Lecture notes on Bargaining and the Nash Program

Hannu Vartiainen

March 5-7, 2012
Higher School of Economics, Moscow

1 Introduction

- Economics is about allocating scarce resources for people with infinite needs
- Typical economists' solution: price mechanism

  ...but this works well only under idealized conditions: large markets, no frictions
- The conditions guarantee that no agent can influence the price and hence no agent has any market power
- Conditions rarely met => outcome is a result of a **bargaining** process where each of the agents has at least some bargaining power
- The bargaining set up: an identifiable group of people choose collectively an outcome and unanimity about the best outcome is lacking
- How should we as outside observers see the situation?

1. How is the outcome determined?
2. Is the outcome normatively good?
3. How do the external factors affect the outcome?
4. Where does the bargaining power come from?
5. How will the number of participants affect the outcome?
6. ...

- The fundamental problem is a feedback loop: my bargaining strategy depends on your bargaining strategy which depends on my bargaining strategy which...

  => the problem is open ended

- Canonical strategic problem - if we can solve this, we can solve "any" strategic problem

- Two leading approaches, both initiated by John Nash (1951, 1953)

  - Cooperative: evaluate the outcome directly in terms of the conditions, "axioms", that a plausible outcome will satisfy
  - Non-cooperative: apply non-cooperative game theory to analyze strategic behavior, and to predict the resulting outcome

- An advantage of the strategic approach is that it is able to model how specific details of the interaction may affect the final outcome

- A limitation, however, is that the predictions may be highly sensitive to those details

1.1 The Nash Program

- Nash (1953): *use cooperative approach to obtain a solution via normative or axiomatic reasoning, and justify this solution by demonstrating that it results in an equilibrium play of a non-cooperative game*

- Thus the relevance of a cooperative solution is enhanced if one arrives at it from very different points of view

- Similar to the microfoundations of macroeconomics, which aim to bring closer the two branches of economic theory, the Nash program is an attempt to bridge the gap between the two counterparts of game theory (axiomatic and strategic)

- Aumann (1997): The purpose of science is to uncover "relationships" between seemingly unrelated concepts or approaches

- Good sources of further reading on the Nash Program are Serrano (2004, 2005)

1.2 These lectures

- Overview of modern bargaining literature

- Emphasis in the interrelation between axiomatic and strategic models

- Centered around the Nash bargaining solution
- Empirical interpretation
- Interpretation of the Nash Program

2 Axiomatic approach

- Let there be a player set $\{1,...,n\} = N$, a joint utility profile from a utility set $U \subseteq \mathbb{R}^n_+$
- The outcome $(0,...,0)$ is the *disagreement point*
- Vector inequalities $u \geq v$ means $u_i \geq v_i$ for all i and $u > v$ means $u_i > v_i$ for all i
- U comprehensive ($u \geq v \geq 0$ and $u \in U$ implies $v \in U$), compact, and convex
- Collection $\mathcal{U}$ of all utility sets U
- Solution is a function $f : \mathcal{U} \to \mathbb{R}^n_+$ such that $f(U) \in U$
- Denote the (weak) Pareto frontier by $P(U) = \{u \in U : u' \geq u$ implies $u' \notin U$ or $u' = u\}$

2.1 Nash's solution

Pareto optimality (PO): $f(U) \in P(U)$, for all $U \in \mathcal{U}$

- Use the notation $aU = \{(a_1u_1, ..., a_nu_n) : (u_1,...,u_n) \in U\}$, for $a = (a_1,...,a_n) \in \mathbb{R}^n$
- Decision theoretically similar problems should induce similar solution

Scale Invariance (SI): $f(aU) = af(U)$, for all $a \in \mathbb{R}^n_{++}$, for all $U \in \mathcal{U}$

- In a symmetric situation, now player should be in an advantageous position and hence the solution should be symmetric

Symmetry (SYM): If $U^\sigma = \{(u_{\sigma(i)})_{i \in N} : u \in U\} = U$, for any permutation $\sigma : N \to N$, then $f_i(U) = f_j(U)$ for all i, j

- Removing outcomes that "do not" affect bargaining should not affect the outcome of the process

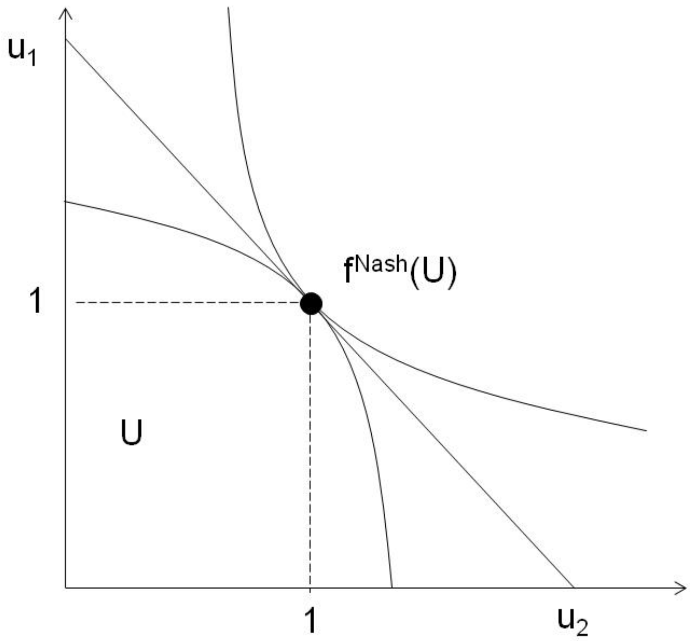
Independence of Irrelevant Alternatives (IIA): $f(U') \in U$ and $U \subseteq U'$ imply $f(U') = f(U)$, for all $U, V \in \mathcal{U}$.

- Thus if pair $f(U)$ is "collectively optimal" in U, and feasible in a smaller domain, then it should be optimal in the smaller domain, too
- Comparable to WARP in the single decision maker situation.
- IIA particularly appropriate under the interpretation that the bargaining solution is proposed by an arbitrator

Theorem 1 *A bargaining solution f satisfies PO, SI, IIA, and SYM on $\mathcal{U}$ if and only if f is the **Nash** bargaining solution f^{Nash} such that*

$$f^{Nash}(U) = \arg\max_{u \in U} \prod_{i=1}^{n} u_i$$

Proof. Necessity: Let f satisfy the axioms. We show that $f(U)$ is the Nash solution. Identify $f^{Nash}(U)$ and find scales $a_1, ..., a_n$ such that $a_i = 1/f^{Nash}(U)$ for all i. Then $f^{Nash}(aU) = (1, ..., 1)$ and $aU \subseteq \Delta := \{v \in \mathbb{R}^n_{++} : v_1 + ... + v_2 \leq n\}$. By PO and SYM, $f^{Nash}(\Delta) = (1, ..., 1)$. By IIA, $f^{Nash}(aU) = (1, ..., 1)$. By SI, $f(U) = f^{Nash}(U)$.



■

- Removing SYM leads to a class of solutions

Theorem 2 *A bargaining solution f satisfies PO, SI and IIA on $\mathcal{U}$ if and only if f is an **asymmetric Nash** bargaining solution f^α such that, for some $(\alpha_1, ..., \alpha_n) \in \mathbb{R}_+$,*

$$f^\alpha(U) = \arg\max_{u \in U} \prod_{i=1}^{n} u_i^{\alpha_i}$$

- The weights $\alpha_1, ..., \alpha_n$ could now be interpreted as a reflection of the players' **bargaining power**
- The higher α_i is, the bigger utility i will receive under the solution

Example 3 *Let $n = 2$ and $U = \{u \in \mathbb{R}^2_+ : u_1 + u_2 \leq 1\}$. Let $\alpha_1 = \beta$ and $\alpha_2 = 1 - \beta$, for $\beta \in (0,1)$. Then $f^\alpha(U) = \arg\max u_1^\beta u_2^{1-\beta}$. At the optimum, $u_2 = 1 - u_2$. The first order condition*

$$\beta u_1^{\beta-1}(1-u_1)^{1-\beta} - (1-\beta)u_1^\beta(1-u_1)^{-\beta} = 0.$$

Thus $f_1^{Nash}(U) = \beta$ and $f_2^{Nash}(U) = 1 - \beta$. That is, the payoff of the agent increases in his bargaining power.

