrak{C}$ with values in $\mathfrak{C}\otimes\mathfrak{C}$; by (3.14) and (3.16) they coincide on the generating vector space $L\oplus\overline{L}$

We conclude that $d = d_{\rm C}$ on $\mathfrak{C} \subset \mathbb{T}(B)$. In particular, this gives the 'low level' description for (3.11) via (3.15).

Thus, we can compute $\mathrm{Ext}_B(\mathbb{C},\mathbb{C})$ as the cohomologies of the complex

$$(C^\bullet(L), d) = \mathrm{Hom}_B(\,\mathfrak{C}(L)\,,\,\mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\Lambda(L),\mathbb{C}) = (\Lambda(L^*), d_{\rm C}^*)\ ,$$

which carries the natural DG algebra structure whose multiplication is induced by the multiplication in the s-exterior algebra and the differential is induced by (3.15). Let us finalize this preliminary discussion as

PROPOSITION 3.4.1. *Let A be an arbitrary commutative Koszul quadratic algebra, $B = A^! = U(L)$ be its dual algebra treated as the universal enveloping algebra for a graded Lie s-algebra L. Then A, considered as DG algebra with the zero differential, admits canonical isomorphism*

$$A \simeq H\left(C^\bullet(L), d\right) \tag{3.18}$$

with the cohomology algebra of DG algebra $(C^\bullet, d_{\rm C}^*)$, which has $C^m = \Lambda^m L^* = S^m \overline{L}^*$, the differential $d_{\rm C}^* : C^m \longrightarrow C^{m+1}$ is acting as

$$d_{\rm C}^*\psi(\overline{e}_1\overline{e}_2\,\cdots\,\overline{e}_m) = \sum_{1\leqslant i<j\leqslant m} \pm\,\psi(\overline{[e_i,e_j]}\,\overline{e}_1\,\cdots\,\underset{i}{\widehat{\ }}\,\cdots\,\underset{j}{\widehat{\ }}\,\cdots\,\overline{e}_m)\ , \tag{3.19}$$

and the multiplication in $C^\bullet$ is given by the shuffle product[35]

$$[\varphi\circ\psi]\,(\overline{e}_1\overline{e}_2\,\cdots\,\overline{e}_{k+m}) = \sum \pm\varphi(\overline{e}_{i_1}\overline{e}_{i_2}\,\cdots\,\overline{e}_{i_k})\cdot\psi(\overline{e}_{j_1}\overline{e}_{j_2}\,\cdots\,\overline{e}_{j_m})\ . \tag{3.20}$$

The isomorphism (3.18) *takes the internal graded component A_i to the i-th internal degree component of the i-th cohomology space. It comes from the quasiisomorphism* (3.9) *and quasiisomorphic embedding* (3.10) *of Chevalley's resolution* (3.11) *into bar resolution* (3.3).

3.5. Differential perturbation lemma. The proof of the main results of the next sect. 3.6 will be based on the lemma influenced by A. Losev's talks on 'enhanced spectral sequences', which is a slight variation on the simplest, degree one, case of the homotopy structure transferring in Kadeashvili's type of thing[36].

Let $(E\,,\ d : E \longrightarrow E\,,\ d^2 = 0)$ be an arbitrary differential S-module over an arbitrary ring S. Assume we are given with the diagram of S-modules and S-linear homomorphisms

$$E \underset{\varrho}{\overset{\lambda}{\rightleftarrows}} H \tag{3.21}$$

together with S-linear homotopy $E \xrightarrow{\varkappa} E$ that satisfy the following properties:

$$\lambda\varrho = 1\ , \qquad \varrho\lambda = 1 + d\varkappa + \varkappa d\ , \tag{3.22}$$

$$\varkappa^2 = d^2 = \lambda d = d\varrho = \lambda\varkappa = \varkappa\varrho = 0 \tag{3.23}$$

(this means that H can be included into E as the retract of d capturing all its homology). We intend to show that under this assumptions and appropriate 'convergency condition' any perturbation $D = d + \delta$ (even not necessary commuting with d) induces some non trivial differential ∂ on H and a perturbation $(\lambda', \varrho', \varkappa')$, of the diagram (3.21) and the homotopy $\varkappa$, such that λ', ϱ' remain to be the inverse

[35] the summation in (3.20) goes over all $I = \{i_1 < i_2 < \cdots < i_k\}$, $J = \{j_1 < j_2 < \cdots < j_m\}$ such that $I \sqcup J = \{1,\,2,\,\ldots\,,\,(m+k)\}$

[36] comp. with [**7**]; see also [**31**] for similar explicit A_∞-formulas most closed to our framework

homotopy equivalences between the complexes (E,D) and (H,∂). The convergency condition in question is the existence of $\varkappa'$, ε_λ, $\varepsilon_\varrho \in \mathrm{End}_S(E)$ defined by the following series

$$\begin{gathered}\varkappa' \stackrel{\text{def}}{=} \varkappa + \varkappa\delta\varkappa + \varkappa\delta\varkappa\delta\varkappa + \varkappa\delta\varkappa\delta\varkappa\delta\varkappa + \dots = \varkappa(1+\varepsilon_\lambda) = (1+\varepsilon_\varrho)\varkappa\,, \\ \varepsilon_\lambda \stackrel{\text{def}}{=} \sum_{m\geqslant 1}(\delta\varkappa)^m = \delta\varkappa'\,, \quad \varepsilon_\varrho \stackrel{\text{def}}{=} \sum_{m\geqslant 1}(\varkappa\delta)^m = \varkappa'\delta\,.\end{gathered} \tag{3.24}$$

Note that these operators are well defined, for example, if $\delta\varkappa \in \mathrm{End}_S(E)$ is locally nilpotent[37].

Lemma 3.5.1. *Under the assumptions* (3.21)–(3.23)*, let*

$$D = d+\delta \,:\, E \longrightarrow E\,, \quad D^2 = 0\,,$$

be another S-linear differential on E. If the operators (3.24) *are well defined then the perturbed operators* $\lambda' \stackrel{def}{=} \lambda(1+\varepsilon_\lambda)$, $\varrho' \stackrel{def}{=} (1+\varepsilon_\varrho)\varrho$ *satisfy the conditions*

(1) $\lambda'\varrho' = \mathrm{Id}_H$, $\varrho'\lambda' = \mathrm{Id}_E + D\varkappa' + \varkappa' D$;
(2) $\partial = \lambda' D\varrho = \lambda'\delta\varrho = \varrho' D\lambda = \varrho'\delta\lambda$ *is actually the same operator on* H;
(3) $\partial^2 = 0$, *i. e.* ∂ *provides* H *with the differential;*
(4) $\partial\lambda' = \lambda' D$ *and* $\varrho'\partial = D\varrho'$, *i. e.* $(E,D) \underset{\varrho'}{\overset{\lambda'}{\rightleftarrows}} (H,\partial)$ *are morphisms of complexes providing the inverse to each other homotopy equivalences.*

Proof. It follows from (3.23) that $\varepsilon_\lambda\varrho = \lambda\varepsilon_\varrho = \varepsilon_\lambda\varepsilon_\varrho = 0$. This implies

$$\lambda'\varrho' = \lambda(1+\varepsilon_\lambda)(1+\varepsilon_\varrho)\varrho = \lambda\varrho = \mathrm{Id}_E\,,$$

which is the first relation in (1). The conditions $d^2=0$ and $D^2=(d+\delta)^2=0$ imply that $\delta^2 = -d\delta - \delta d$. Using this relation and (3.24), we get

$$\begin{aligned}\varepsilon_\varrho\varepsilon_\lambda &= \varkappa(1+\varepsilon_\lambda)\delta^2(1+\varepsilon_\varrho)\varkappa = \\ &= -\varkappa(1+\varepsilon_\lambda)d\delta(1+\varepsilon_\varrho)\varkappa - \varkappa(1+\varepsilon_\lambda)\delta d(1+\varepsilon_\varrho)\varkappa = \\ &= -\varkappa' d\varepsilon_\lambda - \varepsilon_\varrho d\varkappa'\,.\end{aligned} \tag{3.25}$$

Now, to compute $\varrho'\lambda'$, we substitute $\varrho\lambda = 1 + d\varkappa + \varkappa d$ and write the result as a sum of three terms

$$\begin{aligned}\varrho'\lambda' &= (1+\varepsilon_\varrho)\varrho\lambda(1+\varepsilon_\lambda) = (1+\varepsilon_\varrho)(1+d\varkappa+\varkappa d)(1+\varepsilon_\lambda) = \\ &= (1+\varepsilon_\varrho)(1+\varepsilon_\lambda) + (1+\varepsilon_\varrho)d\varkappa(1+\varepsilon_\lambda) + (1+\varepsilon_\varrho)\varkappa d(1+\varepsilon_\lambda)\,,\end{aligned} \tag{3.26}$$

then expand these summands using (3.25) and (3.24) :

$$\begin{gathered}(1+\varepsilon_\varrho)(1+\varepsilon_\lambda) = 1+\varepsilon_\varrho+\varepsilon_\lambda+\varepsilon_\varrho\varepsilon_\lambda = 1+\delta\varkappa'+\varkappa'\delta-\varepsilon_\varrho d\varkappa'-\varkappa' d\varepsilon_\lambda\,, \\ (1+\varepsilon_\varrho)d\varkappa(1+\varepsilon_\lambda) = (1+\varepsilon_\varrho)d\varkappa' = d\varkappa' + \varepsilon_\varrho d\varkappa'\,, \\ (1+\varepsilon_\varrho)\varkappa d(1+\varepsilon_\lambda) = \varkappa' d(1+\varepsilon_\lambda) = \varkappa' d + \varkappa' d\varepsilon_\lambda\,.\end{gathered}$$

