Aggregating and Updating Information

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April 25, 2013

Introduction

- what kind of beliefs / opinions / preferences are commonly held in a society?
- how influential is an individual in a social network?
- often an *aggregate* needed: easier to handle / understand than a set of observations
- or, sometimes, an aggregate is all there is
- what information is transmitted / scrambled by an aggregate?
- in this paper: axiomatic analysis of some aggregation rules
- Nash rule axiomatized

This talk:

1. the formal model and possible interpretations of it
2. axioms
3. results
4. discussion and a (very incomplete) literature review

1. Setup

- finite set of agents N, finite set of states S
- each agent has a measure m_i on S:

$$m_i(E) = \sum_{s \in E} m_i(s) \geq 0, \forall E \subset S$$

- $P = (N, S, m)$ is an *aggregation problem*
- an *aggregation rule* f is a function such that $f(P)$ is a measure on S for each problem $P = (N, S, m)$
- a *probability* aggregation problem: m_i, $f(P)$ are *probability* measures

- an aggregation problem $P = (N, S, m)$ can be viewed a $|S| \times |N|$ matrix P, and $f(P)$ as a column vector
- an example with $N = \{1, 2, 3, 4\}, S = \{s, t, w\}$:

$$P \longmapsto f(P), \quad \text{or}$$

$$\begin{bmatrix} m_1(s) & m_2(s) & m_3(s) & m_4(s) \\ m_1(t) & m_2(t) & m_3(t) & m_4(t) \\ m_1(w) & m_2(w) & m_3(w) & m_4(w) \end{bmatrix} \longmapsto \begin{bmatrix} f(P)(s) \\ f(P)(t) \\ f(P)(w) \end{bmatrix}$$

1.1. Interpretations

1. states s are investment projects; $m_i(s)$ is i's forecast of the gross return per dollar on s
2. states s are wine growing areas; $m_i(s)$ is i's forecast of rainfall in August - September in s
3. states s are sites / locations; $m_i(s)$ is number of times i visits s; $f(P)(s)$ is the popularity of the site s (P is bipartite graph)
4. $S = N$; $m_i(s)$ is the strength of the (directed) link between i and s (from s to i) in network P; $f(P)(s)$ is the importance of s

1.2. Aggregation rules

1. *Average rule*: $f^A(P)(s) = \left[\sum_i m_i(s)\right]/n$
2. *Nash rule*: $f^G(P)(s) = \sqrt[n]{\prod_{i\in N} m_i(s)}$
3. *Median rule*: $f^M(P)(s)$ is the median of $\{m_1(s), \ldots, m_n(s)\}$

   Norm rules:
4. (Euclidean): $f^{EN}(P)(s) = n^{-1/2}\sqrt{m_1(s)^2 + \cdots + m_n(s)^2}$
5. (Sup): $f^{SN}(P)(s) = \sup\{|m_1(s)|, \ldots, |m_n(s)|\}$

To each rule f there is a probability aggregation rule:

$f^{\times}(P)(s) = f(P)(s)/f(P)(S)$

2. Axioms

Axiom 1.

(Anonymity): If the only difference between problems P and P' is that $m_i = m'_k$ and $m_k = m'_i$, then $f(P) = f(P')$.

$$P' = \begin{bmatrix} m_4(s) & m_2(s) & m_3(s) & m_1(s) \\ m_4(t) & m_2(t) & m_3(t) & m_1(t) \\ m_4(w) & m_2(w) & m_3(w) & m_1(w) \end{bmatrix} \longmapsto \begin{bmatrix} f(P)(s) \\ f(P)(t) \\ f(P)(w) \end{bmatrix} = f(P')$$

Axiom 2.

(Neutrality): If the only difference between P and P' is that $m(s) = m'(t)$ and $m(t) = m'(s)$, then $f(P)(s) = f(P')(t)$ and $f(P)(t) = f(P')(s)$ and $f(P)(x) = f(P')(x)$ for other states x.

$$P' = \begin{bmatrix} m_1(w) & m_2(w) & m_3(w) & m_4(w) \\ m_1(t) & m_2(t) & m_3(t) & m_4(t) \\ m_1(s) & m_2(s) & m_3(s) & m_4(s) \end{bmatrix} \longmapsto \begin{bmatrix} f(P)(w) \\ f(P)(t) \\ f(P)(s) \end{bmatrix} = f(P')$$

Axiom 3.