2.2 Other solutions

- The outcome $\mu_i(U)$ is i's *ideal point* in U, defined by $\mu_i(U) = \max\{u_i : u \in U\}$

Individual monotonicity (IMON): If $U \subseteq U'$ and $\mu_i(U) = \mu_i(U')$, then $f_i(U') \geq f_i(U)$ for all i

Theorem 4 *A bargaining solution f satisfies PO, SI, IMON, and SYM on $\mathcal{U}$ if and only if f is the **Kalai-Smorodinsky** bargaining solution f^{KS} such that $f^{KS}(U)$ is the maximal point in the intersection of U and the segment connecting 0 to $(\mu_1(U), ..., \mu_n(U))$*

Proof. Necessity: Let f satisfy the axioms. We show that $f(U)$ is the Kalai-Smorodinsky solution. Identify $f^{KS}(U)$ and find scales $a_1, ..., a_n$ such that $a_i = 1/\mu_i(U)$ for all i. Then $\mu_i(aU) = 1$ for all i. Let T be the convex hull of points $\{(1,0,...,0), (0,1,0,...,0), ..., (0,...,0,1), f^{KS}(aU)\}$. Then, since $\mu_i(T) = \mu_i(aU)$ for all i, $f^{KS}(aU) = f^{KS}(T)$. By PO and SYM, $f(T) = f^{KS}(T)$. Since $T \subseteq aU$, by IMON, $f(aU) = f(T)$. Thus $f(aU) = f^{KS}(aU)$. By SI, $f(U) = f^{KS}(U)$. ■

Strong monotonicity (SMON): If $U \subseteq U'$, then $f_i(U') \geq f_i(U)$ for all i

Theorem 5 *A bargaining solution f satisfies PO, SMON, and SYM on $\mathcal{U}$ if and only if f is the **Egalitarian** bargaining solution f^E such that $f^E(U)$ is the maximal point in U of equal coordinates*

Proof. Necessity: Let f satisfy the axioms. We show that $f(U)$ is the Egalitarian solution. Identify $f^E(U) = (x, ..., x)$. Let V be the convex hull of $\{(x,0,...,0), (0,x,0,...,0), ..., (0,...,0,x), (x,...,x)\}$. By PO and SYM, $f(V) = f^E(V)$. Since $V \subseteq U$, by SMON, $f(U) \geq f(V)$. Since $f(V) = f^E(V)$ is in the boundary of U, we have $f(U) = f^E(V) = f^E(U)$. ■

- However, since the egalitarian solution does not satisfy scale invariance, it is hard to justify on behavioral grounds
- Equivalently, strong monotonicity is too strong a condition

2.3 Population based axiomatization of the Nash solution

- The IIA assumption has received much criticism
  - Outside alternatives may have effect on bargaining via their strategic significance
  - Under strategic bargaining, the single agent connotation not appropriate
- A stability argument due to Lensberg (1988), and Lensberg and Thomson (1991)
- Based on consistency and continuity considerations
- For any $K \subseteq N$, denote by $\mathcal{U}^K$ the set of utility sets restricted to the player set (i.e. $\mathcal{U} = \mathcal{U}^N$)
- Let f be defined for all $\mathcal{U}^K$, i.e. also for subsets the K of N
- This permits drawing connections between problems of different dimension => more tools to restrict the solution
- Denote $u_K = (u_i)_{i \in K}$
- Continuity requires that for any two problems U, U' close to other, the solution should also be close
- If d is a metric on X, then Hausdorff metric d_H of two nonempty subsets Y and Z of X is defined by $d_H(Y, Z) = \max\{d(Y, Z), d(Z, Y)\}$, where $d(y, Z) = \inf_{z \in Z} d(y, z)$ and $d(Y, Z) = \sup_{y \in Y} d(y, Z)$

Continuity (CONT): If sequence $\{U^k\} \subseteq \mathcal{U}^S$ converges in Hausdorff metric to U, then $f(U^k)$ converges to $f(U)$

- Consistency requires that the players continue bargaining even if some of the players "leave" the game with their utility shares
- For $X \subset \mathbb{R}^n_{++}$, denote the projection at u on the player set S by $p^u_S(X) = \{v_S : (v_S, u_{N \setminus S}) \in X\}$

Bilateral stability (STAB): For any $U \subseteq \mathcal{U}^N$, if $p^v_{\{i,j\}}(U) = T$ and $f(U) = v$, then $f_{\{i,j\}}(U) = f(T)$

- Thus, the solution, when restricted to a two-player projection of the game at the solution outcome, must not change the outcome
- Can be extended to the multilateral case
- The following strengthening of the symmetry condition requires that changing the names of the players will not affect the outcome

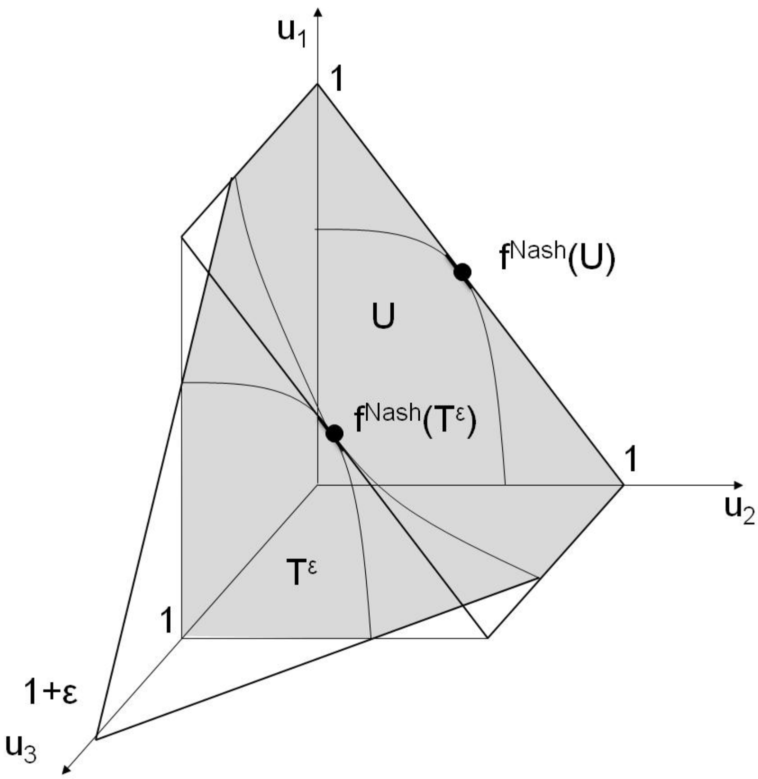
Anonymity (ANON): if $U^{\sigma} = \{(u_{\sigma(i)})_{i \in N} : u \in U\}$, for any permutation $\sigma : N \to N$, then $f(U^{\sigma}) = (f_{\sigma(i)}(U))_{i \in N}$

- In particular, if U is a two-player problem and symmetric, then $f_1(U) = f_2(U)$

Lemma 6 *If a solution f satisfies PO, ANON, CONT, STAB, and SI, then $f_{\{i,j\}}(U) = f^{Nash}(T)$ where $p^{v}_{\{i,j\}}(U) = T$ and $f_{\{i,j\}}(U) = v$*

Proof. (Sketch) Consider $U \in \mathcal{U}^2$ and normalize it such that $f^{Nash}(U) = (0.5, 0.5)$. By SI this normalization is without loss of generality. Our aim is to show that $f(U) = f^{Nash}(U)$. Assume that U contains a nondegenerate line segment ℓ centered around $f^{Nash}(U)$. By CONT, this assumption is without loss of generality. Create player 3 and construct a problem $T = \{u \in \mathbb{R}^3_+ : u_1 + u_2 \leq 1, u_3 \leq 1\}$. For any $\varepsilon \geq 0$, identify the smallest cone C^{ε} that contains $\{(0,0,1+\varepsilon)\} \cup \{(u_1, u_2, 1) : (u_1, u_2) \in U\}$. Finally, let $T^{\varepsilon} = T \cap C^{\varepsilon} \in \mathcal{U}^3$. By PO and ANON, $f(T^0) = f^{Nash}(T^0) = (0.5, 0.5, 1)$. Since $P(T^{\varepsilon})$ around $(0.5, 0.5, 1)$ contains $\{u : u_{\{1,2\}} \in \ell, u_3 = 1\}$ it follows by CONT that $f_{\{1,2\}}(T^{\varepsilon}) \in \ell$ for small enough $\varepsilon > 0$. Thus, since $p^{f(T^{\varepsilon})}_{\{1,2\}}(T^{e}) = U$ for small enough e, STAB implies that $f_{\{1,2\}}(T^{\varepsilon})$ is fixed for small enough e. But then

$f_{\{1,2\}}(T^{\varepsilon}) = (0.5, 0.5)$ by CONT. By STAB, $f(U) = (0.5, 0.5) = f^{Nash}(U)$.



