Q-Analysis and Human Mental Models: A Conceptual Framework for Complexity Estimate of Simplicial Complex in Psychological Space

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I. Introduction

One of main stages in systems studying is associated with analysis of links (relationships) between constituents (elements) of those systems. The process of analysis represents systematic rigorously prescribed procedure (call it «hard» for convenience). Revealing and understanding of systems' structural features is grounded on categories of the whole and its parts, bringing into play various forms of cognitive activity, and apparently observations, reasoning and formalized mathematical representations (models) are among them. Analysis can be viewed as a surjection, i.e. mapping of the observable reality onto the human's brain forming peculiar multilayered mental model(s).

Preliminary conclusions on complexity of system are often drawn on the basis of common observations of system's organization if comprehended or caught on neatly [17,27]. Revealing and understanding of structural peculiarities, classification of systems as simple or complex ones take into consideration several factors, and the most significant of them are variety of elements and connections between them. The latter virtually serves as one of major sources of display of multifaceted complexity [10-13,24].

Diverse mathematical methods used for analysis of system's structure and estimation of its complexity, as well as extensive field of mathematical modeling in whole, apparently imply presence of humans «on the scene», i.e. active implicit use of their habitual and unintelligible cognitive processes like thinking, perceiving, making decisions, etc. Rapidly increasing area of cognitive modeling is not exclusively linked to knowledge fields concerned with «process of thought» in large, but also utilizes mathematical and computer languages to describe and analyze particularities of human information processing [8,35].

The paper undertakes an attempt to elaborate on ideas stated in [14]; it discusses possible approach to modification of system's structural complexity estimate Ψ obtained within the framework of the holist system-theoretic procedure called *Q-analysis* and usually attributed to mathematician Ronald Atkin [2-5]. The modeling approach has been already used for solving problems in range of application domains that cover, for example, street network, geology, GIS analysis, Internet-based teleoperation schemes, transportation, water distribution (list of references can be found in [14]). The gist of the procedure is based on guidelines laid down by C.H.Dowker's paper [16]; in outline it discusses *simplicial complexes* K and L defined by hypothetical relation between elements of two sets, isomorphism of their homology and cohomology groups. In accordance with Q-analysis approach, a structure of system under consideration is utilized with a purpose of obtaining its representation in the form of *geometric simplicial complex* K. It is formed by regularly adjoining faces called *simplexes* –

in other words, the intersection of two simplexes is «either empty, or an iterated (common) face of each» [30]. With regard to topologization as a polyhedra (subspace of the Euclidean space $\mathbb{R}^n$), each q-dimensional simplex, or *q-simplex* Δ_q ($q \geq 0$) is a convex hull of its (q+1) vertices, i.e. simplex Δ_0 is a point, Δ_1 – line segment, Δ_2 is a triangle (with its interior), Δ_3 – tetrahedron, etc. We do not put emphasis on topological spaces, therefore K can be associated with geometric simplicial complex as a realization of abstract complex [30,23].

Dimensionality of complex K (dim(K)) obtained is equal to the maximum of dimensions of its simplexes. The aggregate of such simplexes constitutes formal representation of the system under study, and the analysis of such model is performed consecutively at each dimensional level q, $q = \dim(K), \ldots, 0$, through determining the number of clusters of simplexes joined by *chains of q-connectivity*. Results obtained are presented in the form of K's *structural vector* $Q = (Q_{\dim(K)}, \ldots, Q_1, Q_0)$. Thereafter, a measure of structural complexity $\Psi \equiv \Psi(K)$ of complex K is deduced from numeric components of vector Q on the ground of simple formula [9,13].

Such pure «mechanistic» approach does not allow to unfold (reveal) information masked in vector Q, making it impossible to draw daily human cognitive abilities into the process. If this standpoint is accepted, then the structural complexity (connectivity) estimate Ψ within the scope of used mathematical representation can be more thoroughly expressed on the strength of domain expert's diverse considerations and prerequisites, i.e. George Kelly's *personal constructs*. Propositions of the theory of personal constructs (TPC) go that none of humans «has neutral access to reality», anticipation of ambient events psychologically channelizes person's processes [25,18]. Consequently, the impact of *relativity* and *subjectivity factors* on both results interpretation and carried out formal calculations becomes tangible [26].

