Absence-proofness: A new cooperative stability concept

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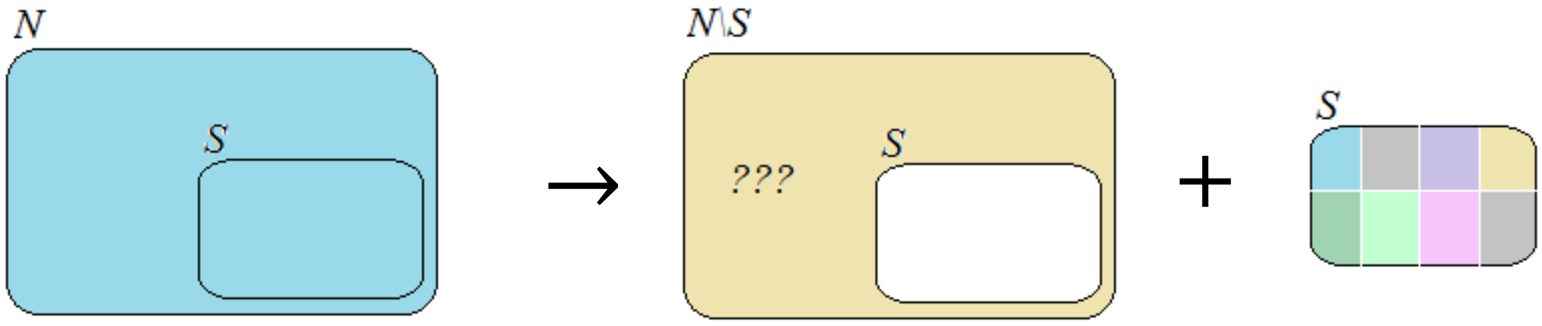
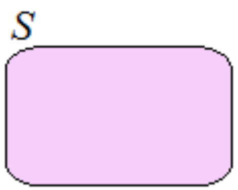
Overview

Core Stability

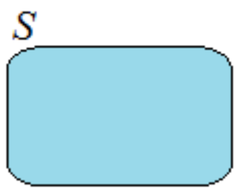


Definition 1: The blue allocation is core stable if for no $S \subseteq N$

a reallocation



Pareto dominates

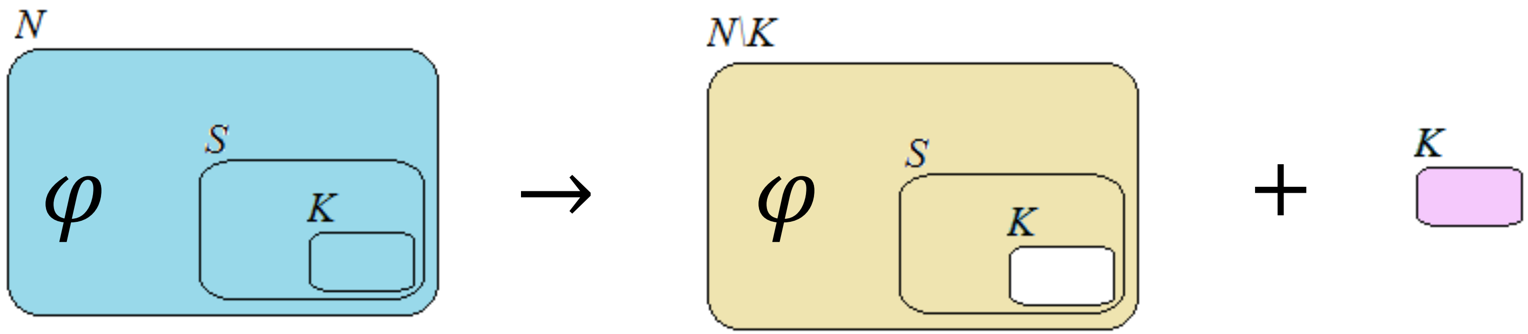


for agents in S.

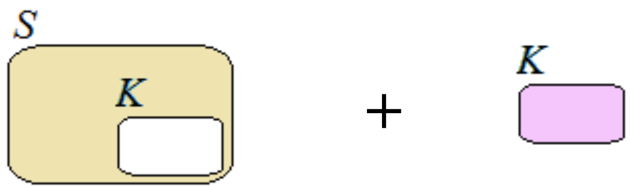
- There is no interaction between S and N \ S if S secedes.
- Agents in *S* do not need to know what happens in N \ S.