■

- Thus the solution must be consistent with the Nash solution in any of its two-player projections

Lemma 7 *If a solution f satisfies PO, ANON, CONT, and STAB, then $f_{\{i,j\}}(U) = f^{Nash}(T)$ such that $p^v_{\{i,j\}}(U) = T$ and $f_{\{i,j\}}(U) = v$ for all $i, j \in N$ implies $f(U) = f^{Nash}(U)$*

Proof. (Sketch) Suppose that the surface of U is differentiable. Then U is supported by a unique hyperplane H at $f(U)$. Let H' be the hyperplane that supports $\{v : \Pi_i v_i \geq \Pi_i f_i(U)\}$. We are done if $H = H'$. Suppose not. Then there is $\{i, j\}$ such that $p^{f(U)}_{\{i,j\}}(H) \neq p^{f(U)}_{\{i,j\}}(H')$. But then $f_{\{i,j\}}(U) \neq f^{Nash}(T)$ such that $p^v_{\{i,j\}}(U) = T$ and $f_{\{i,j\}}(U) = v$, contradicting the assumption. ■

- Thus the solution must be consistent with the Nash solution in any of its two-player projections

- Combining the lemmata, we obtain the modern axiomatization of the Nash solution

Theorem 8 *A bargaining solution f satisfies PO, ANON, CONT, STAB, and SI, on $\mathcal{U}$ if and only if f is the* ***Nash*** *bargaining solution f^{Nash} such that, for all $U \in \mathcal{U}^S$, for all $K \subseteq N$,*

$$f^{Nash}(U) = \arg\max_{u \in U} \Pi_{i \in K} u_i$$

3 Strategic approach

- A fundamental problem with the axiomatic approach to bargaining is that many if not most of the properties of the solution are ultimately normative
- For example, while efficiency is an intuitive outcome of negotiation, what is the procedure that backs up the intuition?
- It is well known that strategic behavior of the players constraints what can be collectively achieved in bargaining scenarios
- When the players have conflicting interests, it is too optimistic to think that the players voluntarily commit to the jointly beneficial course of action
- Instead, they want to enhancing their own view by choosing a negotiation strategy that maximizes their own surplus, even at the expense of the others

3.1 Rubinstein's bargaining game

- There is a set $1, ..., n$ of agents, distributing a pie of size 1
- The present value of i's consumption x_i at time t is

$$u_i(x_i)\delta^t,$$

where (for time being) u_i is assumed increasing, *concave*, and continuously differentiable utility function and $\delta \in (0, 1)$ is a discount factor

- Thus the players' preferences reflect "risk-aversion"
- Possible allocations of the good constitute an $n-1$ -simplex $S = \{x \in \mathbb{R}^n_+ : \Sigma_i x_i \leq 1\}$
- There is a $\Delta > 0$ delay between bargaining stages
- Unanimity bargaining game Γ: At any stage $t = 0, \Delta, 2\Delta, ...,$

  - $i(0) = i$
  - Player $i(t) \in N$ makes an offer $x \in S$, where x_j is the share of player j and all other players accept or reject the offer in the ascending order of their index
    * If all $j \neq i(t)$ accept, then x is implemented
    * If j is the first who rejects, then j becomes $i(t+1)$

- Focus on the **stationary subgame perfect equilibria** where:

1. Each $i \in N$ makes the same proposal $x(i)$ whenever it is his turn to make a proposal.
2. Each i's acceptance decision in period t depends only on x_i that is offered to him in that period.

- Define a function v_i such that

$$u_i(v_i(x_i, t)) = u_i(x_i)\delta^t, \quad \text{for all } x_i \text{ for all } t$$

- Since u is continuous, v_i is continuous
- By the concavity of u_i, $u_i'(x_i)/u_i(x_i)$ is decreasing, strictly positive under all $x_i > 0$, and hence, for all t,

$$\frac{\partial}{\partial x_i} v_i(x_i, t) \in (0, 1)$$

- Equilibrium is a consistency condition - distinct players' proposals must be compatible with one another in a way that all proposals are accepted, given the consequence of the deviation
- No final period from which to start the recursion - equilibrium has to lean on a **fixed point argument**

Lemma 9 *(Krishna and Serrano 1996): Given* $\Delta > 0$, *there is a unique* $d(\Delta) > 0$ *and* $x(\Delta) \in \mathbb{R}^n_{++}$ *such that*

$$\begin{aligned} v_i(x_i(\Delta) + d(\Delta), \Delta) &= x_i(\Delta), \quad \text{for all } i, \\ \sum_{i=1}^{n} x_i(\Delta) + d(\Delta) &= 1. \end{aligned}$$

Proof. Denote $v_i^{-1}(x_i, \Delta) = y_i$ if $v_i(y_i, \Delta) = x_i$. Let

$$c_i(x_i) := v_i^{-1}(x_i, \Delta) - x_i, \quad \text{for all } x_i$$

$c_i(\cdot)$ is strictly positive for $x_i > 0$ and there is $c_i^* \in \mathbb{R}_{++} \cup \{\infty\}$ such that

$$\sup_{x_i \geq 0} c_i(x_i) = c_i^*$$

Since $\partial v_i^{-1}(x_i, \Delta)/\partial x_i = 1/(\partial v_i(x_i, \Delta)/\partial x_i) > 1$, the function $x_i \mapsto v_i^{-1}(x_i, \Delta) - x_i = c_i(x_i)$ is continuous and monotonically increasing. Hence also its inverse

$$x_i(a) := c_i^{-1}(y) = v_i(x_i(a) + a, \Delta), \quad \text{for all } a \in [0, c_i^*),$$

is continuous and monotonically increasing. Since $0 = x_i(0)$ and $\infty = x_i(c_i^*)$, there is, by the Intermediate Value Theorem, a unique $d > 0$ such that

$$\sum_{i=1}^{n} x_i(d) + d = 1.$$

For this d also

$$v_i(x_i(d) + d, \Delta) = x_i(d), \text{ for all } i.$$

■

- Our focus is on stationary SPE in which time does not matter: player i makes the same offer whenever it his turn to make one, and he accepts/rejects the same offer irrespective who makes the offer and when

- Player i's equilibrium offer $x(i) \in S$ maximizes his payoff with respect to this and the resource constraint

- Player i's offer $(x_1(i), ..., x_n(i))$ is accepted in a stationary SPE by j if

$$x_j(i) \geq v_j(x_j(j), \Delta), \quad \text{for all } j \neq i$$

Theorem 10 Γ *has a unique stationary SPE. In this stationary SPE, at any period* t, *(i) the offer made at* t *is accepted, (ii) the player* i *who makes the offer at* t *receives* $x_i(\Delta) + d(\Delta)$ *and a responder* j *receives* $x_j(\Delta)$, *as specified in the previous lemma.*

Proof. In a stationary SPE all proposals are accepted, otherwise the proposing player would speed up the process by making an offer that gives all the other players at least the discounted payoff they get from the offer that is eventually accepted, and little more to himself. At the optimum, all constraints bind:

$$x_j(i) = v_j(x_j(j), \Delta), \quad \text{for all } j \neq i,$$

and

$$\sum_{i=1}^{n} x_i(j) = 1, \quad \text{for all } j.$$

Since i's acceptance not dependent on the name of the proposer, there is x_i such that $x_i = x_i(j)$ for all $j \neq i$. Define d such that

$$d = 1 - \sum_{i=1}^{n} x_i.$$

Since

$$x_i(i) = 1 - \sum_{j \neq i} x_j = x_i + d,$$

it follows that

$$\begin{aligned} x_i &= v_i(x_i + d, \Delta), \quad \text{for all } i, \\ \sum_{j=1}^{n} x_j &= 1 - d. \end{aligned}$$