As already mentioned above, the paper can be viewed as a continuation of [14] in respect of outline of approach aimed at construction of complex K's structural complexity estimate Ψ on the base of some propositions picked up from the field of cognitive science. The rest of the paper is organized as follows: information granules (typical features) obtained through formal procedure of Q-analysis, their consolidation into feature vectors are discussed in Section II. Ideas related to psychological space (P-space), perceived similarity and geometric models of cognition form the basis of Section III. Example that covers proposed computational scheme is considered in Sections IV and V. Conclusion and final remarks are drawn in Section VI.

II. Results of Q-analysis formal procedure. P-space, typical features, idealized cases (IC) and actual estimates of q-connectivity

As it is observed in [14], Q-analysis procedure delivers demonstratively certain calculated values (numeric data); from the epistemological viewpoint, results summarized in structural vector Q can be treated as substantially masked *declarative knowledge* derived from facts and formal representation of system's structural connectivity by means of incidence matrix and corresponding simplicial complex K. It means that we can treat components of vector Q as stimuli that are mentally apprehended, «placed» and processed by humans (domain experts) in certain area of our mind called *psychological space*, or *P-space* for short. Following [25,40], such space serves in the capacity of basis for our inward representations; it doesn't need to pre-exist as a touchable world, but it comes to incorporeal existence in the wake of emerging replications (images) of construing elements under consideration.

In accord with [3,5,9,13], axiomatically introduced complexity estimate $\Psi \equiv \Psi(K)$ is based solely on numeric values of structural vector's elements Q_q, $q = \overline{\dim(K), 0}$. In view of the fact that each particular Q_q stands for the number of connectivity components revealed at the level

$$\Psi(K) = \frac{2}{(N+1)\cdot(N+2)} \sum_{q=0}^{N} (q+1)\cdot Q_q \text{, where } N = \dim(K) \qquad (1)$$

q of complex K analysis, all Q_q utilized in (1) conceal the number of accountable p-simplexes (i.e. non-empty rows of matrix Λ, $p \geq q$). This number is alterable when jumping between q-levels, and such essential information is simply disregarded. Consequently, the following questions can be raised naturally: will experts pay attention to those apparently «missing» pieces of essential information while expressing their opinions about estimate of $\Psi(K)$? Will they attempt to extract all available model's data to bring them into play at the stage of personal constructs formation to replicate these percepts as dimensions in psychological space? Will experts proceed along the path of combining findings into some representation form that is convenient for both perception and comparison (a kind of anchoring), i.e. those mental actions that humans do with confidence? Definitely, the amount of information at the expert's disposal within the scope of Q-analysis procedure is rather scanty, but nevertheless the number of similar questions to ask can be well expanded.

Eventually, Q-analysis approach by itself results not only in obtaining q-connectivity vector. We can assert that «hidden» or concomitant with realized formal steps numeric data of full value are not brought into play properly; these vital information granules are as follows:

1. Q_q – the number of connectivity components at the dimensional level q, i.e. q-th element of structural vector $Q = (Q_N, Q_{N-1}, ..., Q_0)$, $N = \dim(K)$. As it is stressed in [3], members in each such component are joined by *multidimensional «tubes» of simplexes* that in actual fact «embody the local structure of complex K» as system's model,
2. s_q – the number of simplexes having dimension q or greater (all of them are considered when calculating numeric value of component Q_q of the complex K's structural vector),
3. $s(K)$ – total number of non-empty simplexes of all dimensions in complex K,
4. q – current dimensional level of simplicial complex's K analysis ($q = \dim(K), ..., 1, 0$).