Adding up the right sides, we get the homotopy relation required in (1)

$$\varrho'\lambda' = 1 + \delta\varkappa' + \varkappa'\delta + d\varkappa' + \varkappa' d = 1 + D\varkappa' + \varkappa' D\,.$$

Since we have $\lambda D = \lambda\delta$, $D\varrho = \delta\varrho$, $\varepsilon_\lambda\delta = \delta\varepsilon_\varrho$, (2) follows

$$\lambda' D\varrho = \lambda'\delta\varrho = \lambda(1+\varepsilon_\lambda)\delta\varrho = \lambda\delta(1+\varepsilon_\varrho)\varrho = \lambda\delta\varrho' = \lambda D\varrho'\,.$$

[37] i. e. $\forall\, e\in E\ \ \exists\, m = m(e) \in \mathbb{N} : (\delta\varkappa)^m e = 0$

The commutation relations (4) also follow from $\lambda D = \lambda\delta$, $D\varrho = \delta\varrho$ using (1)

$$\partial\lambda' = \lambda D\varrho'\lambda' = \lambda D(1 + D\varkappa' + \varkappa' D) = \lambda(1 + \delta\varkappa')D = \lambda' D \ ,$$
$$\varrho'\partial = \varrho'\lambda' D\varrho = (1 + D\varkappa' + \varkappa' D)D\varrho = D(1 + \varkappa'\delta)\varrho = D\varrho' \ .$$

Finally, $\partial\partial = \lambda D\varrho'\partial = \lambda D^2\varrho' = 0$ gives (3). $\square$

3.6. Chevalley's complex as $S(V^*)$-module. Consider graded Lie ideal

$$L_{\geqslant 2} = \bigoplus_{m\geqslant 2} L_m \subset L$$

and denote by $(C^\bullet(L_{\geqslant 2}), d_{\mathrm{C}}^{\geqslant 2})$ its Chevalley complex. The s-exterior algebra of L splits as the graded algebra into the tensor product

$$C^\bullet(L) = \Lambda(L^*_{\geqslant 2} \oplus L^*_1) = \Lambda(L^*_{\geqslant 2}) \underset{\mathbb{C}}{\otimes} \Lambda(L^*_1) = C^\bullet(L_{\geqslant 2}) \underset{\mathbb{C}}{\otimes} S \tag{3.27}$$

where[38] $S = S(V^*) = \Lambda(L^*_1)$ is the projective coordinate algebra of $\mathbb{P}(V)$. The both sides of (3.27) carry the natural structure of right S-modules[39] coming from the algebra inclusion

$$S(V^*) = \Lambda(L^*_1) \hookrightarrow \Lambda(L^*)$$

and the isomorphism (3.27) is clearly S-linear. Moreover, the Chevalley differential (3.19), acting on the left side, is also S-linear because of $d_{\mathrm{C}}(L_1) = 0$. The right side of (3.27) carries the intrinsic S-linear differential

$$d = d_{\mathrm{C}}^{\geqslant 2} \otimes 1 \tag{3.28}$$

induced by the Chevalley differential for the Lie s-algebra $L_{\geqslant 2}$. We transfer it to the left side preserving the notation d for it. Then on the left side we have

$$d_{\mathrm{C}} = d + \delta \tag{3.29}$$

where d acts on the subalgebra $\Lambda(L^*_{\geqslant 2})$ by the same formula (3.19) and annihilates all the monomials containing L^*_1-factors, and δ is the difference, which is automatically S-linear as well. We are in a position to apply the differential perturbation lemma of 3.5.1.

THEOREM 3.6.1. *Let A be commutative Koszul quadratic algebra,*

$$B = A^! = U(L)$$

be its dual, treated as the universal enveloping algebra of the graded Lie s-algebra L (see 3.1). Then for each $p \geqslant 1$ and any q there exists an isomorphism

$$R_{p,q}(A) \simeq H^{q-p}(L_{\geqslant 2}, \mathbb{C})_q \tag{3.30}$$

between q-th internal degree components of p-th syzygy space of A (see (2.4)) and $(q-p)$-th cohomology space of $L_{\geqslant 2}$.

PROOF. Let us split $C^\bullet(L_{\geqslant 2})$ as the vector space over $\mathbb{C}$ into a direct sum of bigraded subspaces

$$C^\bullet(L_{\geqslant 2}) = H \oplus I \oplus P \tag{3.31}$$

[38] recall that $L_1 = V$ has internal degree 1, thus, its s-exterior algebra is nothing but the ordinary symmetric algebra

[39] since S is a commutative algebra, it does not matter from what side does it act from, but we use the right action to outline that it commutes with the left B-action on the cobar complex

where $I = \operatorname{im} d_{\mathrm{C}}^{\geqslant 2}$ and $H \oplus I = \ker d_{\mathrm{C}}^{\geqslant 2}$. Thus $H \simeq H^\bullet(L_{\geqslant 2})$ and d takes P isomorphically onto I and annihilates $H \oplus I$. Write $\varkappa$ for the operator

$$H \oplus I \oplus P \xrightarrow{\varkappa} H \oplus I \oplus P \tag{3.32}$$

that annihilates $H \oplus A$ and acts on I as $-d^{-1} : I \xrightarrow{\sim} P$. We write

$$C^\bullet(L_{\geqslant 2}) \underset{\varrho}{\overset{\lambda}{\rightleftarrows}} H \tag{3.33}$$

for the embedding and the projection associated with the direct sum decomposition (3.31). So, our $\varkappa$, λ, ϱ satisfy the relations (3.22), (3.23). Tensoring (3.33) by S and combining it with the S-module isomorphism (3.27), we get the diagram of S-modules

$$(C^\bullet(L), d_{\mathrm{C}}(L)) \underset{\varrho}{\overset{\lambda}{\rightleftarrows}} H^\bullet(L_{\geqslant 2}) \underset{\mathbb{C}}{\otimes} S\ , \tag{3.34}$$

and the S-linear map $C^\bullet(L) \xrightarrow{\varkappa} C^\bullet(L)$ that satisfy the relations (3.22), (3.23) as well. Further, the composition $\delta\varkappa$, where δ comes from the decomposition (3.19), is locally a nilpotent operator, because it is clear from (3.19) that δd^{-1} preserves the homological degree and strictly decreases the difference between the total internal degree and degree induced by the homological grading coming from $C^\bullet(L_{\geqslant 2})$. Thus, the differential perturbation lemma from 3.5.1 provides $H^\bullet(L_{\geqslant 2})$ with the differential

$$\partial = \lambda \circ \left(\sum_{m \geqslant 0} (\delta\varkappa)^m \right) \circ \varrho \tag{3.35}$$

such that $(H^\bullet(L_{\geqslant 2}) \otimes S, \partial)$ becomes a complex of free graded S-modules *homotopy equivalent* to the Chevalley complex $(C^\bullet(L), d_{\mathrm{C}}(L))$. Since the latter is quasiisomorphic to A as a DG S-module, we can compute

$$R_p = \operatorname{Tor}_p^S(A, \mathbb{C})$$

as p-th cohomology of the complex obtained by tensoring $(H^\bullet(L_{\geqslant 2}) \otimes S, \partial)$ by $\mathbb{C}$ over S. Because the differential (3.35) strictly increases the internal S-module degree, it will be annihilated by this tensoring and we get the required isomorphism (3.30). $\square$

3.6.2. *Coincidence of two algebra structures on the syzygies.* In §2 we have equipped the space of syzygies by an algebra structure induced by the Koszul DGA resolution (2.8) for A. On the other side, the Lie algebra cohomology $H^\bullet(L_{\geqslant 2})$ also has an algebra structure induced by the Chevalley's DGA resolution. So, the both sides of (3.30) come with the intrinsic algebra structures. In fact this two structures do coincide.

THEOREM 3.6.3. *The isomorphism $R \simeq H^\bullet(L_{\geqslant 2})$ constructed in the previous theorem is an isomorphism of algebras.*

PROOF. We have to compare Chevalley's DG algebra $C^\bullet(L_{\geqslant 2})$ with the DG algebra

$$\mathscr{A} = A \underset{S}{\otimes} K_S\ , \tag{3.36}$$

where $K_S = S^\bullet(V^*) \underset{\mathbb{C}}{\otimes} \Lambda^\bullet(V^*)$ is the Koszul complex (2.5), which is the Koszul resolution for $\mathbb{C}$ as the left module over $S = S(V^*)$. To this aim consider

$$E = C^\bullet(L) \underset{S}{\otimes} K_S = C^\bullet(L) \underset{\mathbb{C}}{\otimes} \Lambda(V^*) \tag{3.37}$$

equipped with the differential $D = d_{\mathrm{C}} \underset{S}{\otimes} 1 + 1 \underset{S}{\otimes} d_{\mathrm{K}}$, where d_{K} is the Koszul differential (2.6). This DG algebra has the compatible structure of right DG module over DG algebra K_S. The S-module isomorphism (3.27) extends obviously to the isomorphism of right K_S-modules

$$C^\bullet(L) \underset{S}{\otimes} K_S = E \ \simeq\ E' = C^\bullet(L_{\geqslant 2}) \underset{\mathbb{C}}{\otimes} K_S\ . \tag{3.38}$$

The right side has the intrinsic S-linear differential $d = d_{\mathrm{C}}^{\geqslant 2} \otimes 1 + 1 \otimes d_{\mathrm{K}}$, which can be transferred to E. Thus, we get two decompositions for D into a sum

$$D = d_{\mathrm{C}} \underset{S}{\otimes} 1 + 1 \underset{S}{\otimes} d_{\mathrm{K}} = d + \delta \tag{3.39}$$

We are going to compute the algebra structure on the cohomology space $H(E, D)$ using the spectral sequences associated with these two decompositions.