By the previous lemma, there is a unique x and d that meet these conditions.
■

3.2 Removing the stationarity restriction

- Stationarity needed for the result when $n \geq 3$
- With history dependent strategies, any allocation x can be supported in SPE
- Construct an SPE in the j punishment mode where i proposes 0 to j and $1/(n-1)$ to all other players
  - All players accept i's offer
  - If i proposes something else, all players reject
- If k is first to deviate, then ℓ who makes the offer next period takes the role of i and k takes the role of j in the next period and the play moves to k punishment mode
- No single player benefits from a one-time deviation
- Any outcome can now be supported in SPE by threatening to move the j -punishment mode if j is the first to deviate
- Stationary strategies simple, and can be motivated by complexity considerations (Chatterjee-Sabourian 2000)
- However, in many set ups, natural strategies *are* history dependent and stationarity is automatically violated
  - Punishment

- Cooperation

- Non-stationary strategies can, in general, be welfare improving
- Krishna-Serrano (1996)
  - Allow accepting players leave the game with their endowment
  - Solution must be multilaterally stable (Lensberg 1983): the equilibrium outcome for $1, ..., k$ remains unchanged when $k+1, ..., n$ leave with their equilibrium shares
  - -> Since stationary not needed for 2-player problems, by stability, it is not needed for 3-player problems, etc.
  - Rubinstein (1982): in the two-player case, stationarity not needed when only two players
  - Recall the definitions of $x(\Delta)$ and $d(\Delta)$ from the previous theorem

Theorem 11 *Let $n = 2$. Then Γ has a unique SPE. In this SPE, (i) all offers are accepted, (ii) player i who makes the offer receives $x_i(\Delta) + d(\Delta)$ and a responder j receives $x_j(\Delta)$.*

Proof. (sketch) The maximum share of the pie that 2 can achieve when making the offer is $\pi_2^0 = 1$.

The minimum share of the pie that 1 can guarantee himself when making the offer is $\phi_1^0 = 1 - v_2(1)$.

The maximum share of the pie that 2 can achieve when making the offer is $\pi_2^1 = 1 - v_1(1 - v_2(1))$.

The minimum share of the pie that 1 can guarantee himself when making the offer is $\phi_1^1 = 1 - v_2(1 - v_1(1 - v_2(1)))$

The maximum share of the pie that 2 can achieve when making the offer is $\pi_2^2 = 1 - v_1(1 - v_2(1 - v_1(1 - v_2(1))))$.

...

Then $\phi_1^{k+1} = 1 - v_2(1 - v_1(\phi_1^k))$ and $\pi_2^{k+1} = 1 - v_1(1 - v_2(\pi_2^k))$ and similarly $\phi_2^{k+1} = 1 - v_1(1 - v_2(\phi_2^k))$ and $\pi_1^{k+1} = 1 - v_2(1 - v_1(\pi_1^k))$, for all $k = 0, 1, ...$

By construction, $\pi_1^k \geq \pi_1^{k+1}$ and $\phi_2^{k+1} \geq \phi_2^k$, and $\phi_1^k \geq \phi_1^{k+1}$ and $\pi_2^{k+1} \geq \pi_2^k$ for all k. In a steady state, there are ϕ_i and π_i such that $\phi_i = 1 - v_j(1 - v_i(\phi_i))$ and $\pi_i = 1 - v_j(1 - v_i(\pi_i))$. By Lemma, there is a unique x^Δ and d^Δ such that $x_i^\Delta = v_i(x_i^\Delta + d^\Delta)$ and $x_1^\Delta + x_2^\Delta + d^\Delta = 1$. Thus $1 - x_i^\Delta = 1 - v_i(1 - v_j(1 - x_i^\Delta))$. This implies that $\pi_i = \phi_i = 1 - x_i^\Delta$. Since π_i is the maximum share of the pie that i can achieve and ϕ_i is the maximum share of the pie that i can guarantee himself, for $i = 1, 2$, x^Δ is the unique SPE of the game. ■

3.3 Relationship to the Nash solution

- Recall in the stationary SPE, i's offers $x_i(\Delta) + d(\Delta)$ to himself and $x_j(\Delta)$ to $j \neq i$ under lag $\Delta > 0$ between two periods

- Since
$$v_i(x_i(\Delta) + d(\Delta), \Delta) = x_i(\Delta)$$
for all Δ, and since $\lim_{\Delta \to 0} v_i(x_i, \Delta) = x_i$ for all $x_i \in [0, 1]$ it follows that $d(\Delta) \to 0$ as $\Delta \to 0$

- Recall that $u_i(x_i)\delta^\Delta = u_i(v_i(x_i, \Delta))$

- For any $\{i, j\} \subseteq N$,
$$\begin{aligned} u_i(x_i(\Delta) + d(\Delta))u_j(x_j(\Delta)) &= \delta^{-\Delta}u_i(x_i(\Delta))u_j(x_j(\Delta)) \\ &= u_i(x_i(\Delta))u_j(x_j(\Delta) + d(\Delta)) \end{aligned}$$

- Thus, in the problem where players i and j share the pie of size $X_{ij}(\Delta) = x_i(\Delta) + x_i(\Delta) + d(\Delta)$, their stationary SPE proposals lie in the same hyperbola (of dimension 2)

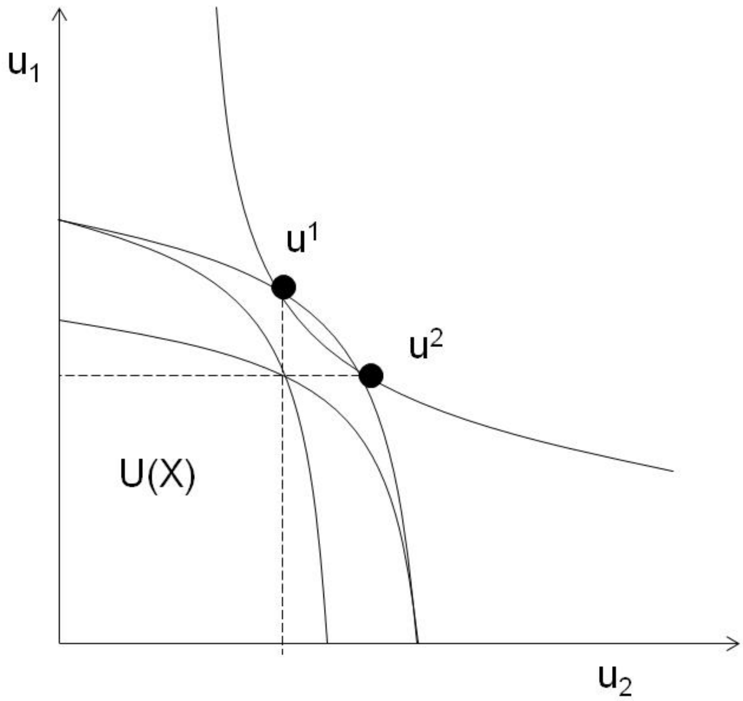
- Since $d(\Delta)$ converges to 0 and $X_{ij}(\Delta)$ converges to some bounded number $\bar{X}_{ij}$ as Δ tends to 0, the stationary SPE proposals converge to the Nash bargaining solution of the two-player problem of sharing pie of size $\bar{X}_{ij}$ (Binmore-Rubinstein-Wolinsky 1986)

- Recall that if an outcome constitutes a Nash bargaining solution in all its two player projections, then it constitutes the Nash solution (Thomson and Lensberg, 1991)

- Denote by $U(X) = \{u(x) : x \in X\}$ the utility set spanned by outcomes in X

- We have proved:

Theorem 12 *The payoff profile resulting from the stationary SPE of* Γ *converges to the Nash bargaining solution* $f^{Nash}(U(X))$ *as* $\Delta \to 0$