It must be specifically admitted that first three characteristics 1÷3 from the aforementioned list can be portrayed as *typical features* of a given dimensional level q, $q = \dim(K), ..., 1, 0$ [14]. Their acquisition is not connected with any additional efforts on the part of domain experts, they are emerging parts that accompany regular steps of Q-analysis procedure. However, the core question to bring forward here is concerned with the way to turn these accessible portions of information into convenient forms qualified for storage and further processing that confine to conventional models used in different research fields of psychology. In the presence of considerable number of levels q ($q = \dim(K), ..., 1, 0$), separate use of characteristics 1÷3 may only lead to complete mess and perplexity. Uttering the word «separate» may involuntarily urge us to impose a handy structured base to simplify comprehension as a *complex mental ability* to represent, understand and interpret accumulated pieces of information. Thereupon, one of considerations that come to mind is related to well-known spatial representation allowing to take also account of grouped entities 1÷4 (i.e. peculiar tokens) as well as domain expert's knowledge. Such outline of mental model(s) can be regarded as a framework that unites tokens and relationships found out in the course of analysis. Both the nature of P-space and its «geometry» are essentially distant from conventional concept of space in mathematics. Nevertheless, the idea to represent objects (stimuli) as points in space and estimate similarity of those stimuli through distance between corresponding points are firmly rooted in psychological studies for many decades. On one hand, such geometrical approach provides easy-to-use mechanism of representation of many practical psychological situations; on the other hand, it also endues considered space with metrics and enables to deviate from its purely dichotomic interpretation based on inherent differentiating/integrating functions [6,25].

As a result, we may cautiously assume that the description of the concept of connectivity (estimate $\Psi(K)$) can be put into effect on the strength of simple term dictionary composed of variable q-level ($q = \dim(K),...,1,0$) feature vectors

$$A_q = \left(\frac{s_q}{s(K)}, \frac{Q_q}{s_q} \right) = (A_q^{(1)}, A_q^{(2)}) , \tag{2}$$

The latter are formed on the basis of granules 1÷4 in view of their rational combination that ensures perception and clear interpretation of vector's A_q elements. The first component $A_q^{(1)}$ in (2) represents the fraction of non-empty simplexes considered at a given q-level, whereas the second one corresponds to the average number of simplexes finding themselves in one q-connectivity component. The output of Q-analysis as well as row vectors (2) are solely numeric (integer and real numbers), so the problem of estimating these numbers in connection with construction of $\Psi \equiv \Psi(K)$ replenishes a range of questions raised above. Publications [15,36] give a detailed account of the human mental ability (*«number sense»*) that is exhibited in everyday calculations. Making emphasis on basic premises and thought regulations that generally prevail in human perception while drawing conclusions, we may depart from the thesis: «... when estimating numbers, most people start with a number (anchor) that comes easily to mind and adjust up or down from that initial state» [42]. In respect to (2), corresponding limiting values can be fixed as follows:

$$\text{if } s_q = s(K)\text{, then } \widetilde{A}_q^{(1)} = 1 \quad | \quad \text{if } Q_q = 1\text{, then } \widetilde{A}_q^{(2)} = 1/s_q \tag{3}$$

What do expressions combined in (3) mean? The left part simply states that the first element of vector A_q is equal to unity, if all non-empty simplexes of complex K are considered without exception at this particular level q. The expression on the right (symbol '|' in (3) serves as a separator) shows the case when all s_q simplexes find themselves in a single connectivity component Q_q. In other words, under the compliance with tight condition $Q_q = 1$, the degree of connectivity used to express the complexity Ψ_q of level q is growing with increase of s_q. We call $\widetilde{A}_q^{(1)}$ and $\widetilde{A}_q^{(2)}$ in (3) *limiting values* because they correspond to the most severe case that can be rarely observed in practice at particular q-level at the stage of complex K analysis. The overall complexity estimate $\Psi(K)$ should be comprised of individual «local» Ψ_q. With this idea in mind [14], we may choose the following implication IC-rules (Idealized Case of connectivity-complexity correspondence at q-levels) in the capacity of peculiar *anchors* mentioned before:

$$\widetilde{A}_q = \left(\widetilde{A}_q^{(1)}, \widetilde{A}_q^{(2)}\right) = \left(1, \frac{1}{s_q}\right)\Bigg|_{Q_q=1,\ s_q=s(K)} \quad \rightarrow \quad \widetilde{\Psi}_q = 1 \tag{4}$$

It can be called Ψ_q-*squiggle rule*. At the same time, the *actual* (i.e. computed) *estimate* $\widehat{\Psi}_q$ of q-connectivity can be directly inferred from IC-rules and forms (Ψ_q-*cap*) shown below

$$\widehat{A}_q = \left(\widehat{A}_q^{(1)}, \widehat{A}_q^{(2)}\right) = \left(\frac{s_q}{s(K)}, \frac{Q_q}{s_q}\right) \rightarrow \widehat{\Psi}_q = \ < \textit{calculated value} > \tag{5}$$