Overview

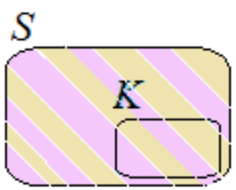
Absence-proofness

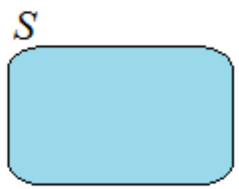


After the allocation process in $N \backslash K$, agents in S ($S \backslash K$ and K) meet behind closed doors.

Total resources of *S* afterwards: $\varphi_{S \backslash K}(N \backslash K) =$ 

Definition 2: An allocation rule φ is *absence-proof* (***AP***) if for no problem, no N, $K \subseteq S \subseteq N$,

a re-allocation 

Pareto dominates $\varphi_S(N) =$ 

for agents in S

TU Games

Basics:

$\boldsymbol{N} \in \boldsymbol{\mathcal{N}}$: Set of agents

$\boldsymbol{v}: \boldsymbol{2^N} \to \mathbb{R}_+$: Characteristic function, $v(S)$ is the maximum surplus $S \subseteq N$ can generate.

$(\boldsymbol{N}, \boldsymbol{v})$: TU cooperative game

Efficient allocation: A distribution of $v(N)$ among agents in N.

$$x = (x_i)_{i \in N} \in \mathbb{R}_+^N \text{ s.t. } \sum_{i \in N} x_i = v(N)$$

Core: x is in the core if $\quad \sum_{i \in S} x_i \geq v(S), \quad \forall S \subseteq N$

Subgame of $(\boldsymbol{N}, \boldsymbol{v})$***:*** (T, v_T) is derived by the restriction of v over agents in $T \subseteq N$.

$$\text{for all } S \subseteq T,\ v_T(S) = v(S)$$

Regular domain of games: A set D of games s.t. for each game (N, v) in D all of its subgames are also in D.

Solution $\boldsymbol{\varphi}$ ***on D*** assigns an efficient allocation to each game (N, v) in D.

TU Games

Motivating Example

$$N = \{1, \dots, 5\} \quad = \quad Workers = \{1,2,3\} \quad \cup \quad Firms = \{4,5\}$$

$v(S) = min\{|S \cap W|, |S \cap F|\}$, for all $S \subseteq N$

$$v(N) = 2, \qquad v(\{1,4,5\}) = 1$$

Unique core allocations for game N and subgame $T = \{1,4,5\}$*:*

$$(N, v) \rightarrow x = (0,0,0,1,1)$$

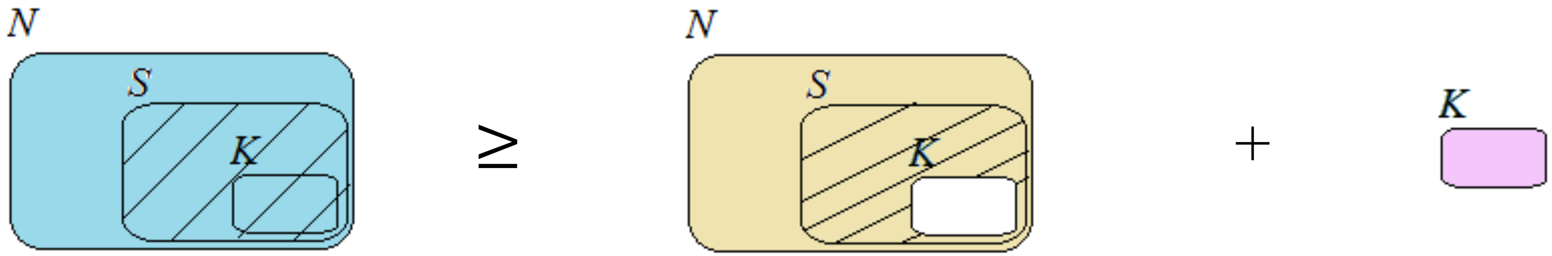
$$(T = \{1,4,5\},\ v_T) \rightarrow x' = (1,*,*,0,0)$$

Then, the set of workers $W = \{1,2,3\}$ can manipulate this core selection by absence of agents 2 and 3 in the job market.

No absence-proof solution on a regular domain D that contains this game!

TU Games

Definition 3: A solution φ is absence-proof on a regular domain D if for all games $(N,v) \in D$, and all $K \subseteq S \subseteq N$,



$$\sum_{i \in S} \varphi_i (N, v) \geq \sum_{i \in S \setminus K} \varphi_i \left(N \setminus K, v_{N \setminus K}\right) + v(K) \qquad (1)$$

Definition 4: (*Sprumont 1990*) **A** solution φ is *population monotonic* (***PM***) on a regular domain D if for all games $(N,v) \in D$ and all $i \in T \subseteq N$

$$\varphi_i(T, v_T) \leq \varphi_i(N, v)$$

TU Games

Proposition 1: If φ is PM on D then it is also AP on D.