3.4 Patience means bargaining power

- Let now the players' discount factors be tailored to each agent, to reflect their relative **(im)patience:** if $\delta_i > \delta_j$, then i is more patient than j
- For example, if $\delta_i = e^{-r}$, then i discounts future by the rate r_i
- The players have potentially different discount factors $\delta_1, ..., \delta_n$
- Note that nothing in the previous analysis concerning the existence and uniqueness of the stationary SPE is changed as different discount factors
- However, the convergence result requires a modification
- Let $\alpha_i = -1/\log \delta_i$
- Then, for any i,

$$\delta_i^{-\alpha_i} = e$$

- Hence, for any $\{i,j\} \subseteq N$,

$$\begin{aligned} u_i(x_i(\Delta)+d(\Delta))^{\alpha_i} u_j(x_j(\Delta))^{\alpha_j} &= \delta_i^{-\Delta\alpha_i} u_i(x_i(\Delta))^{\alpha_i} u_j(x_j(\Delta))^{\alpha_j} \\ &= \delta_j^{-\Delta\alpha_j} u_i(x_i(\Delta))^{\alpha_i} u_j(x_j(\Delta))^{\alpha_j} \\ &= u_i(x_i(\Delta))^{\alpha_i} u_j(x_j(\Delta)+d(\Delta))^{\alpha_j} \end{aligned}$$

- Since $d(\Delta) \to 0$ as $\Delta \to 0$, it follows that in the limit, the outcome is the Nash bargaining solution all the two player projections

Theorem 13 *The payoff profile resulting from the stationary SPE of* Γ *converges to the Nash bargaining solution* $f^{\alpha}(U(X))$ *as* $\Delta \to 0$, *where* $\alpha_i = -1/\log \delta_i$ *for all* i

Increasing δ_i increases the weight of player i and, gives him more **bargaining power**

Example 14 *Let* $n = 2$, *and linear utilities* $u_1(x_1) = x_1$, $u_2(x_2) = x_2$. *The discount factors of the two players are* $\delta_1 = e^{-r_1}$ *and* $\delta_2 = e^{-r_2}$ *for some "discount rates"* r_1 *and* r_2. *In the unique SPE,* i *offers* $x_i(\Delta) + d(\Delta)$ *and* $x_j(\Delta)$ *to* $j \neq i$ *such that*

$$(x_1(\Delta)+d(\Delta))\delta_1^{\Delta} = x_1(\Delta) \text{ and } (x_2(\Delta)+d(\Delta))\delta_2^{\Delta} = x_2(\Delta).$$

Since also $x_1(\Delta) + x_2(\Delta) + d(\Delta) = 1$, *we can solve for* $x_1(\Delta)$ *and* $x_2(\Delta)$:

$$x_1(\Delta) = \frac{\delta_1^{\Delta} - \delta_1^{\Delta}\delta_2^{\Delta}}{1-\delta_1^{\Delta}\delta_2^{\Delta}} \text{ and } x_2(\Delta) = \frac{\delta_2^{\Delta} - \delta_1^{\Delta}\delta_2^{\Delta}}{1-\delta_1^{\Delta}\delta_2^{\Delta}}$$

Finally,

$$\begin{aligned} x_1(\Delta) &\to \frac{\log\delta_2}{\log\delta_1+\log\delta_2} = \frac{r_2}{r_1+r_2} \\ x_2(\Delta) &\to \frac{\log\delta_1}{\log\delta_1+\log\delta_2} = \frac{r_1}{r_1+r_2} \end{aligned}$$

Thus, increasing i*'s personal discount rate* r_i *decreases his payoff but increasing the opponent's* j *discount rate* r_j *increases* i*'s payoff.*

3.5 General utility set

- Let bargaining take place in a compact, convex, comprehensive utility set $U \subseteq \mathbb{R}^n_{++}$
- Discount factors $\delta_1, ..., \delta_n$

Theorem 15 Γ *has a stationary SPE. Any SPE is characterized by the following properties: (i) all offers are accepted, (ii) player i who makes the offer receives payoff* u_i^i *and a responder* $j \neq i$ *receives* u_j^i *such that* $u_i^j = \delta_i^\Delta u_i^i$, *for all i*

- Kultti-Vartiainen (2010): all stationary SPE converge to the (asymmetric) Nash solution where $\alpha_i = -1/\log \delta_i$ if the surface of the Pareto frontier is **differentiable**

Theorem 16 *Let the Pareto frontier of U be differentiable. For any* $\varepsilon > 0$ *there is* $\Delta^\varepsilon > 0$ *such that for all* $\Delta < \Delta^\varepsilon$, *any stationary SPE of* Γ *is in the* $\varepsilon-$*neighborhood of*

$$f^\alpha(U) = \arg\max_{u \in U} \prod_i u_i^{\alpha_i}$$

where $\alpha_i = -1/\log \delta_i$ *for all i.*

Proof. (sketch) Let $u^1, ..., u^n$ be the equilibrium offers. By the equilibrium characterization, for any j

$$\begin{aligned}\prod_i (u_i^j)^{\alpha_i} &= \prod_{i \neq j} \delta_i^{\Delta \alpha_i} \prod_i (u_i^i)^{\alpha_i} \\ &= e^{-\Delta(n-1)} \prod_i (u_i^i)^{\alpha_i}.\end{aligned}$$

Thus all equilibrium offers lie in the same $\alpha-$weighted hyperbola. Since the distance of u^i and u^j shrinks when Δ tends to 0, and they are linearly independent in the limit, all u^i must converge to the point in which a U is separable by a hyperplane from the hyperbola. ■

- Herings and Predtetchinski (2010) generalize this to the general class of problems where the proposing player is chosen by using a Markovian recognition policy

- Smoothness of the Pareto frontier is critical:

Example 17 *Let* $U = \{u \in \mathbb{R}_+^3 : u_1 + \max\{u_2, u_3\} \leq 1\}$. *Stationary SPE offers* $u^1, u^2, u^3 \in U$ *satisfy*

$$\begin{aligned}\delta^\Delta u_1^1 &= u_1^3 = u_1^2, \\ \delta^\Delta u_2^2 &= u_2^1 = u_2^3, \\ \delta^\Delta u_3^3 &= u_3^2 = u_3^1.\end{aligned}$$

Since players do not waste their own consumption possibilities when making offers

$$\begin{aligned}u_1^1 + u_2^1 &= u_1^1 + u_3^1 = 1 \\ u_1^2 + u_2^2 &= 1 \\ u_1^3 + u_3^3 &= 1\end{aligned}$$

Solving the equilibrium offers for players 1, 2 and 3,

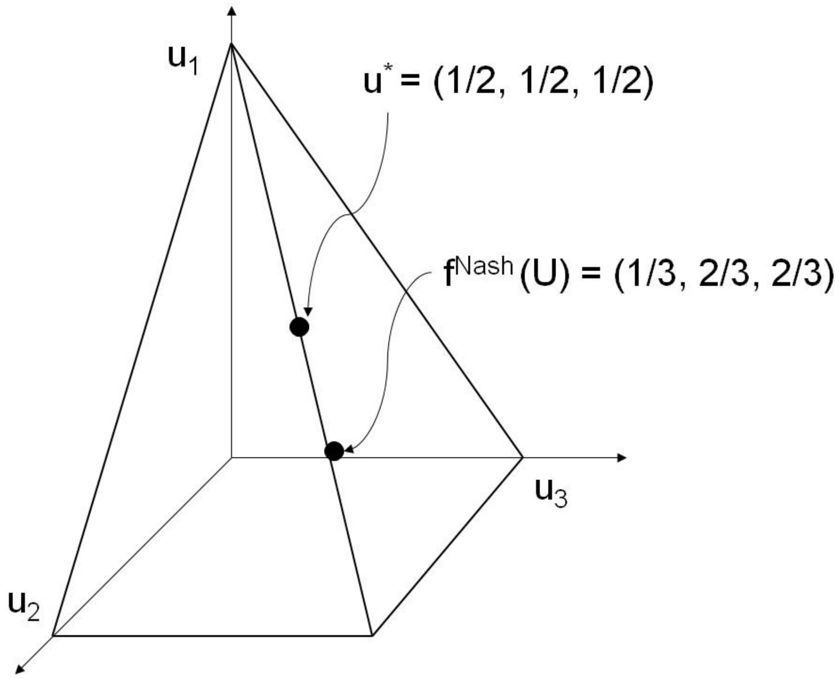
$$
\begin{aligned}
u^1 &= \left(\frac{1}{1+\delta^\Delta}, \frac{\delta^\Delta}{1+\delta^\Delta}, \frac{\delta^\Delta}{1+\delta^\Delta}\right), \\
u^2 &= \left(\frac{\delta^\Delta}{1+\delta^\Delta}, \frac{1}{1+\delta^\Delta}, \frac{\delta^\Delta}{1+\delta^\Delta}\right), \\
u^3 &= \left(\frac{\delta^\Delta}{1+\delta^\Delta}, \frac{\delta^\Delta}{1+\delta^\Delta}, \frac{1}{1+\delta^\Delta}\right).
\end{aligned}
$$

As Δ *tends to* 0*, equilibrium offers converge to*

$$u^* = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

However, the Nash solution is

$$f^{Nash}(U) = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right).$$



3.6 Preference foundations

- The fundamental axiom of economics: only preferences can be observed - utility functions only **represent** preferences
- Hence to relate bargaining outcomes in any meaningful way to the empirical data, the solution has to be defined in terms of the preferences rather than in terms of utilities that represent them

- But what kind of preferences do the intertemporal utilities represent? Does the Nash solution have an interpretation in terms of them?

