MEASUREMENT OF TAX PROGRESSION WITH DIFFERENT INCOME DISTRIBUTIONS

by

Christian Seidl, University of Kiel, Germany

e-mail: seidl@economics.uni-kiel.de

Abstract

This paper reviews methods of comparing income tax progression. Section II deals with local measures of tax progression, Section III with global measures of tax progression, and Section IV with uniform tax progression. It is shown that all of this measures have specific drawbacks: they either ignore the income distribution altogether, aggregate over income intervals with progression and regression, or they require that the same income distribution holds for comparisons of different tax schedules. However, realistic comparisons of tax progression ask for comparisons of different tax schedules associated with different income distributions. This is addressed in Section V for uniform tax progression with different income distributions. A respective condition of higher progression turns out as the sum of elasticities of the tax schedule and the density function of the income distribution. Alas, this is only a sufficient, not a necessary condition. The paper concluded with the challenge to find necessary and sufficient conditions for uniform tax progression with different income distributions.

JEL Classification: H23.

Keywords: income tax progression, measurement of tax progression, comparisons of tax progression.

I Introduction

Several methods of comparisons of income tax progression were proposed. It all started with local measures of tax progression, in particular tax elasticity and residual income elasticity. As these measures relate to the tax schedule only and disregard the income distribution to which a tax schedule should apply, other measures of tax progression were developed. Starting with Dalton (1922) and Musgrave and Thin (1948) global measures of tax progression were proposed. They rely on inequality measures of pre- and post-tax incomes, which can also be expressed as weighted deviations between taxation and a revenue-neutral hypothetical proportional taxation. They have the disadvantage that tax schedules which have for some income intervals declining average tax rates my be categorized as more progressive than tax schedules with increasing average tax rates throughout. Pitfalls like that led to the development of global tax progression which are based on dominance relations of Lorenz curves. Conditions for domination of Lorenz curves may be represented by single-crossing conditions [Hemming and Keen (1983)], or domination of elasticities (as they are known from local measures of tax progression) over the whole income support [Jakobsson (1976); Kakwani (1977a)]. Measures of global tax progression suffer from the restriction that they can only compare tax schedules under the restriction that the same income distribution applies to all tax schedules to be compared.

Section V of this paper develops methods to apply global tax progression to situations in which both the tax schedules and the income distributions are different. This allows to answer questions such as whether the German tax schedule with the German income distribution is more or less progressive than the American tax schedule with the American income distribution. The concept of greater tax progression is based on dominance relations of first moment distribution functions. Conditions dominance of first moment distributions functions are dominance relations of the sums of elasticities of the tax schedules and elasticities of the density functions of the income distributions. This not only gives due attention to the role of income distributions for tax progression, it allows also a much broader scope of analysis of comparisons of tax progression. Alas, only sufficient conditions are available right now. The big challenge is to find necessary and sufficient conditions.

This paper concentrates on the basic problems and avoids all further ramifications.

For this reason it is confined to tax progression in terms of comparisons of gross and net incomes, and keeps the assumption of an equal income support for different income distributions. It may readily be extended to deal with all these ramifications [see Seidl (1994)]. The rub for all of them would be to find necessary and sufficient conditions for uniform tax progression for different income distributions.

This paper employs the following notation: y denotes income such that $y \in [\underline{y}, \bar{y}]$, $f(y) \geq 0$ denotes the density function of the income distribution on the support $[\underline{y}, \bar{y}]$, $\mu := \int_{\underline{y}}^{\bar{y}} y f(y) dy$ denotes mean income, $T(y) : [\underline{y}, \bar{y}] \rightarrow [\underline{y}, \bar{y}]$ denotes the income tax schedule, $t(y) := \frac{T(y)}{y}$ denotes the average income tax schedule with $t(0) = 0$, $\tau(y) := T'(y)$ denotes the marginal income tax schedule, $\theta := \int_{\underline{y}}^{\bar{y}} T(y) f(y) dy$ denotes mean taxation, $x(y) := y - T(y)$ denotes net income, and $\xi := \mu - \theta$ denotes mean net income.

II Local Measures of Income Tax Progression

Local measures of income tax progression just focus on the tax schedule. For the purpose of this paper we single out the two most important and frequently used ones, viz. the **tax elasticity**

$$\varepsilon(y) := \frac{\tau(y)}{t(y)} = \frac{T'(y)}{T(y)} y$$

and the **residual income elasticity**

$$\eta(y) := \frac{1 - \tau(y)}{1 - t(y)} = \frac{x'(y)}{x(y)} y = \frac{d[y - T(y)]/dy}{y - T(y)} y.$$

$\varepsilon(y)$ measures **liability progression**, $\eta(y)$ **residual progression**. According to liability progression a tax schedule is progressive at $\tilde{y}$ if $\varepsilon(\tilde{y}) > 1$; according to residual progression a tax schedule is progressive at $\tilde{y}$ if $0 < \eta(\tilde{y}) < 1$.