The first decomposition $D = d_{\mathrm{C}} \otimes 1 + 1 \otimes d_{\mathrm{K}}$ has commuting summands, i. e. represents E as a double complex. Those spectral sequence that firstly computes the Koszul cohomology degenerates in E_2-term. Its E_1-term is concentrated at the zero row and coincids with the Koszul resolution (3.36). Thus, $H(E, D) \simeq R$ and has the multiplication induced from (3.36).

The second decomposition $D = d + \delta$ corresponds to the natural filtration on E coming from the Serre–Hochschild filtration of the pair $(L, L_{\geqslant 2})$ on $C^\bullet(L)$, where q-th filtered component is dual to the $\mathbb{C}$-linear span of all monomials (3.13) that contain $\leqslant q$ generators $e_i \in \overline{L}_1 \subset \overline{L}$. It is clear from (3.19), (3.20) that this filtration is compatible with D and the product in E. Thus, we conclude that the algebra structure on $H(E, D)$ coincides with the one induced from the DG algebra structure on the E_1-term of the spectral sequence for this filtration.

Obviously this E_1-term is nothing but the right side of (3.38) with its natural algebra structure and differential $d = d_{\mathrm{C}}^{\geqslant 2} \otimes 1 + 1 \otimes d_{\mathrm{K}}$. To compute its homology we can use the fact that it, in its own turn, is the sum of two commuting differentials. Applying the same arguments as above, we conclude that the E_2-term is isomorphic to $H^\bullet(L_{\geqslant 2})$ and has the algebra structure induced by the Chevalley DGA resolution for $L_{\geqslant 2}$. Now Theorem 3.6.1 implies that this spectral sequence also degenerates at E_2-term. We conclude that there is an algebra isomorphism $H(E, D) \simeq H^\bullet(L_{\geqslant 2})$. Thus $R \simeq H(E, D) \simeq H^\bullet(L_{\geqslant 2})$ and we can say that the multiplicative structures on R is induced from $C^\bullet(L_{\geqslant 2})$. □

4. Some geometric examples

4.1. Notation and preliminaries. In this section we use Theorem 3.6.1 to describe the syzygies

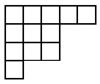
$$R_{p,q}(A) \simeq H^{q-p}(L_{\geqslant 2}, \mathbb{C})_q \tag{4.1}$$

of projective coordinate algebras of certain HW-orbits in $\mathbb{P}(V)$ by computing the cohomologies staying in the right hand side of (4.1). Recall (see 2.1.1) that the bigraded components (4.1) the syzygies of an HW-orbit are equipped with the natural action of $\mathrm{GL}(V)$. Our computations will use irreducible decompositions of (4.1) w.r.t. this action.

4.1.1. *Young diagram notations.* We depict a partition[40]

$$\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$$

(where $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_k$) by Young diagram with k rows of lengths $\lambda_1, \lambda_2, \dots, \lambda_k$ like

(4.2)  (for $\lambda = [5, 3, 3, 1]$)

and write $\lambda' = [\lambda'_1, \lambda'_2, \dots, \lambda'_m]$ for the transposed diagram (say, for (4.2) we have $\lambda' = [4, 3, 3, 1, 1]$). The total number of cells $|\lambda| = \sum \lambda_i$ will be called *a weight* of the diagram. The shorted notation $[\varrho_1^{s_1}, \varrho_2^{s_2}, \dots, \varrho_n^{s_n}]$ means the Young diagram that has s_i rows of length ϱ_i (in (4.2) $\lambda = [5, 3^2, 1]$).

Also we will use the Frobenius notation and write

$$(\alpha_1, \alpha_2, \dots, \alpha_p \,|\, \beta_1, \beta_2, \dots, \beta_p) \tag{4.3}$$

for the Young diagram λ whose main diagonal consists of p cells and

$$\lambda_i = \alpha_i + i \,, \quad \lambda'_i = \beta_i + i \quad \text{for each } i = 1,\, 2,\, \dots,\, p$$

(in (4.2) $\lambda = (4, 1, 0 \,|\, 3, 1, 0)$). Note that in this notation $\alpha_1 > \alpha_2 > \dots > \alpha_p \geqslant 0$ and $\beta_1 > \beta_2 > \dots > \beta_p \geqslant 0$.

4.1.2. *Irreducible* GL*-modules.* With any Young diagram $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$ is associated an irreducible $\mathrm{GL}_k(\mathbb{C})$-module π_λ of the highest weight

$$\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \dots + \lambda_k \varepsilon_k$$

which is simultaneously the irreducible $\mathrm{SL}_k(\mathbb{C}) \subset \mathrm{GL}_k(\mathbb{C})$ - module of highest weight

$$(\lambda_1 - \lambda_2)\, \alpha_1 + (\lambda_2 - \lambda_3)\, \alpha_2 + \dots + (\lambda_{k-1} - \lambda_k)\, \alpha_{k-1}$$

(where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ are the simple roots as in 1.1.3). Recall that π_λ can be constructed by factorizing the space

$$\Lambda^\lambda \stackrel{\text{def}}{=} \Lambda^{\lambda'_1} V \otimes \Lambda^{\lambda'_2} V \otimes \,\cdots\, \otimes \Lambda^{\lambda'_m} V \tag{4.4}$$

(where $V = \mathbb{C}^k$ is the tautological GL_k-module and the exterior powers are the lengths of columns of λ) through *the column exchange relations*[41]. Such a relation can be written for any choice of the following data:

- a filling T of the Young diagram λ by $n = |\lambda|$ vectors $v_1, v_2, \dots, v_n \in V$
- a number i of a column in λ
- a collection I of cells in the next $(i+1)$-th column

Taking an exterior product of the vectors v_ν along each column of λ and tensoring these products together, we get an element $v^T \in \Lambda^\lambda$; then the column exchange relation, which corresponds to T, i, I says that in π_λ

$$v^T = \sum_\sigma v^{\sigma T} \,, \tag{4.5}$$

---

[40] our notations for the Young tableaux and associated symmetric functions agree with [**13**], [**30**] where the reader can find all the formulas we will use below to express the symmetric polynomials through each other

[41] see [**13**]

where σ runs through the permutations of v_ν's providing order preserving exchanges between I-cells and all collections of $\#I$ cells in the previous i-th column[42]. For example, an exchange relation corresponding to a filled diagram $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$ and the only possible choice $i = 1$, $I = \{2\}$ says that $\forall\, v_1, v_2, v_3$ in $\pi_{(2,1)}$ we have

$$(v_1 \wedge v_2) \otimes v_3 = (v_3 \wedge v_2) \otimes v_1 + (v_1 \wedge v_3) \otimes v_2 \ ,$$

which is nothing but the Jacobi relation $[[v_1, v_2], v_3] + [[v_2, v_3], v_1] + [[v_3, v_1], v_2] = 0$.

If we write $e_1, e_2, \dots, e_k$ for the standard basis in $V = \mathbb{C}^k$, then the standard basis for π_λ is formed by classes of elements $e^T \in \Lambda^\lambda$ obtained from fillings T of λ by vectors e_i such that the indexes of e_i weakly increase across each row and strictly increase down each column, i.e. form *a Young tableau* T of the shape λ on the alphabet $[1\,..\,k]$. With any such a tableau T one can associate a monomial $x^T = x_1^{m_1} x_2^{m_2} \dots x_k^{m_k}$, where m_i is number of occurrences of i in T. Then the character of the irreducible GL_k-module π_λ is the Schur polynomial

$$s_\lambda(x_1, x_2, \dots, x_k) = \sum_T x^T \ ,$$

where the sum is running over all Young tableaux. For example, a filling

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 5 \\ \hline 3 & 3 & 4 \\ \cline{1-3} 4 & 4 & 5 \\ \cline{1-3} 5 \\ \cline{1-1}\end{array}$$

of the diagram (4.2) is a valid tableau for GL_5 and contributes monomial $x_1^3 x_2 x_3^2 x_4^3 x_5^2$ into $s_{[5,3,3,1]}$ (this monomial computes the eigenvalue of the standard basic vector coming from $e^T = (e_1 \wedge e_3 \wedge e_4 \wedge e_5) \otimes (e_1 \wedge e_3 \wedge e_4) \otimes (e_1 \wedge e_4 \wedge e_5) \otimes e_2 \otimes e_5$).

4.1.3. *Euler's* GL*-characters.* Since the category of GL-modules is semisimple, each term of any GL-equivariant complex $K^\bullet$ splits as

$$K^\nu = \bigoplus_{\lambda \in \Lambda_\nu} \pi_\lambda \ ,$$

where Λ_ν is the set of highest weights of all irreducible representations appearing in K^ν (counted with multiplicities). We call the alternated sum

$$\chi_{K^\bullet} \stackrel{\text{def}}{=} \sum_\nu (-1)^\nu \sum_{\lambda \in \Lambda_\nu} s_\lambda \tag{4.6}$$

the Euler GL*-character* of $K^\bullet$. Clearly, $\chi_{K^\bullet} = \chi_{H(K^\bullet)}$, where $H(K^\bullet)$ is considered as a complex with zero differentials. Similarly, one can define the Euler GL-characters for graded GL-equivariant commutative and Lie s-algebras. A powerful tool for comparing these characters is provided by the following version of the Koszul duality.

4.1.4. *Quillen's duality.* In discussion before Proposition 3.4.1 we associated the Chevalley complex $C^\bullet(\mathfrak{a})$ with any Lie s-algebra $\mathfrak{a}$. It can be treated as commutative s-algebra $\Lambda^\bullet(\mathfrak{a}^*[-1])$ equipped with the differential whose action on generators is dual to the bracket $\Lambda^2\mathfrak{a} \longrightarrow \mathfrak{a}$. This construction provides a functor from the category of Lie s-algebras to one of commutative DG-algebras.

On the other side, with any commutative s-algebra A one can associate in a similar way the Harrison complex $\mathscr{H}^\bullet(A)$, which is a free Lie s-algebra $\mathscr{L}ie^\bullet(A^*[-1])$ equipped with the differential whose action on generators is dual to the multiplication $S^2A \longrightarrow A$. Thus, we get a functor acting in the opposite direction.