- A crucial assumption in our previous analysis was that the utility function is *concave*

- This guarantees that the derivative of the v_i function is between 0 and 1 which is needed for the fixed point result

- Let $n = 2$

- Let pie be divided at any point of time $T = \mathbb{R}_+$ and denote by $X = \{(x_1, x_2) \in \mathbb{R}_+ : x_1 + x_2 \leq 1\}$ the possible allocations of the pie

- Let (complete, transitive) preferences over $X \times T$ satisfy, for all $x, y \in S$, for all $i \in N$, and for all $s, t \in T$, satisfy (Fishburn and Rubinstein, 1982):

A1. $(x, t) \succeq_i (\mathbf{0}, 0)$

A2. $(x, t) \succeq_i (y, t)$ if and only if $x_i \geq y_i$

A3. If $s > t$, then $(x, t) \succeq_i (x, s)$, with strict preference if $x_i > 0$

A4. If $(x^k, t^k) \succeq_i (y^k, s^k)$ for all $k = 1, ...,$ with limits $(x^k, t^k) \to (x, t)$ and $(y^k, s^k) \to (y, s)$, then $(x, t) \succeq_i (y, s)$

A5. $(x, t) \succeq_i (y, t + \Delta)$ if and only if $(x, 0) \succeq_i (y, \Delta)$, for any $t \in T$, for any $\Delta \geq 0$

- Under A1-A5, there is a function v_i such that $(y, 0) \sim_i (x, t)$ if $v_i(x_i, t) = y_i$, for all x, y

- For such function, $v_i(x_i, t) = y_i$ and $v_i(x_i, s) = z_i$ imply $y_i > z_i$ whenever $s < t$

- Fishburn and Rubinstein (1984) show that time preferences can be represented by a discounting model

Lemma 18 *If time preferences satisfy A1-A5, then, for any $\delta \in (0, 1)$ there is a utility function u_i such that $(y, 0) \sim_i (x, t)$ if and only if $u_i(y_i) = u_i(x_i)\delta^t$*

- However, A1-A5 imply only little restrictions on the shape of u_i

- In particular, u_i need *not* be concave, as was critically assumed in the previous models

- The problem: concavity of u_i does not have natural meaning in terms time-preferences - it cannot be imposed as an axiom

- Hence the convergence results need not hold - in fact the Nash bargaining solution may no longer be well defined (as the induced utility set need not be convex)
- Our aim: to derive a Nash-like solution by using the time-preferences alone, and show the convergence
- Fix δ and let u_1 and u_2 be the representations of the time preferences of 1 and 2
- Then the Nash bargaining solution, if it exists, is defined by $x^* = (x_1^*, x_2^*)$ such that $u_1(x_1^*)u_2(x_2^*) \geq u_1(x_1)u_2(x_2)$ for all $x \in X$
- Alternative interpretation of the Nash solution (cf. Rubinstein-Safra-Thomson 1992):

Theorem 19 *Outcome x^* is the Nash bargaining solution if and only $v_i(x_i, t) > x_i^*$ implies $v_j(x_j^*, t) \geq x_j$ for any x and for any $t > 0$*

- This interpretation reflects *justified envy*: a demand to get more by threatening to delay consumption is not justified if the other player is willing to delay consumption equally if he does not have to give more
- Implication: the Nash bargaining solution that is derived from time-preferences is **not** dependent on the chosen discount factor δ - all representations imply the same Nash bargaining outcome
- Recall that $u_i(x_i)\delta^t \geq u(y_i)$ if and only if $v_i(x_i, t) \geq y_i$

Proof. If: If there is x such that $u_1(x_1)u_2(x_2) > u_1(x_1^*)u_2(x_2^*)$, then there is t such that, for some i,

$$\frac{u_j(x_j)}{u_j(x_j^*)} > \delta^t > \frac{u_i(x_i^*)}{u_i(x_i)}.$$

For such t, $\delta^t u_i(x_i) > u_i(x_i^*)$ and $\delta^t u_j(x_j^*) < u_j(x_j)$.

Only if: Let x^* maximize the Nash product. For any $t > 0$, if $\delta^t u_i(x_i) > u_i(x_i^*)$, then, since $u_i(x_i^*)u_j(x_j^*) \geq u_i(x_i)u_j(x_j)$, also $\delta^t u_j(x_j^*) > u_j(x_j)$. ■

- Questions:
  1. Under which conditions is the Nash bargaining solution well defined (unique)?
  2. Under which conditions does the noncooperative bargaining game yield, in the limit, the Nash bargaining outcome?
- Recall that $\partial v_i(x_i, t)/\partial x_i \in (0, 1)$ was the critical condition for the utility based model, and implied by the concavity of u

- Our approach: assume this

A6. $x_i - v_i(x_i, t)$ is strictly increasing in x_i for all t

- Let x^* be the convergence point such that $x(\Delta) \to x^*$ as $\Delta \to 0$
- Recall that $v_i(x_i^i, t) = x_i(\Delta)$ implies $v_j(x_j^j, t) = x_j(\Delta)$ if and only if $x_i^i = x_i(\Delta) + d(\Delta)$ and $x_j^j = x_j(\Delta) + d(\Delta)$
- Let for any $\lambda \in [0,1]$, $x^\lambda = \lambda x^i + (1-\lambda)x^j$

Lemma 20 *For any y and for any $t > 0$ it holds true that $v_i(y_i, t) > x_i^\lambda$ implies $v_j(x_j^\lambda, t) \geq y_j$*

Proof. $v_i(y_i, t) > x_i^\lambda$ implies, since v_i is increasing, that $y_i > x_i^i$. Thus $x_j^j = 1 - v_i(x_i^i, t) > 1 - v_i(y_i, \Delta)$. By A6,

$$\begin{aligned} y_i - v_i(y_i, t) &> x_i^i - v_i(x_i^i, t) \\ &= x_j^j - v_j(x_j^j, t) \\ &> 1 - v_i(y_i, t) - v_j(1 - v_i(y_i, t), t). \end{aligned}$$

Hence,

$$v_j(1 - v_i(y_i, t), t) > 1 - y_i.$$

Since $v_i(y_i, t) > x_i^\lambda$, we have, by $x_i^\lambda + x_j^\lambda = y_i + y_j = 1$ and since v_j is increasing,

$$\begin{aligned} v_j(x_j^\lambda, t) &> v_j(1 - v_i(y_i, t), t) \\ &> y_j, \end{aligned}$$

as desired. ■

Lemma 21 *For any t, if $v_i(y_i, t) > x_i$ implies $v_j(x_j, t) \geq y_j$ for all y, then there is $\lambda \in [0,1]$ such that $x = \lambda x^i + (1-\lambda)x^j$.*

Proof. It suffices to show $x_i \leq x_i^i$ for all i. Suppose, on the contrary, that $x_i > x_i^i$ for some i. Since v_i is increasing, $x_j^j = 1 - v_i(x_i^i, t) > 1 - v_i(x_i, t)$. Thus, by A6,

$$\begin{aligned} x_i - v_i(x_i, t) &> x_i^i - v_i(x_i^i, t) \\ &= x_j^j - v_j(x_j^j, t) \\ &= 1 - v_i(x_i^i, t) - v_j(1 - v_i(x_i^i, t), t) \\ &> 1 - v_i(x_i, t) - v_j(1 - v_i(x_i, t), t). \end{aligned}$$