The economic meaning of these two local measures of tax progression is simple: $\varepsilon(\tilde{y}) > 1$ means that the tax on an extra monetary unit for a taxpayer with income $\tilde{y}$ exceeds his or her average tax burden for income $\tilde{y}$; $0 < \eta(\tilde{y}) < 1$ means that an extra monetary unit leaves a taxpayer with gross income $\tilde{y}$ less net income, $x'(y)$, than $\frac{x(y)}{y}$, which is the ratio of net income and gross income. Note that both measures are equivalent to the general definition of tax progression, viz. that the average tax rate at $\tilde{y}$ is an increasing function of income:

LEMMA 1:

$$\left.\frac{dt(y)}{dy}\right|_{y=\tilde{y}} > 0 \Leftrightarrow \varepsilon(\tilde{y}) > 1 \Leftrightarrow 0 < \eta(\tilde{y}) < 1.$$

PROOF:

$$\frac{dt(y)}{dy} = \frac{d}{dy}\frac{T(y)}{y} = \frac{T'(y)y - T(y)}{y^2} = \frac{\tau(y) - t(y)}{y} > 0 \Leftrightarrow$$

$$\tau(y) > t(y) \Leftrightarrow \frac{\tau(y)}{t(y)} = \varepsilon(y) > 1 \Leftrightarrow 1 - \tau(y) < 1 - t(y) \Leftrightarrow \frac{1-\tau(y)}{1-t(y)} = \eta(y) < 1.$$

Q.E.D.

To be forewarned is to be forearmed: note that for two tax schedules $T_1(y)$ and $T_2(y)$ it does **not hold** that $\varepsilon_1(\tilde{y}) > \varepsilon_2(\tilde{y})$ is equivalent to $\eta_1(\tilde{y}) < \eta_2(\tilde{y})$. [After all, this property justifies the existence of different measures of tax progression.]

Let us, for the sake of simplicity, restrict our further analysis to residual progression (the analysis for liability progression runs analogous, although it may produce different results). Then tax schedule $T_1(\cdot)$ is more progressive than $T_2(\cdot)$ at $\tilde{y}$ if $0 < \eta_1(\tilde{y}) < \eta_2(\tilde{y}) < 1$. Now we can have two cases:

(i) $\eta_1(y) \leq \eta_2(y)$ for all $y \in [\underline{y}, \bar{y}]$ and a strict inequality sign for a nonempty interval $I \subset [\underline{y}, \bar{y}]$. Then tax schedule $T_1(\cdot)$ is uniformly more progressive than $T_2(\cdot)$.

(ii) There are two[1] nonempty and disjoint income intervals $I_1, I_2 \subset [\underline{y}, \bar{y}]$, $I_1 \cap I_2 = \emptyset$, such that $0 < \eta_1(\tilde{y}) \leq \eta_2(\tilde{y}) < 1$ for all $\tilde{y} \in I_1$ and $1 > \eta_1(\hat{y}) \geq \eta_2(\hat{y}) > 0$ for all $\hat{y} \in I_2$ [one inequality sign strict for a nonempty part of each interval]. Then $T_1(\cdot)$ is more progressive than $T_2(\cdot)$ on the income interval I_1, whereas $T_2(\cdot)$ is more progressive than $T_1(\cdot)$ on the income interval I_2. We cannot say which tax schedule is more progressive on the whole income support $[\underline{y}, \bar{y}]$. How could we compare income tax progression when the master income support $[\underline{y}, \bar{y}]$ holds? Two ways were proposed, first, **global measures** of tax progression, second, conditions for **uniform** tax progression.

[1] The generalization to finitely many income subintervals is obvious.

III Global Measures of Tax Progression

Global measures of tax progression apply another concept of tax progression which is based on income distribution measures of gross and net incomes. In the simplest cases the Gini income inequality measure $G(\cdot)$ is used to construct global measures of tax progression. Examples are, e.g., the measure proposed by Reynolds and Smolensky (1977):

$$RS = G(x) - G(y) \tag{1}$$

If $RS < 0$ the tax schedule is progressive, if $RS = 0$ it is proportional, and if $RS > 0$ it is regressive.

Pechman and Okner (1974) and Okner (1975) normalized RS by $G(y)$:

$$PO = \frac{G(x) - G(y)}{G(y)}, \tag{2}$$

where the tax schedule is progressive for $PO < 0$, proportional for $PO = 0$, and regressive for $PO > 0$.

Musgrave and Thin (1948) proposed

$$MT = \frac{1 - G(x)}{1 - G(y)}. \tag{3}$$

If $MT > 1$ the tax schedule is progressive, if $MT = 1$ it is proportional, and if $MT < 1$ it is regressive.

Many more global measures of tax progression were proposed. Based on work by Formby et al. (1981; 1984), Pfähler (1982; 1983), Kakwani (1984), Kiefer (1984), and Liu (1984), Pfähler (1987) showed that many global measures of tax progression are encompassed by the expression[2]:

$$GP = \frac{1}{\xi} \int_{\underline{y}}^{\bar{y}} \left[x(y) - \left(1 - \frac{\theta}{\mu}\right) y \right] w(y) f(y) dy, \tag{4}$$

where $\frac{\theta}{\mu}$ is the aggregate tax rate of this economy and $w(y)$ is a weighting function. The term in square brackets is the local deviation of actual net income from a hypothetical revenue-neutral net income under a proportional tax with rate $\frac{\theta}{\mu}$. It is normalized for mean net income.

[2] We state only the formula for net incomes; it has a dual in terms of tax schedules.

Note that for constant $w(y)$ GP is zero because the difference between total net income and total net income under a hypothetical revenue-neutral proportional tax is zero. Hence, the value of GP depends on the shape of the weighting function applied. If greater positive GP should indicate higher progression, then $w(y)$ should be decreasing. If smaller negative GP should indicate higher progression, then $w(y)$ should be increasing. Note that $w(y)$ can in both cases be either positive or negative.

Using appropriate weighting functions $w(y)$, GP encompasses many global measures of tax progression, e.g., as proposed by Musgrave and Thin (1948), Hainsworth (1964), Khetan and Poddar (1976), Suits (1977), Kakwani (1977b, 1984, 1987), Pfähler (1987), and Lambert (1988).