Both vectors (2) and left-hand side of (4) can be regarded as low-dimensional stimuli (the number of their components is minimal) having apparently concrete meaning to domain-expert within the framework of Q-analysis procedure. In addition, those stimuli may have handy representation as points in two-dimensional space. As appears from (4), vector $\widetilde{A}_q$ is associated with the expression of maximum (idealized) complexity $\widetilde{\Psi}_q$ manifested through q-connectivity of complex K. According to a theory of memory retrieval, if patterns, i.e. IC-rules (4), are stored in the memory for a time needed to analyze results at particular q-level, $q = N,...,1,0$, then cognizable stimuli in the form of Ψ_q-*cap* association rule resonates with them through «invocation» of (4)-(5) parts and assessment of their similarity. Such view is appreciably simplified version of the basis underlying *retrieval theory* that implies comparison of probe (stimulus) with «each item in the search set simultaneously (i.e. in parallel)» [36].

III. Perceived similarity. Prototypical exemplars. Models of similarity

Perceived similarity is not evidently believed to be invariant, so up to date acknowledged models are proposed to measure similarity between two patterns (exemplars). In particular, Amos Tversky's *contrast model* (CMd) [44] turns measuring of similarity to feature-matching process that provides for weighted accounting common and distinctive features of patterns (entities). In the present case of generalization of Q-analysis results such model is unlikely applicable, since the number of features concerned is too small. *Geometric models* (GMd) that assume inverse distance measures in a metric space as a basis of proximity (similarity) between exemplars remain now among the most influential and commonly used ones [22]. In spite of occasionally mentioned potential problems with geometric models [22,45], it must be admitted that their results in low-dimensional space appear to be quite convincing and interpretable. With regard to scheme (2), (4) and (5) utilizing two-dimensional simple vectors coupled with Ψ_q estimates, the cause to prefer just them seems well-grounded.

In respect to two hypothetical equal-sized objects $x_1^T = (x_1^{(1)},...,x_1^{(n)})$ and $x_2^T = (x_2^{(1)},...,x_2^{(n)})$ having numeric components (or, attributes) $x_1^{(i)}$ and $x_2^{(i)}$, $i = \overline{1,n}$, the proximity (dissimilarity) between them is made conditional on corresponding distances d_i, $i = \overline{1,n}$, between attributes:

$$\begin{aligned} &d_i = \mathrm{dist}(x_1^{(i)}, x_2^{(i)});\ p_i = \mathrm{prox}(x_1^{(i)}, x_2^{(i)}) = f(d_i) \\ &D(x_1^T, x_2^T) = g(d_i) \ \rightarrow\ P(x_1^T, x_2^T) = f(D(x_1^T, x_2^T)) \end{aligned} \tag{6}$$

where p_i are calculated proximities between respective attributes $x_1^{(i)}$ and $x_2^{(i)}$ ($i = \overline{1,n}$), D is a distance between two objects calculated through specified functional transformation(s) of individual d_i and $f(\cdot)$ is a function (mapping) determining chosen similarity model on the respective space [1]. Process of similarity's formalization suggests accounting for manifest entries, viz. (*A*) dimensions vector (object) x_k^T is associated with, and (*B*) values $x_k^{(i)}$ «bound to» those dimensions. For the case (2)-(5) that also deals with vectors $\widetilde{A}_q$ and $\widehat{A}_q$, we may notice certain resemblance to propositions of prototype theory put forward by Eleanor Rosch almost four decades ago [37]. Prototype as mentally represented pattern of knowledge (model of concept) is linked to specific characteristics; in respect to K and its subcomplexes, category «maximally attainable complexity (or, connectivity)» at a given q-level, $q = \overline{\dim(K), 0}$, can be evinced by IC-rules (4). We can treat rules $\widetilde{A}_q \to \widetilde{\Psi}_q$ as an abstract form of comprehension of the aforesaid category; rules can be considered as marked anchors in human comparison/ categorization of objects based on their perceptual resemblance. Each calculated in turn $\widehat{A}_q$ is appraised for the purpose of its proximity to the prototypical exemplar (representation built on $\widetilde{A}_q$). Further verbal valuation of the q-level by force of proximity degree requires utilization of extra conceptual categories of connectivity (e.g. *«marginal»*, *«close to medium»*, to name a few) that don't possess sharp boundaries [7]. This fact serves as illustration of appropriateness of fuzzy sets/logic use as a formal cognitive modeling tool on the basis of concepts and related categories in the analysis of K. The paper does not attempt to form such categories for the reason that it is aimed at discussion of framework of approach's alternatives.