Hence,

$$v_j(1 - v_i(x_i, t), t) > 1 - x_i. \tag{1}$$

Since v_i and v_j are continuously increasing, we can choose y such that $y_i > v_i(x_i, t)$ and such that

$$\begin{aligned} v_j(1 - v_i(x_i, t), t) &> v_j(1 - y_i, t) \\ &> 1 - x_i. \end{aligned} \tag{2}$$

Thus $v_j(y_j, t) > x_j$, violating the statement of the lemma. ■

- Thus, under A1-A6, the Nash solution does exist and is equivalent with there being a maximizer of the product $u_1(x_1)u_2(x_2)$, where (u_1, u_2) is the representation of the time-preferences
- Since functions u_1 and u_2 are continuous, and P is compact, it follows immediately that a Nash product maximizer exists

Theorem 22 *Under A1-A6, the Nash bargaining solution exists, is unique, and coincides with the limit of the SPE outcome of the bargaining game* Γ *as* $\Delta \to 0$

- Convergence without additional assumptions concerning the utility representation - nothing is assumed about the players' risk preferences

3.7 Implementing the other solutions

- But the convergence result approximate: only holds when $\delta \to 1$ (or the time span between offers vanishes)
- Exact implementation of the Nash solution: Howard (1992)
- Implementing the other solutions
  - Kalai-Smorodinsky: Moulin (1984)
  - Shapley: Gul (1989), Perez-Castrillo-Wettstein (2001)
  - The Core: Serrano-Vohra (1997), Lagunoff (1994)
  - Bargaining set: Einy-Wettstein (1999)
  - Nucleoulus: Serrano (1993)
  - etc...
- Do strategic considerations put any restrictions on what can be implemented?

4 Implementation foundations

- The most natural notion of strategic interaction: the **Nash equilibrium**
- Which solutions can be implemented in Nash equilibrium?
- Implementation theory: studies general conditions under which an outcome functions - e.g. a bargaining solution - can be implemented non-cooperatively

4.1 Nash implementation - impossibility

- Let $n = 2$
- There is a pie of size 1, to be shared among the two players with $x \in [0, 1]$ denoting a typical share of player 1, and $1 - x$ the share of player 2
- U comprises all continuous and strictly increasing vNM utility functions $u_i : [0, 1] \to \mathbb{R}$ normalized such that $u_i(0) = 0$ for all $u_i \in U$
- Denote the set of lotteries on $[0, 1]$ by Δ
- Expected payoff from a lottery $p \in \Delta$

$$\begin{aligned} u_1(p) &= \int_{[0,1]} p(x)u_1(x)dx \\ u_2(p) &= \int_{[0,1]} p(x)u_2(1-x)dx \end{aligned}$$

- Bargaining solution (BS) $f : U^2 \to [0, 1]$ specifies an outcome for each pair of utility functions where $f(u)$ is the share of player 1 and $1 - f(u)$ the share of 2 under profile $u = (u_1, u_2)$
- A **game form** $\Gamma = (M_1 \times M_2, g)$ consists of **strategy sets** M_1 and M_2, and an **outcome function** $g : M_1 \times M_2 \to \Delta$
- Given $u = (u_1, u_2)$, the pair (Γ, u) constitutes a **normal form game** with the set of Nash equilibria $NE(\Gamma, u)$
- Mechanism Γ **Nash implements** bargaining solution f if, for all $u \in U^2$,

$$g(NE(\Gamma, u)) = f(u)$$

- Denote the **lower contour set** of i at $q \in \Delta$ under $u \in U^2$ by

$$L_i(q, u) = \{p \in \Delta : u_i(q) \geq u_i(p)\}$$

- BS f is **Maskin monotonic** if for all pairs u, u', if $x \in f(u')$ and $L_i(x, u') \subseteq L_i(x, u)$, for $i = 1, 2$, then $x \in f(u)$
- Maskin (1977): f Nash implementable only if it is Maskin monotonic
- Which bargaining solutions are Maskin monotonic?
- Maskin monotonicity implies that there has to be a preference reversal from u to u' if $x \in f(u) \backslash f(u')$
- BS f is **scale invariant** if $f(u) = f(\alpha u)$, for all $\alpha \in \mathbb{R}^2_{++}$, for all $u \in U^2$

Lemma 23 *Any Maskin monotonic BS f is scale invariant*

- Thus BS f Nash implementable only if it scale invariant
- BS f is *symmetric* if $u_1(f(u_1, u_2)) = u_2(1 - f(u_1, u_2))$ whenever $(w_1, w_2) = u(x)$ for some x implies that there is x' such that $(w_2, w_1) = u(x')$
- Note that, as we require that no pie is wasted, our BS f is automatically Pareto optimal: $f_1(u) + f_2(u) = 1$ for all $u \in U^2$
- Nash bargaining solution

$$f^{Nash}(u) = \arg\max_{[0,1]} u_1(x) u_1(1 - x)$$

Lemma 24 *Let f be a (Pareto optimal and) symmetric BS. If f can be Nash implemented, then $f^{Nash}(u) = f(u)$ for all $u \in U^2$.*

- Proof: Given that f must be scale invariant, replace IIA with Maskin monotonicity in the proof of Nash's theorem (see Vartiainen 2007 for details)
- Can the Nash bargaining solution be Nash implemented?

Example 25 *Let $u_1(x) = x$ and $u_2(1 - x) = 1 - x$. Then $f^{Nash}(u) = 1/2$. Perform a Maskin monotonic transformation of 1's utility by choosing $u_1^\varepsilon(x) = x$ for $x \in [0, 1/3]$, and $u_1^\varepsilon(x) = 1/3 + \varepsilon(x - 1/3)$ for $x \in (1/3, 1]$. For small enough $\varepsilon > 0$, $f^{Nash}(u_1^\varepsilon, u_2) = 1/3$*

Lemma 26 *The Nash bargaining solution f^{Nash} cannot be Nash implemented*

Theorem 27 *No Pareto optimal and symmetric BS f can be Nash implemented*

4.2 Virtual implementation - possibility

- Thus basically nothing relevant can be implemented by using the most appealing solution concept
- How far must one go in extending the mechanism, to implement something relevant?
- Our aim: to construct a mechanism (cf. Moore-Repullo 1988; Dutta-Sen1988)) that "almost" Nash implements any BS
- Let $\Gamma^* = (M^*, g^*)$ satisfy $M_1^* = M_2^* = U^2 \times \Delta \times \mathbb{N}$ with typical elements (u^1, q^1, k^1) and (u^2, q^2, k^2), respectively, and

  1. $g^*(m_1, m_2) = f(u)$ if $u^1 = u^2 = u$
  2. $g^*(m_1, m_2) = q^i$ if $q^i \in L_i(f(u^j), u^j)$, $u^1 \neq u^2$, and $k^i > k^j$
  3. $g^*(m_1, m_2) = 1$ if $k^1 > k^2 > 0$ and $g^*(m_1, m_2) = 0$ if $k^2 > k^1 > 0$
  4. $g^*(m_1, m_2) = (0, 0)$, in all other cases

- We claim that Γ^* Nash implements any Maskin monotonic BS
- Let $u = (u_1, u_2)$ be the true utility profile
  - It cannot be the case that (1) holds under $u^1 = u^2 = u' \neq u$ and $k^1 = k^2 = 0$ since, by (2), there would be $q^i \in L_i(f(u'), u') \backslash L_i(f(u), u)$ such that $k^i > 0$ that would constitute a profitable deviation for i
  - It cannot be the case that (2) holds since, by (3), $k^j > k^i > 0$ would constitute a profitable deviation for j
  - It cannot be the case that (3) holds since one of the players would have a profitable deviation
  - It cannot be the case that (4) holds since, as a strictly individually rational BS chooses $f(u) \in (0, 1)$ and, hence, by (1) i would have a profitable deviation $u^i = u^j$
- Thus the only possible Nash equilibrium is $u^1 = u^2 = u$ which implements $f(u)$

Lemma 28 *Any strictly individually rational BS f can be Nash implemented if it is Maskin monotonic*

- Take any strictly individually rational f and let f^ε satisfy

$$\begin{aligned} f^\varepsilon(u) \;=\; & \text{implement } f(u) \text{ with probability } 1-\varepsilon \\ & \text{and the uniform lottery over } [0,1] \text{ with prob. } \varepsilon \end{aligned}$$