Global measures of tax progression not only serve to categorize tax schedules into progressive, proportional and regressive, but serve also to derive an **ordering** of tax progression. They allow calling tax schedule $T_1(\cdot)$ as **more progressive** than $T_2(\cdot)$ if the global measure applied shows a higher value for $T_1(\cdot)$ than for $T_2(\cdot)$, which indicates more progression for $T_1(\cdot)$ than for the tax schedule $T_2(\cdot)$.

Global measures of tax progression have several advantages. First, they work for different tax schedules and different income distributions. This means that interregional and intertemporal comparisons of tax progression can be effectuated. Second, equation (4) exhibits a double weighting, both by $w(y)$ and by $f(y)$. That is, particular characteristics of a tax schedule gain more (less) weight if more (less) taxpayers are affected. Third, global measures of tax progression are able to compensate income subintervals with opposite properties of tax schedules by appropriate weighting and subsequent aggregation.

However, this last advantage turns at the same time out to be a major handicap of global measures of tax progression. Aggregating the effects of tax schedules over all income intervals may lead to the result that $T_1(\cdot)$ is categorized as being more progressive than $T_2(\cdot)$, although $T_1(\cdot)$ has a decreasing average tax rate throughout some interval, while $T_2(\cdot)$'s average tax schedule is increasing throughout the whole income support. Alternatively, suppose that a tax schedule is progressive for the lower incomes and regressive for the higher incomes. This may lead to a Lorenz curve which intersects the Lorenz curve of pre-tax income. In this case we cannot exclude the same value of the two Gini inequality measures, which would indicate a proportional tax schedule under the RS, PO, and MT measures of tax progression, although this tax schedule is far from being proportional. The second handicap of global progression measures is rooted in the aggregation procedure of equation (4), which presupposes comparability of the tax burden across all income strata. This is much related to the assumption of interpersonal comparability of utility. These handicaps led to the development of uniform tax progression to which we turn in the next section.

IV Uniform Tax Progression

Uniform tax progression adopted yet another concept of progression. It can be formulated in terms of taxes or in terms of net incomes. For the presentation in this paper we shall stick to the formulation in terms of net incomes. Uniform tax progression defines tax schedule $T_1(\cdot)$ to be more progressive than tax schedule $T_2(\cdot)$ if the Lorenz curve of the net incomes resulting from $T_1(\cdot)$, viz. the Lorenz curve of $x_1(y)$, dominates the Lorenz curve of the net incomes resulting from $T_2(\cdot)$, viz. the Lorenz curve of $x_2(y)$. This very definition reveals the main purpose of this analysis, namely to compare income tax schedules according to their progression ordering rather than to categorize them as progressive, proportional, or regressive.

Uniform tax progression purports to derive conditions of tax schedules such that greater progression of one tax schedule is established. This was achieved in two ways, either by way of **single-crossing conditions**, or by way of **elasticity properties** of the tax schedules. As both ways turn out to produce related conditions, we start with the single-crossing condition and show its relationship with elasticity properties. We follow Hemming and Keen (1983) [see also Dardanoni and Lambert (1988, Section 4)]:

Theorem 2: $T_1(\cdot)$ is more progressive than $T_2(\cdot)$ for all income distributions which raise the same revenue if and only if $x_2(y)$ single-crosses $x_1(y)$ from below on $[\underline{y}, \bar{y}]$ at y^*.

Proof: Sufficiency: for $\hat{y} \in [\underline{y}, y^*]$ Lorenz dominance of $x_1(y)$ follows directly from the definition of single-crossing. For $\hat{y} \in [y^*, \bar{y}]$ we have

$$(5)\quad \int_{\underline{y}}^{\hat{y}} [x_1(y) - x_2(y)] f(y) dy = \int_{\underline{y}}^{\bar{y}} [x_1(y) - x_2(y)] f(y) dy + \int_{\hat{y}}^{\bar{y}} [x_2(y) - x_1(y)] f(y) dy \geq 0$$

by $T_1(\cdot)$ and $T_2(\cdot)$ raising the same revenue and by $x_2(y) \geq x_1(y)$ for all $y \in [y^*, \bar{y}]$.

Necessity: by contradiction suppose that $x_1(y)$ and $x_2(y)$ cross more than once. Then there exists at least one subinterval on which $x_2(y)$ single-crosses $x_1(y)$ once from below, and at least one on which the reverse holds. By the sufficiency part of the proof it suffices to show that if $x_2(\cdot)$ single-crosses $x_1(\cdot)$ from below on a subinterval $I_1 \subset [\underline{y}, \bar{y}]$, then there exists an income distribution $f_1(y)$ for which $T_1(\cdot)$ and $T_2(\cdot)$ raise the same revenue on $f_1(y)$. For the second subinterval $I_2 \subset [\underline{y}, \bar{y}]$ on which $x_1(y)$ single-crosses $x_2(y)$ from below, an income distribution $f_2(y)$ exists for which $T_1(\cdot)$ and $T_2(\cdot)$ raise the same revenue on $F_2(y)$. Then $T_1(\cdot)$ is more progressive than $T_2(\cdot)$ on I_1 and less progressive on I_2. Q.E.D.

Next we may address the problem that $T_1(\cdot)$ and $T_2(\cdot)$ do not raise the same revenue. Then we simply normalize $x_1(y)$ by ξ_1, $\tilde{x}_1(y) := \frac{x_1(y)}{\xi_1}$, and $x_2(y)$ by ξ_2, $\tilde{x}_2(y) := \frac{x_2(y)}{\xi_2}$, and work for Lorenz curves proper. The analysis remains basically the same.