Making emphasis on psychological space (P-space) endued with metric, we give preference to specific geometric model (GMd) of cognition. Inherently it focuses on analysis of similarity (difference) of data objects complying with Roger Shepard's 'universal law of generalization' that suggests to take a view of similarity as «a function of the distance between psychological representations» [28,39]. Calculated vector (stimulus) $\widehat{A}_q$ (5) is represented on two emerged dimensions in P-space thus enabling to attribute $\widehat{A}_q^{(1)}$ and $\widehat{A}_q^{(2)}$ to corresponding values on psychological dimensions. As an option, category «maximal complexity (or, connectivity)» of K at a given q-level (4) may correspond to psychological representation in the form of point ($\widetilde{A}_q^{(1)}, \widetilde{A}_q^{(1)}$). The similarity (6) between two vectors (4)-(5) can be determined [22] inversely by way of the parameterized distance measures (r-Minkowski metric) as

$$\begin{aligned} D_q^{[r]}(\widetilde{A}_q, \widehat{A}_q) &= d(q) = \left[\sum_{k=1}^{2} \left| \widetilde{A}_q^{(k)} - \widehat{A}_q^{(k)} \right|^r \right]^{1/r} \\ P_q(\widetilde{A}_q, \widehat{A}_q) &= \exp\left[-a \cdot (D_q^{[r]}(\widetilde{A}_q, \widehat{A}_q))^n \right], \; q = \dim(K), \ldots, 1, 0 \end{aligned} \tag{7}$$

Following [14], value of $\widehat{\Psi}_q$ can be obtained on the basis of (7) under the assumption of existence of latent dependency $\widehat{\Psi}_q = P_q(\widetilde{A}_q, \widehat{A}_q) = f(D_q^{[r]})$ characterized by conditions: (1) if $D_q^{[r]} = 0$, then the complexity $\widehat{\Psi}_q$ is equal exactly to $\widetilde{\Psi}_q = 1$, (2) if $D_q^{[r]} = d(q) > 0$, then $\widehat{\Psi}_q$ is decreasing gradually towards zero with the growth of d. Gaussian function $P = \exp(-a \cdot d^n)$ with sensitivity parameter $a > 0$ (determined by domain experts) and $n = 2$ can be viewed as a potential candidate for determining complexity estimate $\widehat{\Psi}_q$, $q = N, \ldots, 0$ [19,32,39]. Generally speaking, geometric model ensures adequate form of information representation in cognitive science, and corresponding vectors (stimuli) $\widehat{A}_q$ and $\widetilde{A}_q$ are set in motion in the capacity of geometrical structures (points) [22,29].

Discussions and arguments covered by some research studies accentuate consistently the impossibility to «catch» stimulus generalization gradients by means of single parameter *a* (7) often used in cognitive science and psychology. Following tendency of those publications that discuss geometrical models of similarity, our approach makes stress on simple transition forms $D_q^{[r]} \to P_q = \widehat{\Psi}_q$ having clear expression from the viewpoint of representation of human's information perception and processing. Steps of action chain «data obtained from Q-analysis procedure – grouping for the purpose of representation in low-dimensional conceptual space – conferring metric upon space – forming proximity (connectivity) estimates followed by their verbal interpretation» also have something in common with individual *Gestalt principles of perception* (similarity, proximity, etc.) in the sense that Q-analysis data are «organized into a stable and coherent form» named feature vectors (2)-(4)-(5) [43].