[42] if $\#I = m$ there are totally $\binom{\lambda'_i}{m}$ such permutations

Applying to these functors the general result proven in [**17**, th. 4.2.5] for any pair of Koszul dual operads, we get

LEMMA 4.1.5. *Two functors described above are homotopy inverse to each other, i. e. for any pro-nilpotent Lie s-algebra* $\mathfrak{a}$ *and any s-commutative algebra* A *there exist natural quasi-isomorphisms* $C^\bullet(\mathscr{H}^\bullet(A)) \xrightarrow{\sim} A$, $\mathscr{H}^\bullet(C^\bullet(\mathfrak{a})) \xrightarrow{\sim} \mathfrak{a}$. □

COROLLARY 4.1.6. *Let* $\mathfrak{a}$ *be a pro-nilpotent graded Lie s-algebra equipped with* GL_k*-action preserving the graded Lie s-algebra structure. Then the Euler* GL_k*-characters of* $\mathfrak{a}$ *and* $H = H^\bullet(\mathfrak{a}, \mathbb{C})$ *are related by*

$$\chi_H = \sum_{n\geqslant 0} (-1)^n e_n \circ \chi_{\mathfrak{a}} , \tag{4.7}$$

$$\chi_{\mathfrak{a}} = \Big(- \sum_{m\geqslant 1} \mu(m) \ln\big(1 - (-1)^m p_m\big)\Big) \circ \chi_H , \tag{4.8}$$

where μ *is the Möbius function,* e_n *are the elementary symmetric polynomials,* p_m *are Newton's sums of powers, and* $\circ$ *means the plethysm of symmetric functions.*

PROOF. It is well known that GL-character of $\Lambda^n(V)$ equals e_n (see [**30**]) and GL-character of $\mathscr{L}ie^n(V)$ equals $\frac{1}{n}\sum_{d|n}\mu(d)p_d^{\frac{n}{d}}$ (see [**27**]). The signs in (4.7) and (4.8) come from the grading shift. □

4.1.7. *Free Lie algebras.* If $\mathfrak{a} = \mathscr{L}ie(W)$ be a free Lie (graded, s-) algebra generated by a vector space W, then its universal enveloping algebra $U(\mathfrak{a})$ is a free associative algebra generated by W and the trivial $U(\mathfrak{a})$-module $\mathbb{C}$ admits a short free resolution

$$0 \longrightarrow U(\mathfrak{a}) \otimes W \longrightarrow U(\mathfrak{a}) \longrightarrow \mathbb{C} \longrightarrow 0 . \tag{4.9}$$

For any subalgebra $\mathfrak{b} \subset \mathfrak{a}$ an isomorphism of $\mathfrak{b}$-modules $U(\mathfrak{a}) \simeq U(\mathfrak{b}) \otimes S(\mathfrak{a}/\mathfrak{b})$ shows that (4.9) is a free resolution for $\mathbb{C}$ in the category of $U(\mathfrak{b})$-modules as well. It is well known that pro-nilpotent Lie algebra $\mathfrak{b}$ is free iff $H^i(\mathfrak{b}, \mathbb{C}) = 0$ for $i > 1$. In particular, applying $\mathrm{Hom}_{U(\mathfrak{b})}(\,*\,, \mathbb{C})$ to the short resolution (4.9), we get

LEMMA 4.1.8. *A subalgebra of any free Lie (super)algebra is free.* □

4.2. The most singular case. To begin with, consider the commutative quadratic algebra with the maximal possible space of quadratic relations, i. e.

$$A = S(V)/(S^2V) .$$

Its Koszul dual Lie s-algebra is the free graded Lie algebra $L = \mathscr{L}ie\,(V[-1])$ generated by the vector space V situated in degree 1. The existence of resolution (4.9) (written for $\mathfrak{a} = L$) and criteria from sect. 3.3 imply that L and A are Koszul. By Lemma 4.1.8 the Lie subalgebra $L_{\geqslant 2} \subset L$ is free as well. This implies that the syzygies (4.1) vanish for $(q - p) > 1$. Thus, non trivial syzygies are described by

PROPOSITION 4.2.1. *For* $A = S(V)/S^2(V)$ *the component* $R_{p,(p+1)}$ *of syzygies* (4.1) *is the irreducible* $\mathrm{GL}(V)$*-module* $\pi_{[2,1^{p-1}]}$. *All the other syzygies vanish.*

PROOF. In this case the Koszul complex (2.7) takes extremely simple form and can be written as the tensor product $K^\bullet = (\mathbb{C} \oplus V) \otimes \Lambda^\bullet(V[-1])$. Thus, its Euler

$\mathrm{GL}(V)$-character has the following expression in terms of the elementary symmetric polynomials e_k

$$\chi_{K^\bullet} = (1+e_1)\cdot\sum_{k\geqslant 0}(-1)^k\,e_k = 1+\sum_{k\geqslant 1}(-1)^k\,(e_1e_k - e_{k+1})\ . \tag{4.10}$$

By the Frobenius character formula[43], the latter multiplier of degree $(k+1)$ coincides with the irreducible $\mathrm{GL}(V)$-character

$$s_{[2,1^{k-1}]} = \det\begin{pmatrix} e_k & e_{k+1}\\ e_0 & e_1\end{pmatrix}$$

of the irreducible GL-module $[2,1^{k-1}]$. Since for each $q>0$ there is only one non zero component $R_{p,q}$ and it has $p=q-1$, we conclude that $[2,1^{k-1}] = R_{k,(k+1)}$. $\square$

4.3. Syzygies of the Veronese curve. The Veronese embedding takes

$$\mathbb{P}_1 = \mathbb{P}(U) \xhookrightarrow{u\mapsto u^d} \mathbb{P}(S^nU) = \mathbb{P}_n\ .$$

In appropriate coordinates on $\mathbb{P}(S^nU)$ it sends $(u_0:u_1)\in\mathbb{P}_1$ to

$$(x_0 : x_1 : \ \dots\ : x_n) = (u_0^n : u_0^{n-1}u_1 : u_0^{n-2}u_1^2 : \ \dots\ : u_1^n)\ .$$

The image is described by the quadratic equations $x_ix_j = x_kx_m$ (for all possible choices of i, j, k, m with $i+j=k+m$), so, we get the quadratic algebra

$$A = \mathbb{C}[x_0,\ \dots\ , x_n]/(x_ix_j - x_kx_m \mid i+j=k+m)\ .$$

In the dual coordinates x^i on $V^* = S^nU^*$ the relations for the Koszul dual Lie s-algebra take a form

$$\sum_{i+j=k}[x^i,x^j] = 0\ ,\quad\text{where } k = 0,\,1,\dots,2n\ . \tag{4.11}$$

Let us order the generators in the following non-standard way

$$x^0 > x^n > x^{n-1} > \dots > x^1\ .$$

This ordering induces the filtration on L. Since the leading Lie monomial of k-th relation in (4.11) is $[x^0,x^k]$ for $k\leqslant n$ and is $[x^n,x^k]$ for $k\geqslant n$, associated with this filtration graded Lie algebra L' is isomorphic to the direct sum of the abelian Lie algebra generated by x^0, x^n and the free Lie algebra $\mathscr{L}ie(W)$, where W is the linear span of remaining $(n-1)$ variables $x^1,\dots,x^{(n-1)}$. Since the abelian part makes no contribution in the Chevalley complex for $L_{\geqslant 2}$, we have $C^\bullet(L_{\geqslant 2}) = C^\bullet\big(\mathscr{L}ie(W)_{\geqslant 2}\big)$ and can repeat word for word the computations of the previous section taking W instead of V.

COROLLARY 4.3.1. *The only non zero syzygies of the Veronese curve in $\mathbb{P}_n$ are $R_{p,p+1} = \pi_{[2,1^{p-1}]}$ with*[44] $\dim R_{p,p+1} = p\cdot\binom{n}{p+1}$. $\square$

[43] see, for example, [**30**, ch. 1.3, formula (3.5)]

[44] we consider $R_{p,p+1}$ as $\mathrm{GL}(W)$-modules and apply the hook length formula (see [**12**], [**30**])

4.4. Syzygies of the grassmannian $\mathbf{Gr(2, N)}$. This computation generalizes the examples 1.1.3, 2.3.3. The Plücker embedding

$$\mathrm{Gr}(2,N) \xrightarrow{\sim} \mathbb{P}(G \cdot v_{\mathrm{hw}}) \subset \mathbb{P}(V_{\omega_2}) \tag{4.12}$$

realizes the grassmannian as the SHW-orbit of $G = \mathrm{SL}(N,\mathbb{C})$ with the index (1.2) equal to N. Moreover, the natural action of $\mathrm{GL}_N(\mathbb{C})$ on $\mathbb{P}(V_{\omega_2})$ preserves the grassmannian and is kept on the syzygies as well. Write $W = V_{\omega_1}^*$ for dual to the standard N-dimensional representation of GL_N. It follows from Propositions 1.3.1 and 1.3.2 that the projective coordinate algebra of the Plücker embedding (4.12) has the following GL_N-module decomposition:

$$A = S^\bullet(\Lambda^2 W)/(\Lambda^4 W) = \bigoplus_{k=0}^{\infty} \pi_{[k,k]} \ . \tag{4.13}$$

Our computation of the syzygies of A will be organized as follows. In section 4.5 we compute the Euler GL-character of the Koszul complex

$$K^\bullet = \Lambda^\bullet\left((\Lambda^2 W)[-1]\right) \otimes A \ . \tag{4.14}$$

Then, in section 4.6, we completely describe the generators of $L_{\geqslant 2}$, i.e. compute $H^1(L_{\geqslant 2},\mathbb{C})$. It turns out that these generators form a graded GL-module[45]

$$\bigoplus_{2\leqslant q\leqslant (N-2)} \pi_{((q-2)\,|\,(q+1))} \quad \text{with} \quad \pi_{((q-2)\,|\,(q+1))} = R_{q-1,q} = H^1(L_{\geqslant 2},\mathbb{C})_q \ .$$

Moreover, we will show that $\pi_{((q_1-1),\dots,(q_p-1)\,|\,(q_1+2),\dots,(q_p+2))}$ appears with multiplicity one in the syzygy space $R_{p,(q_1+\dots+q_p)}$.