Elasticity properties of the tax schedule were used by Jakobsson (1976) to characterize the progression ordering:

Theorem 3: $T_1(\cdot)$ is more progressive than $T_2(\cdot)$ for all income distributions on $[\underline{y}, \bar{y}]$ if and only if $\eta_1(y) \leq \eta_2(y)$ for all $y \in [\underline{y}, \bar{y}]$ and $\eta_1(y) < \eta_2(y)$ for a nonempty income subinterval.

Proof: Jakobsson (1976, 165).

Notice that if $\eta_1(y) < \eta_2(y)$ holds for all $y \in [\underline{y}, \bar{y}]$, then $T_1(\cdot)$ generates systematically lower net incomes than $T_2(\cdot)$ if the same income distribution holds for $T_1(\cdot)$ and $T_2(\cdot)$. This means that $T_1(\cdot)$ raises more revenue than $T_2(\cdot)$. In other words, Jakobsson's theorem is inconsistent with the assumption that $T_1(\cdot)$ and $T_2(\cdot)$ can raise the same revenue. $T_1(\cdot)$ causes a more equal distribution of net incomes than $T_2(\cdot)$, however bought at the price of a higher tax burden for all. The poor have only the satisfaction that the rich are pinched relatively more under the tax schedule $T_1(\cdot)$.

Concerning the only-if part of his proof, Jakobsson (1976) has to show that when $\eta_1(y) \geq \eta_2(y)$ except for a subintervall where $\eta_1(y) < \eta_2(y)$, then $T_1(\cdot)$ is not more progressive than $T_2(\cdot)$. He argues that "we could always choose an income distribution before tax that lies completely within the latter interval." [Jakobsson (1976, 165).] The tax schedule $T_2(\cdot)$ would then be more progressive than $T_1(\cdot)$ for this latter interval.

Jakobsson's (1976) method consists of the consideration of the properties of two tax schedules such that one of them is more progressive than the other for **all** income distributions.

Another route was chosen by Kakwani (1977a, 729, Theorem 1). He worked with **concentration curves**, i.e. Lorenz curves based on continuously differentiable functions $g(y) \geq 0$. Reduced to $g(y) = x(y)$ we can restate his theorem in the following form:

THEOREM 4: $T_1(\cdot)$ is more progressive than $T_2(\cdot)$ if $\eta_1(y) < \eta_2(y)$ for all $y \in [\underline{y}, \bar{y}]$.

PROOF: Kakwani (1977a, 720).

Comparing Theorem 4 with Theorem 3, it is striking that Theorem 4 states a sufficient condition only, whereas Theorem 3 states a sufficient and necessary condition. This is due to Kakwani's different point of departure. Kakwani considers the income distribution as **given** and asks for conditions for tax schedules such that the Lorenz curve of $x_1(y)$ dominates the Lorenz curve of $x_2(y)$. The condition $\eta_1(y) < \eta_2(y)$ for all $y \in [\underline{y}, \bar{y}]$ is only a **sufficient** condition for the Lorenz dominance of the Lorenz curve of $x_1(y)$ over the Lorenz curve $x_2(y)$. Conversely, the Lorenz curve of $x_1(y)$ may well dominate the Lorenz curve of $x_2(y)$, even if $\eta_2(y) > \eta_1(y)$ for some nonempty subinterval I of $[\underline{y}, \bar{y}]$.

Finally, there is a relationship between dominance of the residual income elasticity and the single-crossing condition. Suppose $\tilde{x}_2(y)$ crosses $\tilde{x}_1(y)$ from below at y^*. Then we have at $y \in (y^*, \bar{y}]$:

$$\frac{\tilde{x}_1(y)}{\tilde{x}_2(y)} = e^{\int_{y^*}^{y} \frac{[\eta_1(v)-\eta_2(v)]}{v} dv}. \tag{6}$$

When $\eta_1(y) \leq \eta_2(y)$ for all $y > y^*$ and the inequality sign is strict for some nonempty interval $I \subset (y^*, \bar{y}]$, then $\tilde{x}_1(y) < \tilde{x}_2(y)$ for all $y \in (y^*, \bar{y}]$. In other words, the single-crossing condition holds. Hence, the condition $\eta_1(y) \leq \eta_2(y)$ for all $y \in [\underline{y}, \bar{y}]$ is sufficient for the single-crossing condition to hold. Conversely, when the single-crossing condition holds, this does not imply that $\eta_1(y) \leq \eta_2(y)$ for all $y \in [\underline{y}, \bar{y}]$. $[\eta_1(y) - \eta_2(y)] \geq 0$ may well hold for an interval $I \subset [\underline{y}, y^*)$, or it may change sign for some $y \in [y^*, \bar{y}]$, while leaving the integral term in equation (6) negative.

Uniform tax progression starts with a different concept of progression, viz. Lorenz dominance of the net incomes. Interestingly enough, the residual income elasticity, a local measure of tax progression, becomes the centerpiece of uniform tax progression.[3] Dominance of residual income elasticity constitutes a sufficient condition for the dominance of Lorenz curves of the net incomes. This applies to Kakwani's (1977a) theorem and implies also Hemming and Keen's (1983) single-crossing condition. For Jakobsson's (1976) theorem it becomes a sufficient and necessary condition when it should hold for arbitrary income distributions. Hence, uniform tax progression led us back to a local measure of tax progression. (Remember that there exists a dual analysis in terms of tax elasticity.)