Gaussian type function $\exp\left[-a \cdot (D_q^{[r]})^n\right]$ (7) is commonly utilized in cognitive modeling [39,43], whereas corresponding dimensions in psychological (or, conceptual) space can be weighted (i.e. d(q) turns into $d^w(q) = \left[\sum_{k=1}^{2} w_k \cdot \left|\widetilde{A}_q^{(k)} - \widehat{A}_q^{(k)}\right|^r\right]^{1/r}$); weights w_k can be treated as peculiar bias parameters stating certain aspects of selective attention or additional knowledge on values of arising dimensions [34]. Particularly, fuzzy numbers (intervals) are candidates to express approximate values of w_k $(i = 1, 2)$.

Within the diversity of developed formal approaches to classification and categorization [31,33] there are rather simple ones including light *connectionist* version of *prototype model* [8]. The computational process $\left\{\widetilde{A}_q^{(k)}, \widehat{A}_q^{(k)}\right\}_{k=1,2} \to D_q^{[r=2]} \to \widehat{\Psi}_q$ as described above (Fig.1); so

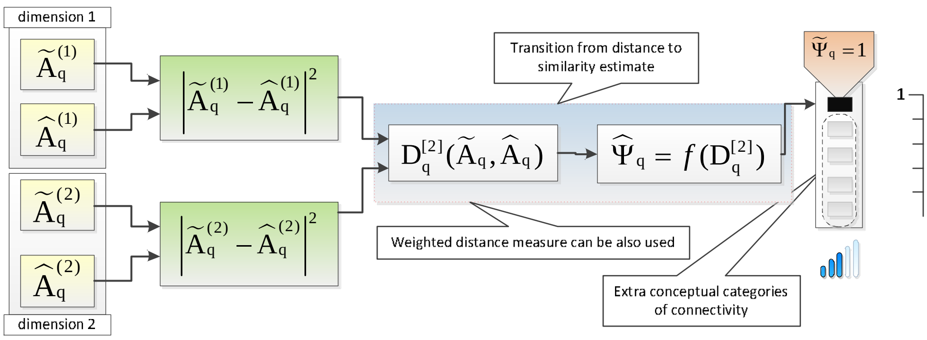

Figure 1. Calculation of q-level connectivity estimate based on connectionist model

long as both dimensions of $\widehat{A}_q$ and $\widetilde{A}_q$ describe peculiarities of particular level's connectivity, we treat them as integral ones – in the broad sense, those are defining attributes that serve as a single holistic descriptor of the level q quality in terms of its structural complexity [19,21]. Under such provisions, metric $D_q^{[r=2]} = \left[(s(K) - s_q)^2 / s^2(K) + (1 - Q_q)^2 / s_q^2\right]^{0.5}$ is recommended for use in contrast to $D_q^{[r=1]}$ (7) in view of capturing both dimensions together [20,22].

IV. Example (part I) – Aggregated complexity estimate

To illustrate idea conveyed above consider binary (incidence) matrix Λ representation of the following relation $\lambda \subset X \times Y$, where $X = \{x_i,\ i=\overline{1,5}\}$ and $Y = \{y_j,\ j=\overline{1,6}\}$, correspondingly:

$$\Lambda = \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array}$$

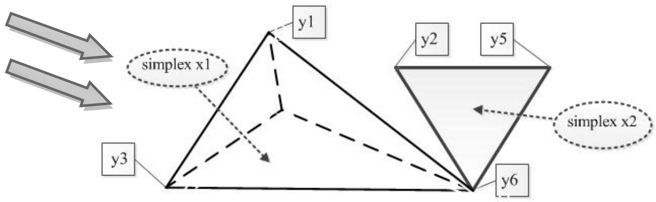

Figure 2. Fragment (sub-complex) of simplicial complex K

It can be noticed that $s(K) = 5$, and results of Q-analysis can be summarized as follows:

level $q = 3$	$Q_3 = 1\ \ (s_3 = 1)$	$\{x_1\}$
level $q = 2$	$Q_2 = 3\ \ (s_2 = 3)$	$\{x_1\}, \{x_2\}, \{x_4\}$
level $q = 1$	$Q_1 = 2\ \ (s_1 = 5)$	$\{x_1, x_2, x_4, x_5\}, \{x_3\}$
level $q = 0$	$Q_0 = 1\ \ (s_0 = 5)$	$\{\text{all simplices } x_i \text{ of complex K},\ i = 1,\ldots,5\}$