This allows to guess the shape of answer and forces to introduce a bigraded skew commutative GL_N-equivariant s-algebra

$$\mathsf{A} = \bigoplus_{p,q} \mathsf{A}_{p,q} \ , \quad \text{where}$$

$$\mathsf{A}_{p,q} = \bigoplus_{\substack{(N-2)\geqslant i_1>\dots>i_p\geqslant 2\\ i_1+\dots+i_p=q}} \pi_{((i_1-2),\dots,(i_p-2)\,|\,(i_1+1),\dots,(i_p+1))} \ . \tag{4.15}$$

We define A as a skew commutative s-algebra generated by the graded vector space

$$\mathsf{A}_1 = \bigoplus_q \mathsf{A}_{1,q} \ , \text{ where } \mathsf{A}_{1,q} = \begin{cases} \pi_{(q-2|q+1)} \ , & \text{for } 2 \leqslant q \leqslant (N-2)\,, \\ 0\,, & \text{otherwise} \end{cases} \tag{4.16}$$

By the definition, the multiplication map $\mathsf{A}_{1,q_1} \otimes \dots \otimes \mathsf{A}_{1,q_p} \longrightarrow \mathsf{A}_{p,(q_1+\dots+q_p)}$ is given by the projection onto irreducible component

$$\pi_{((q_1-2),\dots,(q_p-2)\,|\,(q_1+1),\dots,(q_p+1))} \subset \pi_{(q_1-2|q_1+1)} \otimes \dots \otimes \pi_{(q_p-2|q_p+1)} \ , \tag{4.17}$$

if $q_1 > \dots > q_p$, and vanishes when some of q_i's coincide. Since the component (4.17) has the multiplicity one, the corresponding projection is unique up to proportionality and we actually get well defined associative algebra A with the components (4.15). This algebra gives a particular example of GL-equivariant *hook algebra*.

General properties of hook algebras will be discussed systematically in the next §5. In particular, in section 5.1.2 we show that A is quadratic and Koszul.

Using this result, we show in section 4.7 below that a Lie s-algebra L, Koszul dual to A, is isomorphic to $L_{\geqslant 2}$. This implies the coincidence

$$\mathsf{A} = H(\mathsf{L},\mathbb{C}) = H(L_{\geqslant 2},\mathbb{C}) = R \ .$$

[45]we are using here the Frobenius notations (4.3)

THEOREM 4.4.1. *The syzygies of the Grassmannian* $\mathrm{Gr}(2,N)$ *form a bigraded skew commutative Frobenius quadratic Koszul algebra isomorphic to the algebra* A *defined in* (4.15)–(4.17)*:* $R_{p,q} = \mathsf{A}_{q-p,q}$. □

4.5. Koszul complex. Euler's GL-character of $K^\bullet = \Lambda^\bullet\left((\Lambda^2 W)[-1]\right) \otimes A$ is

$$\chi_{K^\bullet} = \chi_{\Lambda^\bullet(\Lambda^2 W[-1])} \cdot \Big(\sum_j s_{[j,j]}\Big) = \\ = \Big(\sum_k (-1)^k e_k \circ e_2\Big) \;\cdot\; \Big(\sum_j h_j^2 - h_{j+1}h_{j-1}\Big)\,, \tag{4.18}$$

where h_k are the complete symmetric functions, $\circ$ denotes the plethysm of symmetric functions, and

$$s_{[j,j]} = \det\begin{pmatrix} h_j & h_{j+1} \\ h_{j-1} & h_j \end{pmatrix}$$

by the Frobenius character formula[46]. The right hand side of (4.18) has the following expansion in terms of Schur polynomials:

LEMMA 4.5.1. $\chi_K = \sum\limits_{\substack{p\geqslant 0 \\ (N-3)\geqslant i_1 > \cdots > i_p > 0}} (-1)^{i_1+\cdots+i_p} s_{((i_1-1),\ldots,(i_p-1)\,|\,(i_1+2),\ldots,(i_p+2))}$.

PROOF. We deduce it from the following generalized Littlewood formula established in [**20**, Th. 4.4, eq. (2)]:

$$\sum_{\substack{p\geqslant 0 \\ n\geqslant i_1>\cdots>i_p\geqslant 0}} (-1)^{p+i_1+\cdots+i_p} s_{((i_1+r),\ldots,(i_p+r)\,|\,i_1,\ldots,i_p)} = \\ \prod_{i\leqslant j}^{n}(1-x_ix_j) \;\cdot\; \prod_{i=1}^{n} x_i^{\frac{(r-1)}{2}} \cdot \frac{\det\left(x_i^{j+\frac{r-1}{2}} - x_i^{-j-\frac{r-1}{2}}\right)_{1\leqslant i,j\leqslant n}}{\Delta_{C(n)}}\,, \tag{4.19}$$

where the Weyl determinant $\Delta_{C(n)}$ in the denominator can be expressed as

$$\Delta_{C(n)}(x_1,x_2,\ldots,x_n) = \det\left(x_i^j - x_i^{-j}\right)_{1\leqslant i,j\leqslant n} = \\ = \prod_{i=1}^{n} x_i^{-n} \;\cdot\; \prod_{i=1}^{n}(x_i^2-1) \;\cdot\; \prod_{i<j}^{n}(x_i-x_j)(1-x_ix_j)\,. \tag{4.20}$$

We take $r=3$ and note that $\det\left(x_i^{j+1} - x_i^{-j-1}\right)_{1\leqslant i,j\leqslant n}$ staying in the numerator of (4.19) can be written as

$$\Delta_{C(n)} \;\cdot\; \prod_{i=1}^{n} x_i^{-1} \;\cdot\; \Big(\sum_j e_j^2 - e_je_{j+2}\Big)\,. \tag{4.21}$$

Indeed, consider the $(n+1)$-th order Weyl determinant $\Delta_{C(n+1)}(x_1,\ldots,x_n,q)$ as a Laurent polynomial in q and compute its coefficient at q^1 in two ways: by the straightforward expansion of the determinant in the middle of (4.20) and using triple product expansion from the last term of (4.20). In the first case we get

$$(-1)^n \det\left(x_i^j - x_i^{-j}\right)_{\substack{1\leqslant i\leqslant n \\ 2\leqslant j\leqslant n+1}} = (-1)^n \det\left(x_i^{j+1} - x_i^{-j-1}\right)_{1\leqslant i,j\leqslant n}$$

[46] see [**13**] or [**30**, ch. 1.3, formula (3.4)]

In the second case we have

$$\prod_{i=1}^n x_i^{-n-1}\prod_{i=1}^n(x_i^2-1)\prod_{i<j}(x_i-x_j)(1-x_ix_j)q^{-n-1}(q^2-1)\prod_{i=1}^n(x_i-q)(1-x_iq)=$$
$$=\Delta_{C(n)}\cdot\prod_{i=1}^n x_i^{-1}q^{-n-1}(q^2-1)\cdot\Big(\sum_{i=0}^n(-1)^i e_{n-i}q^i\Big)\cdot\Big(\sum_{i=0}^n(-1)^i e_i q^i\Big),$$

whose coefficient at q^1 is (4.21) multiplied by $(-1)^n$. Now, substituting (4.21) in the numerator of (4.19), we get the identity

$$\sum_{\substack{p\geqslant 0\\ i_1>\cdots>i_p\geqslant 0}}(-1)^{p+i_1+\cdots+i_p}s_{((i_1+3),\ldots,(i_p+3)\,|\,i_1,\ldots,i_p)}= \tag{4.22}$$
$$=\prod_{i\leqslant j}^n(1-x_ix_j)\cdot\Big(\sum_j e_j^2-e_{j+1}e_{j-1}\Big)=$$
$$=\Big(\sum_{k=0}^{\frac{n(n+1)}{2}}(-1)^k e_k\circ h_2\Big)\cdot\Big(\sum_{j=0}^n e_j^2-e_{j+1}e_{j-1}\Big).$$

Since it holds for all n, we can consider (4.22) as an identity in the complete ring of symmetric functions (in the infinite set of variables) and apply to (4.22) the ω-involution, which exchanges $s_\lambda\leftrightarrow s_{\lambda'}$, $e_k\leftrightarrow h_k$. So, we get

$$\sum_{\substack{p\geqslant 0\\ i_1>\cdots>i_p\geqslant 0}}(-1)^{p+i_1+\cdots+i_p}s_{(i_1,\ldots,i_p\,|\,(i_1+3),\ldots,(i_p+3))}=$$
$$=\Big(\sum_k(-1)^k e_k\circ e_2\Big)\cdot\Big(\sum_j h_j^2-h_{j+1}h_{j-1}\Big),$$

whose right hand side coincides with (4.18) □

Corollary 4.5.2. *For any decreasing sequence $(N-3)\geqslant i_1>\cdots>i_p>0$ the irreducible representation $\pi_{((i_1-1),\ldots,(i_p-1)\,|\,(i_1+2),\ldots,(i_p+2))}$ appears in the both GL_N-modules*

$$\Lambda^{i_1+\cdots+i_p}(\Lambda^2W)\otimes A_{(p)}\subset\bigoplus_{k\geqslant 0}\Lambda^k\left(\Lambda^2W\right)\otimes A$$

with multiplicity one and comes down into $R_{p,q}=H^p(K^\bullet_{\mathrm{Gr}})_{(q)}$ with $q=p+i_1+\cdots+i_p$.