V Uniform Tax Progression for Different Income Distributions

Although uniform tax progression is an appealing approach, it cannot answer some problems which we encounter in most comparisons of tax progression. Uniform tax progression associates two tax schedules with the **same income distribution**. We have seen in the preceding section that the income distribution can vary, but — and this is crucial — it has to vary simultaneously for **both** tax schedules to be compared. In particular, this renders uniform tax progression especially applicable to comparing candidates for possible tax reforms where one tax schedule is replaced by another one, while leaving the income distribution intact. It may also be used for abstract characterizations of progression properties od tax schedules which hold for a large variety of income distributions.

However, uniform tax progression is unable to perform comparisons of tax progression when both the tax schedules **and** the associated income distributions are **different**. Such comparisons should answer questions of whether the German income tax schedule associated with the German income distribution is more or less progressive than the American income tax schedule associated with the American income distribution. Such are the questions of real political interest. Traditional uniform tax progression could only analyze

[3] For some weighting functions $w(y)$ residual income elasticity may also imply a global measure of tax progression.

whether the German income tax schedule is more or less progressive than the American income tax schedule given that either the German of the American income distribution holds in **both** countries. This may yield opposite results. On top of that, such analyzes cannot command interest as the German income distribution does not hold in America and the American income distribution does not hold in Germany.

In this section we shall again stick to the analysis in terms of net incomes and consider different tax schedules and different income distributions defined on the same support $[\underline{y}, \bar{y}]$.[4] The concept of tax progression used in this section is the dominance relation of the first moment distribution function of net incomes:

$$F_{y-T}(y) := \frac{1}{(1-\theta)\mu} \int_{\underline{y}}^{y} [v - T(v)] f(v) dv. \tag{7}$$

$F_{y-T}(y)$ indicates the share of net incomes of income recipients with gross incomes of no more than y in total net income. $T_1(\cdot)$ is more progressive than $T_2(\cdot)$ if $F^1_{y-T_1}(y) \geq F^2_{y-T_2}(y)$, $y \in [\underline{y}, \bar{y}]$, with a strict inequality sign for a nonempty interval $I \subset [\underline{y}, \bar{y}]$. The next theorem gives us a sufficient condition of greater progression:

THEOREM 5: $F^1_{y-T_1}(y) \geq F^2_{y-T_2}(y)$, $y \in [\underline{y}, \bar{y}]$ with the inequality sign strict for a nonempty interval $I \subset [\underline{y}, \bar{y}]$ if $\eta_1(y) + \varphi_1(y) \leq \eta_2(y) + \varphi_2(y)$ with the inequality sign strict for a nonempty interval $I \subset [\underline{y}, \bar{y}]$, where $\varphi_i := \frac{f_i'(y)}{f_i(y)} y$ denotes the elasticity of the density function of income distributions $i = 1, 2$ with respect to income.

PROOF: For the proof we work with derivatives of the **relative** concentration curve. The relative concentration curve is constructed from $F^1_{y-T_1}(y)$ and $F^2_{y-T_2}(y)$ by a curve which poses for each y the value of $F^2_{y-T_2}(y)$ at the abscissa and the value of $F^1_{y-T_1}(y)$ at the ordinate (see Figure 1). If this relative concentration curve is concave, then $F^1_{y-T_1}(y) \geq F^2_{y-T_2}(y)$, i.e., for all income levels holds that $T_1(\cdot)$ associated with $f_1(\cdot)$ provides a higher share of aggregate net incomes than does $T_2(\cdot)$ associated with $f_2(\cdot)$. The first derivative

[4] These assumptions are made for the sake of simplicity of exposition. The analysis can also be carried out in terms of taxation and for different supports of the income distributions involved. For empirical comparisons of tax progression for different countries or time periods, the assumption of the same support for different income distributions is unserviceable. One has to work with functions defined on transformed domains. However, the basic problem of having only a sufficient condition remains. For more details see Seidl (1994).

of the relative concentration curve is, of course, positive:

$$\frac{dF^1_{y-T_1}(y)}{dF^2_{y-T_2}(y)} = \frac{dF^1_{y-T_1}(y)/dy}{dF^2_{y-T_2}(y)/dy} = \frac{(1-\theta_2)\mu_2[y-T_1(y)]f_1(y)}{(1-\theta_1)\mu_1[y-T_2(y)]f_2(y)} > 0.$$