Vectors $\widehat{A}_q\ (q = \overline{0,3})$ are calculated on the basis of values shown above, i.e. $\widehat{A}_3 = (0.2, 1)$, $\widehat{A}_2 = (0.6, 1)$, $\widehat{A}_1 = (1, 0.4)$ and $\widehat{A}_0 = (1, 0.2)$. In the process, limiting values in (3)-(4) form $\widetilde{A}_q$ that are $\widetilde{A}_3 = (1,1)$, $\widetilde{A}_2 = \left(1, \dfrac{1}{3}\right)$ and $\widetilde{A}_1 = \widetilde{A}_0 = (1, 0.2)$ for respective values of q. On condition that $r = n = 2$ and $a = 1.8$ in (7), we obtain the following values of $D_q^{[r]}$ and $\widehat{\Psi}_q = P_q(\widetilde{A}_q, \widehat{A}_q)$:

$$D^{[r=2]} = \left(D_3^{[2]}, \ldots, D_0^{[2]}\right) = (0.8, 0.78, 0.2, 0) \ \rightarrow\ \left(\widehat{\Psi}_3, \ldots, \widehat{\Psi}_0\right) = (0.32, 0.34, 0.93, 1)$$

Exact-valued estimates do not seem informative with regard to explanations and discussions held within group of domain experts; on top of that, parameter a's natural moderate variability that is typical under its judgmental choice leads to slightly floating $\widehat{\Psi}_q$ values (e.g. if $a = 2.1$, then $\widehat{\Psi}(a, n, D^{[r=2]}) = (0.2608, 0.281, 0.9194, 1)$; the issue of choosing «appropriate» value (or, range of values) of sensitivity parameter a is discussed partially in [14]). Turning acceptable range of calculated $\widehat{\Psi}_q = f(D_q^{[r]})$, $q = \overline{\dim(K), 0}$, into unit interval affords good opportunity for interpretation of resultant numeric values by means of verbal terms (e.g. *«weak (low)»*, *«medium»*, *«rather strong (high)»*, *«strong (high)»*, *«extremely strong (high)»*). They can be associated with fuzzy intervals that enable here to characterize connectivity (complexity) of complex K as being likely *«weak (low)»* at first two higher q-levels, *«rather strong (high)»* or even closer to *«strong (high)»* at the level $q = 1$ and *«extremely strong (high)»* at the lowest level of K's analysis. Hypothetical partitioning of the unit discourse into meaningful fuzzy sets is shown by gray rectangles in the block under $\widetilde{\Psi}_q = 1$ anchor in Fig.1.

Values of components of structural vector $Q^* = (2,3,1)$ for conjugate K^* ($\dim(K^*) = 2$) that represents the relation $\lambda^* \subset Y \times X$ result in obtaining $D^{[2]}$ and $\widehat{\Psi}_q\big|_{a=1.8}$ as $(0.833, 0.333, 0)$ and $(0.287, 0.819, 1)$ giving reasonable cause to label $\widehat{\Psi}_2$ and $\widehat{\Psi}_1$ as *«low»* and *«high (strong)»*, correspondingly. Besides, aggregation of individual $\widehat{\Psi}_q$ values into single relative (subjective) estimate $\widehat{\Psi}_{agg}(K)$ of K's complexity can simply utilize standardized weights $\overline{\mathbf{w}}_q = sf_q \Big/ \sum_{k=0}^{N} sf_k$, $q = \overline{N = \dim(K), 0}$, derived from significance factors sf_q of q-levels. To calculate the latter, well-interpretive function $\varphi(q) = \tilde{c} \cdot q^2 + 1$ can be used [14]; the value of $\tilde{c}$ is directly related to psychological level of human perception of q-levels significance that «manifests» in course of simplicial complex's analysis. For instance, under $\tilde{c} = 0.05$, $\widehat{\Psi}_{agg}(K)$ and $\widehat{\Psi}_{agg}(K^*)$ are equal to appr. 0.6 and 0.68. Does it really mean that K^* is more complex if compared to K?