(Common scale covariance): If the only difference between P and P' is that $m' = am$ for some real number $a > 0$, then $f(P') = af(P)$.

Axiom 4.

(Individual scale covariance): If the only difference between P and P' is that $m'_i = am_i$, for some i, for some number $a > 0$, then $f(P') = \alpha(a)f(P)$. The function $\alpha(\cdot)$ is increasing, positive, continuous, and may depend on i and P.

Axiom 4 (Individual scale covariance):

$$P' \longmapsto f(P') = \alpha(a) f(P), \text{ or}$$

$$\begin{bmatrix} m_1(s) & am_2(s) & m_3(s) & m_4(s) \\ m_1(t) & am_2(t) & m_3(t) & m_4(t) \\ m_1(w) & am_2(w) & m_3(w) & m_4(w) \end{bmatrix} \longmapsto \begin{bmatrix} \alpha(a) f(P)(s) \\ \alpha(a) f(P)(t) \\ \alpha(a) f(P)(w) \end{bmatrix}$$

- if i's measure m_i is multiplied by $a > 0$, the ratios $f(P)(s)/f(P)(t)$ remain unchanged

Let $p_{|E}$ be the restriction of measure p on the event $E \subset S$: $p_{|E}(A) = p(A \cap E)$.

Axiom 5.

(Updating): If $P = (N, S, m)$ *and* $Q = (N, E, m_{|E})$ *are such that* $E \subset S$*, then* $f(Q) = f(P)_{|E}$*.*

Axiom 6.

(Bayesian updating): Let $P = (N, S, p)$ *and* $Q = (N, E, q)$ *be probability aggregation problems such that each* q_i *is obtained from* p_i *by Bayesian updating on* E*. Then* $f(Q)$ *is obtained from* $f(P)$ *by Bayesian updating on* E*.*

3. Results

Lemma 1.

The Average rule, the Nash rule, the Median rule, and the Norm rules satisfy Anonymity, Neutrality, Common scale covariance, and Updating. Among these five rules, only the Nash rule satisfies Individual scale covariance.

Lemma 2.

On the class of probability aggregation problems, the Nash rule $f^{G\times}$ satisfies Bayesian updating. The other four rules do not have this property.

Def. Full support: $m_i(s) > 0, \forall i \in N, \forall s \in S$.

Theorem 3.

Assume full support and fix the agent set N. The "scaled" Nash rules $af^G, a \geq 0$, are the only rules satisfying Anonymity, Neutrality, Common scale covariance, Individual scale covariance, and Updating.

Drop *Updating* and the following rule satisfies the other axioms:

$$f(P) = \left[\prod_i m_i(S)\right] \left[\frac{1}{n} \sum_i \left(\frac{m_i}{\sum_s m_i(s)}\right)\right]$$

Axiom 7.

(Unanimity): If in a problem P all agents have the same measure μ, then $f(P) = \mu$.

Def. Common support: $m_i(s) > 0 \iff m_k(s) > 0, \forall i, k, s$.

Theorem 4.

Assume common support. The Nash rule f^G is the only rule satisfying Anonymity, Neutrality, Common scale covariance, Individual scale covariance, Updating, and Unanimity.

Theorem 5.

If f satisfies the assumptions of Theorem 4, and the corresponding probability measure $f^\times$ satisfies Bayesian updating on the class of probability aggregation problems, then $f^\times = f^{G\times}$.

Axiom 8.

(Expert proofness): Suppose an agent e (an expert) is added to the problem $P = (N, S, m)$, $e \notin N, m_e = f(P)$, *then the aggregate* $f(P)$ *doesn't change.*

Intuition: If an expert tells the agents what they already know, the aggregate beliefs do not change. (Closely related properties in voting context: faithful, consistent; positively involved by Young, Saari *etc*)

Proposition 1.

All the rules discussed in this paper satisfy Expert proofnes.

4. Discussion

- the (weighted) average rule was proposed by Stone 1961; (weighted) geometric average (Nash rule) by Hammond (and Kaneko 1979 in social choice); median rule by Balinski and Laraki 2007 (social choice)
- Genest (1984) has shown that (weighted) Nash rule is the only probabilistic aggregation rule satisfying Bayesian updating (called there ”externally Bayesian”), but he uses a much more restrictive framework