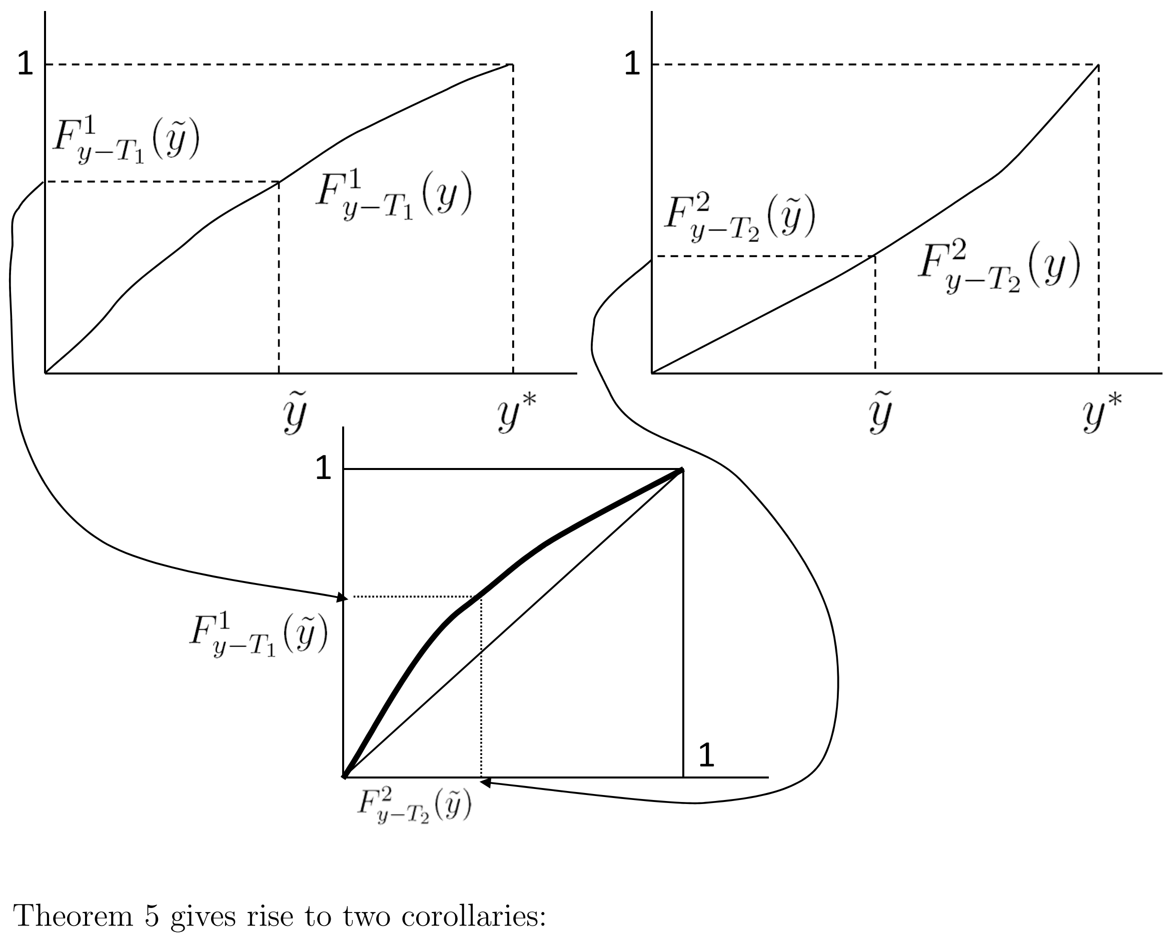
The second derivative of a concave function is nonpositive. The second derivative is:

$$\frac{d^2F^1_{y-T_1}(y)}{dF^2_{y-T_2}(y)^2} = \frac{\eta_1(y)+\varphi_1(y)-\eta_2(y)-\varphi_2(y)}{\frac{(1-\theta_1)\mu_1}{[(1-\theta_2)\mu_2]^2}\frac{\{[y-T_2(y)]f_2(y)\}^2}{[y-T_1(y)]f_1(y)}y}.$$

As the denominator of this ratio is positive, the value of the second derivative is nonpositive if $\eta_1(y)+\varphi_1(y)-\eta_2(y)-\varphi_2(y) \le 0$. This is equivalent to $\eta_1(y)+\varphi_1(y) \le \eta_2(y)+\varphi_2(y)$.

Q.E.D.

Figure 1: a graphical illustration of Theorem 5



Theorem 5 gives rise to two corollaries:

Corollary 6: If $f_1(y) \equiv f_2(y)$ in Theorem 5, then $F_{y-T_1}(y) \ge F_{y-T_2}(y)$ if $\eta_1(y) \le \eta_2(y)$ for all $y \in [\underline{y}, \bar{y}]$ with the inequality sign strict for a nonempty interval $I \subset [\underline{y}, \bar{y}]$.

COROLLARY 7: If $T_1(y) \equiv T_2(y)$ in Theorem 5 then $F^1_{y-T}(y) \geq F^2_{y-T}(y)$ if $\varphi_1(y) \leq \varphi_2(y)$ for all $y \in [\underline{y}, \bar{y}]$ with the inequality sign strict for a nonempty interval $I \subset [\underline{y}, \bar{y}]$.

Theorem 5 shows the interplay between the residual income elasticities of the tax schedules and the elasticities of the density functions of the income distributions. Properties of the tax schedule can be overcompensated by properties of the income distribution, and vice versa. Only the sum between the two parameters matter. For example, if $\eta_1(y) > \eta_2(y)$ for a nonempty interval $I \subset [\underline{y}, \bar{y}]$, then $T_1(\cdot)$ can still be more progressive than $T_2(\cdot)$ when $\varphi_2(y)$ is so much higher than $\varphi_1(y)$ on his interval I that it overcompensates the shortfall of $\eta_2(y)$ of $\eta_1(y)$ on the interval I. Analogously, $\varphi_1(y) > \varphi_2(y)$ on I can be overcompensated by $\eta_1(y) < \eta_2(y)$ on I when the difference $\eta_2(y) - \eta_1(y)$ is high enough.