Proof. Using the Weyl formula[47] we can write χ_K as

$$\chi_{\Lambda^\bullet(\Lambda^2W[-1])}\cdot\chi_A=\sum_{\substack{p\geqslant 0\\ N\geqslant i_1>\cdots>i_p>0}}s_{((i_1-1),\ldots,(i_p-1)\,|\,i_1,\ldots,i_p)}\cdot\sum_{j\geqslant 0}s_{[j,j]}\,. \tag{4.23}$$

The classical Littlewood–Richardson rule[48] implies that

$$s_{((i_1-1),\ldots,(i_p-1)\,|\,(i_1+2),\ldots,(i_p+2))}$$

(staying in Lemma 4.5.1) could appear in (4.23) only as the product

$$s_{((i_1-1),\ldots,(i_p-1)\,|\,i_1,\ldots,i_p)}\cdot s_{[p,p]}\,.$$

[47] comp. with [**30**, ch. 1.5, example 9(a)]

[48] see [**13**], [**30**]

Thus, the irreducible component $\pi_{((i_1-1),\dots,(i_p-1)\,|\,(i_1+2),\dots,(i_p+2))}$ comes into

$$H^p(K^\bullet_{\mathrm{Gr}})_{(p+i_1+\dots+i_p)}$$

exactly from $\Lambda^{i_1+\dots+i_p}(\Lambda^2 W)\otimes A_{(p)}$, where it sits with multiplicity 1. $\square$

4.6. Dual Lie s-algebra. Let us fix some coordinates $x^1,\dots,x^N$ on W and write $x^{ij}=x^i\wedge x^j$ for the corresponding basis of $L_1=\Lambda^2W^*$, which is the generating space for the graded Lie s-algebra L Koszul dual to A in the sense of 3.1. Since $\mathrm{Sym}^2(\Lambda^2W^*)\simeq\pi^*_{[2,2]}\oplus\pi^*_{[1^4]}$, it follows from (3.1), (3.2) that $L_2=\Lambda^4W^*$ and the relations for L have the form

$$[x^{ij},x^{k\ell}]=-[x^{ik},x^{j\ell}]=[x^{i\ell},x^{jk}]\,. \tag{4.24}$$

Let us write shortly $L'_n=L_n/\left([L_{\geqslant 2},L_{\geqslant 2}]\cap L_n\right)=H^1(L_{\geqslant 2},\mathbb{C})_{(n)}=R_{n-1,n}$ for the space of degree n generators in subalgebra $L_{\geqslant 2}$.

Lemma 4.6.1. *$L'_n\simeq\pi^*_{(n-2|n+1)}$ for each n in range $2\leqslant n\leqslant N-2$.*

Proof. Induction on n. For $n=2$ we have $L'_2=L_2=\Lambda^4W^*$ generated by commutators (4.24). For $n>2$ we have the surjective commutator map

$$L'_{n-1}\otimes\Lambda^2W^*\xrightarrow{a\otimes x^{ij}\mapsto[a,x^{ij}]}L'_n\,.$$

By the inductive assumption and the Littlewood–Richardson rule, the left hand side has the following irreducible GL_N-module decomposition

$$\begin{aligned}\pi^*_{[1^2]}\otimes\pi^*_{(n-3|n)}&\simeq\\ &\simeq\pi^*_{(n-2,0|n,0)}\oplus\pi^*_{(n-2|n+1)}\oplus\pi^*_{(n-3,0|n,1)}\oplus\pi^*_{(n-3,0|n+1,0)}\oplus\pi^*_{(n-3|n+2)}\,.\end{aligned}\tag{4.25}$$

The s-Jacobi identity implies that $[a,[x^{ij},x^{k\ell}]]=[[a,x^{ij}],x^{k\ell}]+[[a,x^{k\ell}],x^{ij}]$. Since the left hand side here vanishes in $L'_n=\left(L_{\geqslant 2}/[L_{\geqslant 2},L_{\geqslant 2}]\right)_{(n)}$, the skew symmetrization operator taking

$$\sum_\nu c_\nu\,[[a_\nu,x^{i_\nu j_\nu}],x^{k_\nu\ell_\nu}]\longmapsto\frac{1}{2}\sum_\nu c_\nu\left([[a_\nu,x^{i_\nu j_\nu}],x^{k_\nu\ell_\nu}]-[[a_\nu,x^{k_\nu\ell_\nu}],x^{i_\nu j_\nu}]\right)$$

acts on L'_n as the identity. On the other hand, it annihilates all the irreducible summands of (4.25) except for the second one, which definitely has to appear in $R_{n-1,n}=H^1(L_{\geqslant 2},\mathbb{C})_{(n)}=L'_n$ by the corollary 4.5.2. $\square$

4.7. Proof of Theorem 4.4.1. Since A is (super) skew commutative and Koszul, its quadratic dual algebra $\mathsf{B}=U(\mathsf{L})$ is an universal enveloping algebra for some graded Lie s-algebra L such that $H^\bullet(\mathsf{L},\mathbb{C})=\mathsf{A}$. It follows from Corollary 4.5.2 and Lemma 4.6.1 that there is a surjective homomorphism of associative algebras $H^\bullet(L_{\geqslant 2},\mathbb{C})\twoheadrightarrow\mathsf{A}$. Since 2-th Lie algebra cohomologies describe the relations, we conclude that the relations of $L_{\geqslant 2}$ contain the ones of L. Because the both algebras are generated by the same vector space, there is a surjective GL-module homomorphism $\mathsf{L}\xrightarrow{\psi}L_{\geqslant 2}$. It follows from Corollary 4.1.6 that $\chi_{L_{\geqslant 2}}=\chi_{\mathsf{L}}$, i.e. the Euler GL-character of $\ker\psi$ vanishes. This forces $\ker\psi$ to have each irreducible GL-module equal number of times in its even and odd parts. But it is impossible, because the Young diagrams appearing in odd and even parts of L have different number of cells modulo 4 (they consist of $4k+2$ and $4k$ cells respectively). Thus, $\ker\psi=0$ and theorem 4.4.1 is completely proven.

5. Appendix: koszulity of hook algebras

In this section we use the notations fixed in sec. 4.1.

5.1. Hook algebras. We call Γ-shaped diagram $\Gamma = (\alpha|\beta)$ *a hook* of *width* $\alpha+1$ and *hight* $\beta+1$. We say that two hooks $\Gamma_1 = (\alpha_1|\beta_1)$, $\Gamma_2 = (\alpha_2|\beta_2)$ are compatible and write $\Gamma_1 > \Gamma_2$, if their union $\Gamma_1 \sqcup \Gamma_2 = (\alpha_1, \alpha_2|\beta_1\beta_2)$ is a valid Young diagram, i. e. has $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$.

Let us fix an ordered collection of pairwise compatible hooks

$$\Gamma_1 > \Gamma_2 > \cdots > \Gamma_m \tag{5.1}$$

equipped with some *internal parities* $|\Gamma_i| \in \mathbb{Z}/(2)$. We assume that the heights of all hooks are bounded by k and write $\pi_i = \pi_{\Gamma_i}$ for the corresponding irreducible GL_k-modules. For any increasing collection of indexes $I = (i_1, i_2, \dots, i_s) \subset [1\,..\,m]$ we denote by

$$\Gamma_I = \Gamma_{i_1,i_2,\dots,i_s} = \Gamma_{i_1} \sqcup \Gamma_{i_2} \sqcup \cdots \sqcup \Gamma_{i_s} \tag{5.2}$$

the Young diagram build from the corresponding hooks and write $\pi_I = \pi_{\Gamma_I}$ for the associated irreducible GL_k-module. It follows from the Littlewood–Richardson rule that π_I appears in the irreducible decomposition of $\pi_{i_1} \otimes \pi_{i_2} \otimes \cdots \otimes \pi_{i_s}$ with multiplicity one. Thus, there is a canonical GL_k-equivariant projection

$$\mu_I : \pi_{i_1} \otimes \pi_{i_2} \otimes \cdots \otimes \pi_{i_s} \twoheadrightarrow \pi_I\ . \tag{5.3}$$

Hence, GL_k-module $\mathsf{A} = \bigoplus_I \pi_I$ admits a GL_k-equivariant associative algebra structure whose multiplication satisfies the relations

(1) $x \cdot y = (-1)^{|\Gamma_i|\cdot|\Gamma_j|}\, y \cdot x$ for any $x \in \pi_i$, $y \in \pi_j$ and any choice of $i \neq j$ in the range $[1\,..\,m]$;
(2) $x \cdot y = 0$ for any $x, y \in \pi_i$ and any choice of $i \in [1\,..\,m]$;
(3) $x_1 \cdot x_2 \cdot \cdots \cdot x_s = \mu_I(x_1, x_2, \dots, x_s)$ for any choice of strictly increasing indexes $I = (i_1, i_2, \dots, i_s) \subset [1\,..\,m]$ and any collection of $x_\nu \in \pi_{i_\nu}$.

In other words, A is s-commutative w. r. t. the internal parity, it is generated by $\mathbb{Z}/(2)$-graded vector space $\mathsf{A}_1 = \bigoplus_{i=1}^{m} \pi_{\Gamma_i}$, and all non zero multiplication maps in A are induced by projections (5.3). There is also *an external grading*

$$\mathsf{A} = \bigoplus_{s=0}^{m} \mathsf{A}_s\ , \quad \text{where} \quad \mathsf{A}_s = \bigoplus_{\#I=s} \pi_{\Gamma_I}\ , \quad \mathsf{A}_0 = \mathbb{C}\ .$$

We call A *a hook algebra* associated with hooks (5.1) and write $\mathsf{A}(\Gamma_1, \Gamma_2, \dots, \Gamma_m)$, if the precise reference on the hooks is important. In sec. 5.1.3–5.1.5 we will prove

THEOREM 5.1.1. *Any hook algebra* A *is quadratic and Koszul.*

Since the algebra A described in (4.16)–(4.15) is a hook algebra build from the hooks $\Gamma_i = (i-1|i+2)$, $1 \leqslant i \leqslant (N-3)$ of parities $|\Gamma_i| = i \pmod 2$, we get

COROLLARY 5.1.2. *The algebra* A*, used in sec. 4.4, is quadratic and Koszul.*

5.1.3. *Proof of theorem 5.1.1.* For each $\pi_i = \pi_{\Gamma_i}$ we fix the standard basis labeled by the Young tableaux T_{Γ_i} of shape Γ_i on the alphabet $[1\,..\,k]$ (see sec. 4.1.2).