- The expected payoff to i

$$\begin{aligned} u_1(f^\varepsilon(u)) &= (1-\varepsilon)u_1(f(u)) + \varepsilon \int_{[0,1]} u_1(x)dx \\ u_2(f^\varepsilon(u)) &= (1-\varepsilon)u_2(f(u)) + \varepsilon \int_{[0,1]} u_2(1-x)dx \end{aligned}$$

- We argue that f^ε is Maskin monotonic for any $\varepsilon > 0$
- Since any implementable BS is scale invariant, we may normalize the situation such that $u_i(0) = 0$ and $u_i(1) = 1$
- Take $u_1 \neq u_1'$ and find an open interval $(a, b) \subseteq [0, 1]$ such that $u_1(x) > u_1'(x)$ for all $x \in (a, b)$, or $u_1(x) < u_1'(x)$ for all $x \in (a, b)$
- Assume, for simplicity, that $a = 0$ and $b = 1$ (otherwise, modify f^ε only under (a, b) and not under $(0, 1)$)
- We shall show that $L_1(f^\varepsilon(u), u)\backslash L_1(f^\varepsilon(u), u_1', u_2)$ is not empty, implying that f^ε automatically satisfies Maskin monotonicity
- There are two cases to consider

Case $u_1 > u_1'$: Find $\xi \in (0, 1)$ such that

$$\int_{[0,1]} u_1(x)dx = u_1(\xi)$$

- Construct a lottery

$$q_\xi = \text{implement } f(u) \text{ with prob. } 1-\varepsilon \text{ and } \xi \text{ with prob. } \varepsilon$$

- By construction $q_\xi \in L_1(f^\varepsilon(u), u)$ and $q_\xi \notin L_1(f^\varepsilon(u), u_1', u_2)$

Case $u_1 < u_1'$: Find $\pi \in (0, 1)$ such that

$$\int_{[0,1]} u_1(x)dx = \pi$$

- Construct a lottery

$$q_\pi = \text{implement } f(u) \text{ with prob. } 1-\varepsilon \text{ and } 1 \text{ with prob. } \pi\varepsilon\}$$

- By construction $q_\pi \in L_1(f^\varepsilon(u), u)$ and $q_\pi \notin L_1(f^\varepsilon(u), u_1', u_2)$
- Thus in either case, f^ε satisfies Maskin monotonicity

- Since $\varepsilon > 0$ is arbitrarily small, any strictly individually rational BS can be virtually implemented - with arbitrary precision
- Problems:
  - Optimally small deviation from exact implementation?
  - Mechanism uses an integer construction, and is hence "unreasonable"

4.3 Exact implementation with a reasonable mechanism

- Miyagawa (2002): simple mechanism that implements a large class of solutions
- Define a solution f^W by

$$f^W(u) = \arg\max_{x \in X} W(u_1(x), u_2(x))$$

  where $W : [0,1]^2 \to \mathbb{R}$ is continuous, monotonic and quasi-concave

- The set of functions W satisfying these conditions is denoted by $\mathcal{W}$
- The function W may be interpreted as the objective function of the arbitrator
- E.g. Nash, Kalai-Smorodinsky
- Mechanism Γ^W

1. In stage 1, agent 1 announces a vector $p \in [0,1]^2$ such that $p_1 + p_2 \geq 1$
2. Having observed p, agent 2 makes a counter-proposal $p' \in [0,1]^2$ such that $W(p_1, p_2) = W(p_1', p_2')$
3. The agent who moves in the next stage, i, is then determined based on whether 2 agrees ($p = p'$) or disagrees ($p \neq p'$)
   - If 2 agrees, then he moves next ($i = 2$)
   - Otherwise, 1 moves next ($i = 1$)
4. Agent i then chooses either "quit" or "stay," and then announces a lottery a_i
   - If he chooses to "quit," then the game ends with $p' a_i$ as the outcome
   - If agent i chooses to "stay," then agent $j \neq i$ either "accepts" a_i, in which case the outcome is a_i, or he selects another lottery a_j' in which case the outcome is $p_j' a_j'$

Theorem 29 *For each $W \in \mathcal{W}$, game form Γ^W implements solution f^W in subgame-perfect equilibrium.*

- Thus any reasonable solution can be implemented
- The true test is not whether a solution is consistent with rational play, but whether its implementation can be justified with a intuitively appealing (= simple, used in the real world,...) mechanism
- But then the question of finding a good solution is changed to one finding a good mechanism - do the problems really differ

5 Concluding points

- Bargaining is a fundamental form of economic activity, and hence it is central to economic theory to understand how it works
- The problem with bargaining is that players' behavior is fundamentally interrelated, there is a feedback loop from one's actions to one's own behavior which makes the problem open ended and "hard"
- The solution has to come inside the model, and be based on consistency properties of the problem or a fixed point argument
- The former approach called as axiomatic and the latter strategic
- The Nash bargaining solution at the epicenter of both modelling traditions
- The Nash program seeks to motivate an axiomatic solution on strategic grounds
- The problem: without any restrictions of the form the game can be, **any** solution can implemented in a strategic equilibrium
- Thus the strategic dimension as such does not restrict feasible solutions at all
- Thus to obtain any bite to the strategic approach, one has to assume that some game forms are not appropriate - or are unnatural
- But what game forms are unnatural?

Conclusion: a researcher cannot externalize the responsibility of good modelling to an outside principle

References

[1] Aumann, R. J. (1997), On the State of the Art in Game Theory, an interview, in Understanding Strategic Interaction, Essays in Honor of Reinhard Selten, W. Albers et al. (ed.), Springer, Berlin, 8-34

[2] Binmore K., Rubinstein A, and A.Wolinsky (1986), The Nash Bargaining Solution in Economic Modeling, The Rand Journal of Economics 17, 176-188

[3] Chatterjee, K. and H. Sabourian (2000), Multiperson Bargaining and Strategic Complexity, Econometrica 68, 1491-1510

[4] Fishburn P. and A. Rubinstein (1982), Time Preference, International Economic Review 23, 677-694.

[5] Gul, F. (1989), Bargaining Foundations of the Shapley Value, Econometrica 57, 81-95.

[6] Herings J-J and A. Predtetchinski (2010) On the Asymptotic Uniqueness of Bargaining Equilibria, Maastricht University RM/10/010

[7] Kalai, E. and M. Smorodinsky (1975), Other Solutions to Nash's Bargaining Problem, Econometrica 43, 513-518

[8] Krishna, V. and R. Serrano (1996), Multilateral Bargaining, Review of Economic Studies 63, 61-80

[9] Kultti, K. and H. Vartiainen (2007), Von Neumann-Morgenstern stable sets, discounting, and Nash bargaining, Journal of Economic Theory 137, 721-728

[10] Kultti, K. and H. Vartiainen (2010), Multilateral non-cooperative bargaining in a general utility space, International Journal of Game Theory 39, 677-689

[11] Lagunoff, R. (1994), A Simple Non-Cooperative Core Story, Games and Economic Behavior 7, 54-61

[12] Lensberg, T. (1988), Stability and the Nash Solution, Journal of Economic Theory 45, 330-341

[13] Maskin, E. (1999), Nash equilibrium and welfare optimality, Review of Economic Studies 66:23-38.

[14] Nash, J. F. (1950), The Bargaining Problem, Econometrica 18, 155-162

[15] Nash, J. F. (1953), Two Person Cooperative Games, Econometrica 21, 128-140

[16] Moulin, H. (1984), Implementing the KalaiSmorodinsky Bargaining Solution, Journal of Economic Theory 33, 32-45.

[17] Pérez-Castrillo, D. and D. Wettstein (2001), Bidding for the Surplus: a Non-Cooperative Approach to the Shapley Value, Journal of Economic Theory 100, 274-294

[18] Rubinstein, A. (1982), Perfect Equilibrium in a Bargaining Model, Econometrica 50, 97-109

[19] Rubinstein, A. , Safra Z. and W.Thomson (1992), On the Interpretation of the Nash Bargaining Solution, Econometrica 60, 1171-1186

[20] Serrano, R. (2004), Fifty Years of the Nash Program, 1953-2003, Lecture delivered at the XXVIII Spanish Symposium on Economic Analysis

[21] Serrano, R. (2005), Nash Program, The New Palgrave Dictionary of Economics, 2nd edition, McMillan

[22] Thomson,W. and T. Lensberg (1989), Axiomatic Theory of Bargaining with a Variable Number of Agents, Cambridge University Press