Theorem 5 demonstrates that for comparisons of progression both the tax schedule and the income distribution play crucial roles. This marks a sharp contrast to Theorems 2, 3, and 4 in Section IV. Although these theorems draw not only on tax schedules, but also, by way of Lorenz dominance as constituents of comparisons of progression, on income distributions, no parameters referring to income distributions occur in the respective conditions. This is a consequence of the assumption of the same income distribution for the situations to be compared. If we assume the same income distribution for both tax schedules in Theorem 5, then it degenerates to Corollary 6, which is precisely Kakwani's (1977a) Theorem 4. If we assume the same tax schedule for both income distributions in Theorem 5, then it degenerates to Corollary 7, which shows vividly that the progressive effect of a tax schedule is also determined by the associated income distribution. Transplanting a given tax schedule into another society may produce a more or less progressive tax system depending on the prevailing income distribution. Note that progression comparisons are invalidated for the same tax schedules holding in different societies if there is no dominance relation of the elasticities of the density functions of the respective income distributions. This is tantamount to the lack of dominance of residual income elasticities for the case of identical income distributions.

This approach of progression comparisons exerts a strong appeal because of its realistic attitude. Actual problems ask for comparisons of two **given tax schedules** each associated with a **given income distribution**. It is these two given situations which should be evaluated and compared, neither comparisons for all possible income distributions, nor comparisons for all possible tax schedules.

Alas, this method has a severe drawback: it provides only a **sufficient** condition for greater progression, not a **necessary** condition. In particular, the relative concentration curve may have a shape as depicted in Figure 2.

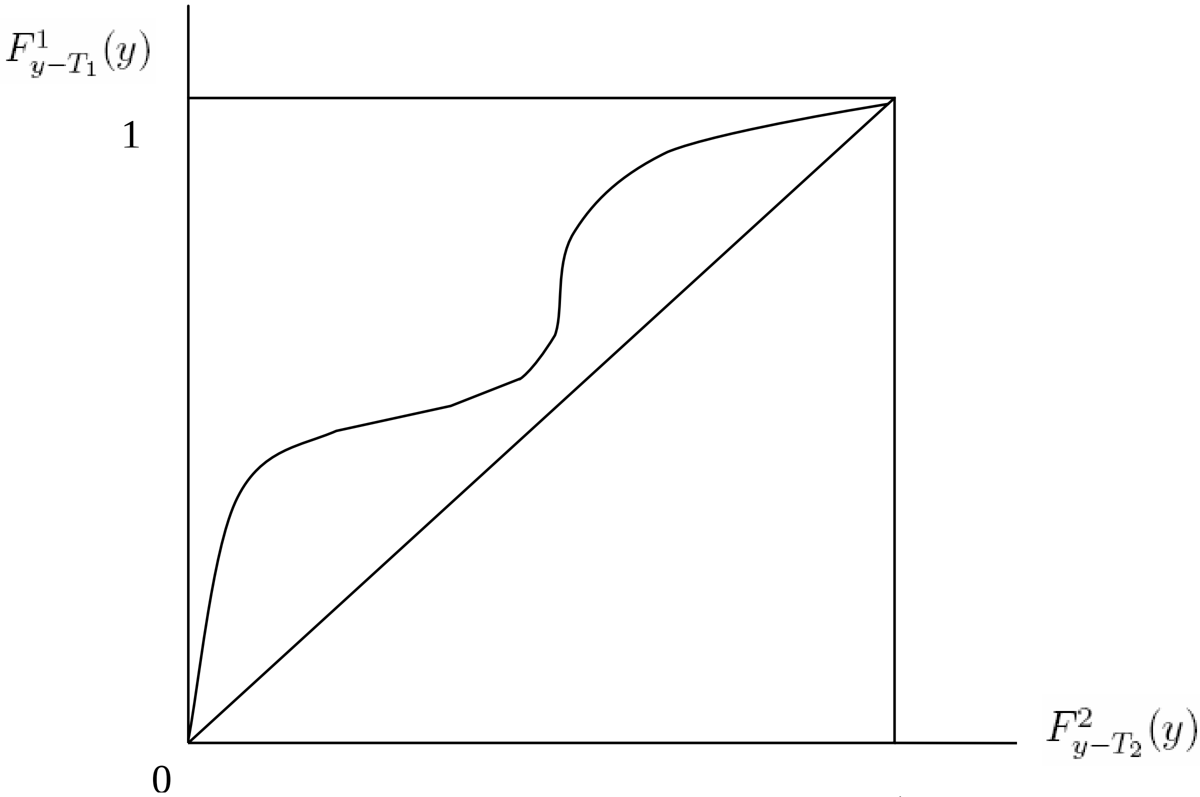


Figure 2

The relative concentration curve as depicted in Figure 2 results from first moment distribution functions such that $F^1_{y-T_1}(y)$ dominates $F^2_{y-T_2}(y)$. They produce a relative concentration curve which, although it lies above the diagonal of the unit square [which means that $T_1(y)$ with income distribution $f_1(y)$ leaves all taxpayers earning no more than y a relatively higher aggregate net income than $T_2(y)$ with income distribution $f_2(y)$], is not concave.

Dealing with such cases would require a **necessary and sufficient condition**. There seem to be several ways to achieve this:

1. We can perhaps conjecture that a general solution of this problem does not exist. It may turn out to be one of Hilbert's problems for which Kolmogorov and Arnold

showed that no solution exists.[5]

2. We max look for conditions on the first moment distribution functions such that the resulting relative concentration curve is concave. This escape is the least satisfactory one. In view of Theorem 5 we may restrict eligibility of the tax schedules and/or the income distributions such that $\eta_1 + \varphi_1 \leq \eta_2 + \varphi_2$ holds. Hence, this escape is immediate, but trivial. Moreover, the problem as outlined in Figure 2 persists.

3. We may look for conditions such that the relative concentration curve does not cross the diagonal. In mathematical terms, one can look for algebraic conditions on the relative concentration curve such that only two degenerate fixed points exist in the unit square, viz. (0,0) and (1,1). It is dubious whether such a theorem exists.