Let $\widetilde{\mathsf{A}}$ be an algebra spanned by A_1 and satisfying only the first two sets of the relations for A, i. e. $\widetilde{\mathsf{A}}$ is s-commutative w. r. t. the internal parity and satisfies $\pi_i \cdot \pi_i = 0$ for all i. Thus, $\widetilde{\mathsf{A}}$ is a quadratic monomial algebra, in particular, it is automatically Koszul[49]. With respect to the GL_k-action, the graded components of $\widetilde{\mathsf{A}}$ are decomposed as

$$\widetilde{\mathsf{A}}_s = \bigoplus_{\#I=s} \pi_{i_1} \otimes \pi_{i_2} \otimes \cdots \otimes \pi_{i_s}$$

where the sum runs over all strictly increasing collections $I = (i_1, i_2, \ldots, i_s) \subset [1\,..\,m]$. Tensor products of the standard basic vectors from π_i form a basis for $\widetilde{\mathsf{A}}_s$. We call these products *standard basic monomials*. They are numbered by Young diagrams (5.2) filled by numbers from range $[1\,..\,m]$ in such a way that each hook $\Gamma_{i_\nu} \subset \Gamma_I$ is a valid Young tableau but the whole Γ_I may be not. Let us call these filled diagrams *hooked tableaux* or h-tableaux for shortness.

We write $x_T \in \widetilde{\mathsf{A}}$ for the standard basic monomial corresponding to an h-tableau T. Note that $x_S \cdot x_T = 0$, if the underlying Young diagrams contain common hooks. Otherwise, $x_S \cdot x_T = \pm x_{S\cdot T}$, where $S \cdot T$ is build from S, T by rearranging their hooks in strictly decreasing order. Thus, x_T do actually behave as monomials.

The hook algebra can be presented as $\mathsf{A} = \widetilde{\mathsf{A}}/J$, where $J = \bigoplus_s J_s$ is a graded ideal whose components split w. r. t. the GL_k-action as

$$J_s = \bigoplus_{\#I=s} J_I \ , \quad \text{where } J_I = \ker\left(\pi_{i_1} \otimes \pi_{i_2} \otimes \cdots \otimes \pi_{i_s} \xrightarrow{\mu_I} \pi_I\right) \ .$$

To show that A is quadratic Koszul algebra, we will equip the set of all standard basic monomials $x_T \in \widetilde{\mathsf{A}}$ with a preorder $\preccurlyeq$ satisfying following two PBW-type[50] conditions:

(5.4) $\qquad x_S \preccurlyeq x_T \Longrightarrow x_{R\cdot S} \preccurlyeq x_{R\cdot T}$ for any h-tableaux R, S, T ;

(5.5) $\qquad$ for any h-tableau T which is not a valid tableau there exist an element $h_T \in (J_2 \cdot \widetilde{\mathsf{A}}) \cap J_I$ (uniquely determined by T) such that

$$h_T - x_T = \sum_{\substack{S \prec T \\ I(S)=I(T)}} c_S x_S \quad \text{for some} \quad c_S \in \mathbb{Z} \ .$$

The second condition implies that each basic monomial x_T is congruent modulo the elements h_T to some monomial $x_{T'}$ whose h-tableau T' is a valid Young tableau. Since the images of the latter monomials form a basis for the vector space $\mathsf{A} = \bigoplus_I \pi_I = \widetilde{\mathsf{A}}/J$, we conclude that the elements h_T generate J as a vector space. Because h_T lay in the ideal (J_2) spanned by the quadratic component of J, we get $J = (J_2)$. Thus, A is quadratic.

Further, let J° be a monomial ideal spanned by the leading monomials of the elements from J. By the same reasons as above, J° is generated by all x_T such that T is not a valid tableau. Now the same arguments as in [**36**, ch. 3] show that koszulity of the monomial quadratic algebra $\mathsf{A}^\circ = \widetilde{\mathsf{A}}/J^\circ$ implies the koszulity of A.

49 see [**36**]

50 see [**36**] for non-commutative version of the Poincare–Birkhof–Witt theory

Indeed, the multiplicative condition (5.4) implies that the preorder in question induces a filtration on the bar complex of the hook algebra A such that the associated graded complex is the bar complex of the Koszul algebra A°. Computing $\mathrm{Ext}_{\mathsf{A}}(\mathbb{C},\mathbb{C})$ via the spectral sequence associated with this filtration, we get $\mathrm{Ext}_{\mathsf{A}^\circ}(\mathbb{C},\mathbb{C})$ as the first term of the sequence. It shows that $\mathrm{Ext}^{i,j}_{\mathsf{A}}(\mathbb{C},\mathbb{C}) = 0$ for $i \neq j$ (see details in [**36**, ch. 3]).

Thus, to finish the proof of theorem 5.1.1, it remains to equip the set of h-tableaux with a preorder satisfying the PBW-properties (5.4)–(5.5).

5.1.4. *PBW-preorder on h-tableaux.* Consider an h-tableau T whose Young diagram is the union of strictly decreasing hooks $\Gamma_{i_1} > \Gamma_{i_2} > \cdots > \Gamma_{i_s}$. For any $\mu \in [1\,..\,k]$, $\nu \in [1\,..\,m]$ let $\chi^T(\mu,\nu)$ be a number of times the element μ does appear in the ν-th hook Γ_{i_ν} of T. For a fixed μ we consider the numbers $\chi^T(\mu,\nu)$ as the components of m-dimensional vector

$$\xi^T_\mu = \left(\chi^T(\mu,1), \chi^T(\mu,2), \dots, \chi^T(\mu,m)\right)$$

where $\chi^T(\mu,\nu) = 0$ when $\nu > s$. For example, if we deal with GL_4-equivariant hook algebra built from $m = 3$ hooks, then 2-hooked tableau

$$T = \begin{array}{|c|c|c|c|}\hline 1&1&2&4\\ \hline 3&1&2&2\\ \hline 4&3\\ \cline{1-2}\end{array}$$

produces four 3-component vectors

$$\begin{aligned} \chi^T_1 &= (2,1,0) \quad & \chi^T_3 &= (1,1,0) \\ \chi^T_2 &= (1,2,0) \quad & \chi^T_4 &= (2,0,0)\,. \end{aligned} \tag{5.6}$$

Note that a diagram T is not uniquely defined by the collection of k vectors

$$\chi^T = (\chi^T_1, \chi^T_2, \dots, \chi^T_k)\,.$$

For example, collection (5.6) also comes from the h-tableau

$$T' = \begin{array}{|c|c|c|c|}\hline 1&1&4&4\\ \hline 2&1&2&3\\ \hline 3&2\\ \cline{1-2}\end{array}$$

and some others.

We will compare the vectors χ^T_μ using inverse right lexicographic ordering, i. e. we say that $\left(\chi^S_\mu(1), \chi^S_\mu(2), \dots, \chi^S_\mu(m)\right) < \left(\chi^T_\mu(1), \chi^T_\mu(2), \dots, \chi^T_\mu(m)\right)$, if

$$\chi^S(\mu,\nu_0) > \chi^T(\mu,\nu_0) \quad \& \quad \forall\, \nu > \nu_0 \quad \chi^S(\mu,\nu) = \chi^T(\mu,\nu)\,. \tag{5.7}$$

We say that $S \prec T$, if $(\chi^S_1, \chi^S_2, \dots, \chi^S_k) < (\chi^T_1, \chi^T_2, \dots, \chi^T_k)$ w. r. t. the inverse right lexicographic ordering, i. e. if $\chi^S_\mu = \chi^T_\mu$ for all $\mu > \mu_0$ and $\chi^S_{\mu_0} > \chi^T_{\nu_0}$ in the sense (5.7). By the definition, the condition $S \preccurlyeq T$ means either the strong inequality $S \prec T$ or the coincidence $(\chi^S_1, \chi^S_2, \dots, \chi^S_k) = (\chi^T_1, \chi^T_2, \dots, \chi^T_k)$.

Thus, the relation $\preccurlyeq$ gives a preorder on the set of h-tableaux and two h-tableaux are equivalent w. r. t. this preorder iff their fillings differ by a permutation preserving the content of each hook. This preorder evidently satisfies the multiplicative condition (5.4). It remains to construct special elements h_T satisfying the PBW-condition (5.5).

5.1.5. *PBW-basis for J.* Consider an arbitrary hooked tableau T which is not a valid tableau. Let the Young diagram of T consist of hooks $\Gamma_1 > \Gamma_2 > \cdots > \Gamma_s$. We take minimal i such that a hooked subtableau of T formed by $\Gamma_{i+1} \sqcup \Gamma_{i+2} \sqcup \cdots \sqcup \Gamma_s$ is a valid tableau. Then a subtableau $D \subset T$ formed by $\Gamma_i \sqcup \Gamma_{i+1}$ is not valid. This can happen by two reasons (we write $\delta_{p,q}$ for an element staying in p-th row and q-th column of D):

(A) $\exists\ k \geqslant 2\ :\ \delta_{k,2} < \delta_{k,1}$, i.e. the wrong inequality appears in some row of D (but not in the first);

(B) condition (A) fails but $\exists\ \ell, k \geqslant 2\ :\ \delta_{1,\ell} \geqslant \delta_{k,\ell}$, i.e. all rows of D are valid and a wrong inequality appears in some column (but not in the first).