V. Example (part II) – Comparing estimates. Saaty's scale of priorities

Calculated estimate $\widehat{\Psi}_{agg}(K)$ dissembles important K's feature – namely, its dimensionality that domain expert(s) should definitely draw attention at while performing analysis and comparing $\widehat{\Psi}_{agg}$ values obtained. With such purpose in mind, the *scale of verbal priorities* proposed by Thomas Saaty [38] can be utilized to express judgments concerning perceived difference between dimensionalities of arbitrary K_1 and K_2. The scale uses five base marks (numbers 1,3,5,7 and 9) that reflect the human ability to perform confident differentiations and four transitional, more «diffused» or compromise as compared to base ones, comparative marks (2,4,6 and 8). Base scale ticks ($ex_i, i = 1,3,5,7,9$) represent differences as *«virtually absent»* (1), *«insignificant»* (3), *«non-negligible»* (5), *«substantial»* (7) and *«absolute»* (9). In essence, such approach exploits rough classification of stimula by three key signs «rejection-indifference-acceptance», each of which is endowed with specializing shades «low-average-high» in line with known trichotomy principle. The use of limited discrete ticks preserves the possibility to link numbers with meaning and ease of their processing.

With account taken of Saaty's scale, we can introduce a semblance of adjusting coefficient $\frac{1}{ex_i}$ to be applied to aggregated estimate of structural complexity. Assuming $N_1 = \dim(K_1)$, $N_2 = \dim(K_2)$ and $\operatorname{diff}(N_1, N_2) = \frac{1}{ex_i} < 1$ ($i = \overline{2,9}$), the adjusted $\widehat{\Psi}_{agg}$ can be deduced on the base of simple multiplication operation as

$$\widehat{\Psi}_{agg}^{(corr.)}(K_1) = \operatorname{diff}(N_1, N_2) \cdot \widehat{\Psi}_{agg}(K_1) \quad (8)$$

With regard to the example under consideration, $N = \dim(K) = 3$, $N^* = \dim(K^*) = N - 1 = 2$. Although almost insignificant, the difference between dimensionalities N and N^* is observed, e.g. $\operatorname{diff}(N^*, N) = 1/2$ ($ex_2 = 2$ is a transitional mark on the scale) in (8). Procedural memory helps to carry out standard mathemat. operations, but attempts to retrieve «familiar» enough results of calculations (e.g. $6/2 \to 3$, $66/2 \to 33$, etc.) from long-term memory may bring approximate output (range) [41]. The latter can be considered in the capacity of acceptable one in respect to a given problem, that is (as an alternative) $\mathrm{diff}(N^*, N)\cdot\widehat{\Psi}_{agg}(K^*) \to\sim [0.32, 0.35]$. Consequently, the interval $[0.32, 0.35]$ can be associated with support of output fuzzy set, and adjusted complexity of K^* can be interpreted roughly as *«weak (low)»*. Most likely, it is the case, because at the level $q = 2$ two simplexes y_4 and y_6 form 2 connectivity components, and at the following level of analysis ($q = 1$) all six simplexes y_i of complex K^*, $i = \overline{1,6}$, are also not connected tightly (like it or not, but 3 components in total).

VI. Conclusion. Final remarks

Actually, integration of information granules (features) obtained at the stage of performing Q-analysis into compact and interpretive form clears the way to form (calculate) structural complexity $\Psi(K)$ estimate making partial feasible use of perception theory and ideas derived from cognitive science. As shown in the paper, development of geometric models proves their adequacy when dealing with similarity that is a base for human cognitive abilities. Such models are intuitive and have a good potential in being adopted in formal models of cognition (e.g. connectionist type of models). To the opinion of the author, even simple and general scheme presented in the paper sets up a weighty milestone in thorough elaboration of different approaches (fuzzy sets and systems, abstract mechanisms of human information processing described by cognitive models, pairwise comparison of simplicial complexes K_1, K_2 followed by making sound decisions concerning their observed differences (similarities)). Within the scope of Q-analysis procedure that manipulates rather scanty amount of available data, any endeavors to «enrich» it with both formal and intuitive procedures to expose essential (hidden) structural features of complex (system's model) can be considered as a step towards development of fully fledged software tool to study multidimensional connectivity of structures.

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