4. We may take the pedestrian way and check numerically whether $[F^1_{y-T_1}(y) - F^2_{y-T_2}(y)] \geq 0$ for all $y \in (\underline{y}, \bar{y})$, or for respective transformations of the income domain.

Note that this section focused on the simplest case of uniform tax progression for different income distributions. A necessary and sufficient condition found for this case would readily carry over to the analysis in terms of tax schedules and to different supports of the income distributions involved [see Seidl (1994)]. The rub is to find them.

VI Conclusion

This paper reviewed methods of comparing income tax progression. Section II dealt with local measures of tax progression, Section III with global measures of tax progression, and Section IV with uniform tax progression. All of this measures have specific drawbacks: they either ignore the income distribution altogether, they aggregate over income intervals with progression and regression, or they require that the same income distribution holds for comparisons of different tax schedules. However, realistic comparisons of tax progression ask for comparisons of different tax schedules associated with different income distributions. A respective condition turns out as the sum of elasticities of the tax schedule and the density function of the income distribution. Alas, this is only a sufficient, not a necessary condition. This paper concluded with the challenge to find necessary and sufficient conditions.

[5] I owe the hint to this conjecture to Fuad Aleskerov.

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References

Dalton, H. (1922), *Principles of Public Finance*, Routledge & Kegan Paul, London.

Dardanoni, V. and Lambert P.J. (1988), Welfare Rankings of Income Distributions: A Rôle for the Variance and Some Insights for Tax Reform, *Social Choice and Welfare* 5, 1-17.

Formby, J.P, Seaks, T.G. and Smith, W.J. (1981), A Comparison of two New Measures of Tax Progressivity, *The Economic Journal* 91, 1015-1019.

Formby, J.P, Seaks, T.G. and Smith, W.J. (1984), Difficulties in the Measurements and Comparison of Tax Progressivity: The Case of North America, *Public Finance/Finances Publiques* 39, 297-313,

Hainsworth, G.B. (1964), The Lorenz Curve as a General Tool of Economic Analysis, *The Economic Record* 40, 426-441.

Hemming, R. and Keen, M.J. (1983), Single-Crossing conditions in Comparisons of Tax Progressivity, *Journal of Public Economics* 20, 373-380.

Jakobsson, U. (1976), On the Measurement of the Degree of Progression, *Journal of Public Economics* 5, 161-168.

Kakwani, N.C. (1977a), Applications of Lorenz Curves in Economic Analysis, *Econometrica* 45, 719-727.

Kakwani, N.C. (1977b), Measurement of Tax Progressivity: An International Comparison, *The Economic Journal* 87, 71-80.

Kakwani, N.C. (1984), On the Measurement of Tax Progressivity and Redistributive Effect of Taxes with Applications to Horizontal and Vertical Equity, in: R.L. Basmann and G.F. Rhodes (eds.), *Advances in Econometrics, Vol. 3: Economic Inequality: Measurements and Policy*, JAI Press, Greenwich CO and London, 149-168.

Kakwani, N.C. (1987), Measures of Tax Progressivity and Redistribution Effect: A Comment, *Public Finance/Finances Publiques* 42, 431-434.

Khetan, C.P. and Poddar, S.N. (1976), Measurement of Income Tax Progression in a Growing Economy: The Canadian Experience, *Canadian Journal of Economics* 9, 613-629.

Kiefer, D.W. (1984), Distributional Tax Progressivity Indexes, *National Tax Journal* 37, 497-513.

Lambert, P.J. (1988), Net Fiscal Incidence Progressivity: Some Approaches to Measurement, in: W. Eichhorn (ed.), *Measurement in Economics. Theory and Application of Economic Indices*, Physica-Verlag, Heidelberg, 519-532.

Liu, P.-W. (1984), A Note on two Summary Measures of Tax Progressivity, *Public Finance/Finances Publiques* 39, 412-419.

Musgrave, R.A. and Thin, T. (1948), Income Tax Progression, 1929-48, *Journal of Political Economy* 56, 498-514.

Okner, B.A. (1975), Individual Taxes and the Distribution of Income, in: J.D. Smith (ed.), *The Personal Distribution of Income and Wealth*, National Bureau of Economic Research, New York, 45-73.

Pechman, J.O. and Okner, B.A. (1974), *Who Bears the Tax Burden?* The Brookings Institution, Washington D.C.

Pfähler, W. (1982), Zur Messung der Progression und Umverteilungswirkung der Steuern, *Zeitschrift für Wirtschafts- und Sozialwissenschaften* 102, 77-96.

Pfähler, W. (1983), Measuring Redistributional Effects of Tax Progressivity by Lorenz Curves, *Jahrbücher für Nationalökonomie und Statistik* 198, 237-249.

Pfähler, W. (1987), Redistributive Effects of Tax Progressivity: Evaluating a General Class of Aggregate Measures, *Public Finance/Finances Publiques* 32, 1-31.

Reynolds, M. and Smolensky, E. (1977), Post Fisk Distributions of Income in 1950, 1961, and 1970, *Public Finance Quarterly* 5, 419-438.

Seidl, C. (1994), Measurement of Tax Progression with Nonconstant Income Distributions in: W. Eichhorn (ed.), Models and Measurement of Welfare and Inequality, Springer-Verlag, Berlin, 337 - 360.

Suits, D.B. (1977), Measurement of Tax Progressivity, *The American Economic Review* 67, 747-752.