We will construct h_T separately for each case. However, in the both cases we construct h_T as an element of the space

$$\pi_1 \otimes \cdots \otimes \pi_{i-1} \otimes J_{i,i+1} \otimes \pi_{i+2} \otimes \cdots \otimes \pi_s\ , \quad \text{where} \tag{5.8}$$

$$J_{i,i+1} = \ker \left(\pi_{\Gamma_i} \otimes \pi_{\Gamma_{i+1}} \xrightarrow{\mu_{\Gamma_i,\Gamma_{i+1}}} \pi_{\Gamma_i \sqcup \Gamma_{i+1}} \right) \subset J_2\ . \tag{5.9}$$

It follows from the Littlewood–Richardson rule that the representation π_T, which is the target of the multiplication map (5.3), comes with multiplicity one in the space $\pi_1 \otimes \cdots \otimes \pi_{i-1} \otimes \pi_{\Gamma_i \sqcup \Gamma_{i+1}} \otimes \pi_{i+2} \otimes \cdots \otimes \pi_s$, i.e. the multiplication (5.3) is factorized through this space and we can canonically include (5.8) into J_s. Further, we will present h_T as $h_T = \pm x_S \cdot h_D \cdot x_R$, where S and R are formed by hooks Γ_ν with $\nu < i$ and $\nu > i+1$ respectively, $D = \Gamma_i \sqcup \Gamma_{i+1} \subset T$ is the subdiagram formed by i-th and $(i+1)$-th consequent hooks, and h_D lies in $J_{i,i+1}$ from (5.9).

Starting from this moment, we restrict ourself by this subdiagram D. Let its shape be $\delta = (\alpha_1, \alpha_2 | \beta_1, \beta_2)$, i.e. $D = \Gamma_1 \sqcup \Gamma_2$, where $\Gamma_1 = (\alpha_1|\beta_1)$, $\Gamma_2 = (\alpha_2|\beta_2)$. Consider the diagrams:

$$\begin{aligned} \Gamma_1' &= (0|\beta_1) \quad (\text{i.e. the first column of } \Gamma_1)\ , \\ \overline{\Gamma}_1 &= (\alpha_1 - 1|0) = \Gamma_1 \smallsetminus \Gamma_1'\ , \\ \overline{\Gamma}_2 &= (\alpha_1 - 1, \alpha_2 - 1|\beta_2 + 1, 0) = \overline{\Gamma}_1 \sqcup \Gamma_2 \end{aligned}$$

(i.e. we split the first hook $\Gamma_1 = \Gamma_1' \sqcup \overline{\Gamma}_1$ into the first column and remaining part of the row and form $\overline{\Gamma}_2$ by putting this row on top of Γ_2). Recall that we write Λ^λ for the tensor product of exterior powers $\Lambda^{\lambda_i'}V$ corresponding to the columns of a given Young diagram λ (see (4.4)). The factorization through the column exchange relations $\Lambda^D \twoheadrightarrow \pi_D$ can be considered as hauling down through the diagram (5.10) below, where $\pi_{\Gamma_1} \otimes \pi_{\Gamma_2} \xrightarrow{\mu} \pi_D$ is the multiplication map, ε stays for particular factorizations through the column exchange relations, the map α : $\Lambda^{\overline{\Gamma}_2} \longrightarrow \Lambda^{\overline{\Gamma}_1} \otimes \Lambda^{\Gamma_2}$ is the alternation in columns of $\overline{\Gamma}_2$, and the maps

$$\begin{aligned} \varrho : \pi_{\overline{\Gamma}_2} &\hookrightarrow \pi_{\overline{\Gamma}_1} \otimes \pi_{\Gamma_2}\ , \quad & \sigma : \pi_{\overline{\Gamma}_1} \otimes \pi_{\Gamma_2} &\twoheadrightarrow \pi_{\overline{\Gamma}_2} \\ \tau : \pi_{\Gamma_1'} \otimes \pi_{\overline{\Gamma}_1} &\twoheadrightarrow \pi_{\Gamma_1}\ , \quad & \eta : \pi_{\Gamma_1'} \otimes \pi_{\overline{\Gamma}_2} &\twoheadrightarrow \pi_D \end{aligned}$$

are canonical projections onto and an inclusion of an irreducible submodule of multiplicity one. Indeed, it follows from the Littlewood–Richardson rule that $\pi_{\overline{\Gamma}_2}$ has multiplicity one in the product $\pi_{\overline{\Gamma}_1} \otimes \pi_{\Gamma_2}$ and π_{Γ_1} has multiplicity one in the product $\pi_{\Gamma_1'} \otimes \pi_{\overline{\Gamma}_1}$. Similarly, π_D has multiplicity one in the product $\Lambda^{\Gamma_1'} \otimes \pi_{\overline{\Gamma}_1} \otimes \pi_{\Gamma_2}$ and this implies that the bottom rhombus of the diagram (5.10) is commutative up to multiplication by non zero scalar factor. A straightforward computation (but quite improper for typesetting and too long for being reproduced here) shows that the composition

$$\Lambda^{\overline{\Gamma}_2} \xrightarrow{\alpha} \Lambda^{\overline{\Gamma}_1} \otimes \Lambda^{\Gamma_2} \xrightarrow{\varepsilon \otimes \varepsilon} \pi_{\overline{\Gamma}_1} \otimes \pi_{\Gamma_2}$$

annihilates all the column exchange relations in $\Lambda^{\overline{\Gamma}_2}$, i.e. the top rhombus in (5.10) is commutative up to multiplication by non zero scalar factor as well.

$$
\begin{tikzcd}
 & & \Lambda^D = \Lambda^{\Gamma_1'} \otimes \Lambda^{\overline{\Gamma}_2} \arrow[dl, "1\otimes\alpha"'] \arrow[dr, "1\otimes\varepsilon"] & & \\
 & \Lambda^{\Gamma_1'} \otimes \Lambda^{\overline{\Gamma}_1} \otimes \Lambda^{\Gamma_2} \arrow[dl, equal] \arrow[dr, "1\otimes\varepsilon\otimes\varepsilon"] & & \Lambda^{\Gamma_1'} \otimes \pi_{\overline{\Gamma}_2} \arrow[dl, "1\otimes\varrho"'] \arrow[dd, equal] & \\
\Lambda^{\Gamma_1} \otimes \Lambda^{\Gamma_2} \arrow[dr, "\varepsilon\otimes\varepsilon"'] & & \pi_{\Gamma_1'} \otimes \pi_{\overline{\Gamma}_1} \otimes \pi_{\Gamma_2} \arrow[dl, "\tau\otimes 1"'] \arrow[dr, "1\otimes\sigma"] & & \\
 & \pi_{\Gamma_1} \otimes \pi_{\Gamma_2} \arrow[dr, "\mu"'] & & \pi_{\Gamma_1'} \otimes \pi_{\overline{\Gamma}_2} \arrow[dl, "\eta"] & \\
 & & \pi_D & &
\end{tikzcd}
\tag{5.10}
$$

Thus, the whole diagram (5.10) is commutative up to rescales at the nodes. Now we are ready to describe the elements h_D.

In case (B) we take the rightmost column with the wrong inequality $\delta_{1,\ell} \geqslant \delta_{k,\ell}$ and consider an element $e^D + e^S \in \Lambda^{\Gamma_1} \otimes \Lambda^{\Gamma_2} = \Lambda^{\Gamma_1'} \otimes \Lambda^{\overline{\Gamma}_1} \otimes \Lambda^{\Gamma_2}$, where $S \prec D$ is obtained from D by transposing the entries $\delta_{1,\ell}$, $\delta_{k,\ell}$. Since the image of this element in $\pi_{\Gamma_1'} \otimes \pi_{\overline{\Gamma}_2}$ is zero, it follows from the commutativity of the bottom rhombus in (5.10) that $h_D = x_D + x_S = \varepsilon \otimes \varepsilon(e^D + e^S) \in \pi_{\Gamma_1} \otimes \pi_{\Gamma_2}$ lies in $\ker \mu \subset J_2$ as required.

In case (A) we take the maximal k such that $d_{k,2} < d_{k,1}$ (i.e. the lowest row with the wrong order) and consider an exchange relation (4.5) that exchanges first k elements of the second column in D with all k-element ordered subsets of the first column, i.e. an element

$$
\widehat{h}_D = e^D - \sum_{\sigma} e^{\sigma D} = e^D - \sum_{S} a_S e^S \in \Lambda^D = \Lambda^{\Gamma_1'} \otimes \Lambda^{\overline{\Gamma}_2} ,
$$

where $S \prec D$ are obtained from D by all the exchanges σ in question. Since the image of $\widehat{h}_D$ in π_D is zero, the class of an element $1 \otimes \alpha(\widehat{h}_D) \in \Lambda^{\Gamma_1} \otimes \Lambda^{\Gamma_2}$ in the factor $\pi_{\Gamma_1} \otimes \pi_{\Gamma_2}$ belongs to $\ker \mu \subset J_2$. At the same time the difference

$$
\Bigl(x_D - \sum_{S} a_S x_S\Bigr) - \varepsilon \otimes \varepsilon\bigl(1 \otimes \alpha(\widehat{h}_D)\bigr) \in \pi_{\Gamma_1} \otimes \pi_{\Gamma_2}
$$

is a sum of $\varepsilon \otimes \varepsilon$-images of elements from $\Lambda^{\Gamma_1} \otimes \Lambda^{\Gamma_2}$ that have a form considered above in the case (B). In particular, this difference lies in J_2 as well. We conclude that $h_T = x_D - \sum\limits_{S} a_S x_S$ satisfies the required properties.

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