Marginal contribution of agent i: $MC_i(T) = v(T \cup \{i\}) - v(T)$

Convex games: (N, v) s.t. $MC_i(T) \leq MC_i(T')$ for all $T \subseteq T' \subseteq N$ and $i \notin T'$.

Corollary 1: The Shapley value and the Dutta-Ray egaliatarian solution are absence-proof on the set of convex games.

Proposition 2: The nucleolus is not absence-proof on convex games.

TU Games

Sprumont (1990): "Coalition formation is a complex process. Our concern is to guarantee that once a coalition N has decided upon an allocation of $v(N)$, no player will ever be tempted to induce the formation of a coalition smaller than N by using his bargaining skills or by any other means."

Let $N = \{1,2,3,4\}$, $S = \{1,2,3\}$ $S' = \{4\}$

$\underline{v(S) = 9}$	$\underline{v(4) = 3}$	$\underline{v(N) = 14}$
$\varphi(S, v_S) = (3,3,3)$	$\varphi(4, v_4) = 3$	$\varphi(N, v) = (4,4,1,5)$

φ is not PM as agent 3's share decreases from 3 to 1 when agent j shows up.

Suppose $\underline{v(3) = 0}$. Even if agent 3 is tempted to not allow agent 4 to join group as he cannot pay enough to convince him to stay away.

TU Games

A Normative Approach to the Core and AP Based on Merging Coalitions

$$S,\ S' \subset N \quad \text{s.t.} \quad S \cap S' = \emptyset$$

S	$\cup$	S'		$=$	N
$v(S) = 9$	+	$v(S') = 3$	+ $sy(S,S')(= 2)$	=	$v(N) = 14$

Core stability

$\sum_{i \in S} \varphi_i(S, v_S) = v(S) = 9.$

If $\sum_{i \in S} \varphi_i(N, v) = 12 > 9 + 2$, $\varphi(N)$ is not a core allocation

Absence-proofness $K \subseteq S$

Say $\sum_{i \in K} \varphi_i(S, v_S) = 6$ Then, all the rest $(N \setminus K)$ gets $12 - 6 = 6$ before the merger

If $\sum_{i \in K} \varphi_i(N, v) = 9 > 6 + 2.$ Then $(N \setminus K)$ gets $14 - 9 = 5$ after the merger

TU Games

Proposition 3:

(i) φ is core stable on D *iff* for all (N, v) and partition S, S' of N

$$\textstyle\sum_{i\in S}\left(\varphi_i(N,v) - \varphi_i(S,v_S)\right) \quad \leq \quad sy(S,S') \tag{2}$$

Increase in the total share of agents in S is not more than the synergy

(ii) φ is AP on D *iff* for all (N, v), for all partition S, S' of N, and $K \subseteq S$

$$\textstyle\sum_{i\in K}\left(\varphi_i(N,v) - \varphi_i(S,v_S)\right) \quad \leq \quad sy(S,S') \tag{3}$$

Increase in the total share of agents in $K \subseteq S$ is not more than the synergy

Exchange Economies

Exchange Economies: Allocation problems where agents exchange private endowments.

Remark on manipulation: When agents in K stay out, they just stay at home wait for $S \setminus K$ to bring their allocation at the reduced problem.

Proposition 4: There is no absence-proof allocation rule in Böhm-Bawerk's horse market and in house assignment problems (Shapley and Shubik (1971)).

Example 1: (Single seller auction): Seller has one good. His reservation price is 0.

4 buyers with valuations: $b_1 = 1$, $b_2 = 1$, $b_3 = 4$, $b_4 = 4$.

	All buyers in	**Buyers 3 and 4 are out**
Core allocations:	Buyer 3 or 4 gets the good pays \$4 Others pay nothing	Buyer 1 or 2 gets the good pays \$1 Others pay nothing

Set of all buyers can manipulate by absence of buyers 3 and 4.

Exchange Economies

House assignment problem (Shapley and Shubik 1971):

- Each agent owns one house
- Agents have valuations for each house
- Utilities are quasilinear in money
- Monetary transfers possible (balanced transfers adds up to 0)

Example: $W = \{1,2,3\}$, $F = \{4,5\}$

	h_1	h_2	h_3	h_4	h_5
$u_1(h_i)$	0	0	0	1	1
$u_2(h_i)$	0	0	0	1	1
$u_3(h_i)$	0	0	0	1	1
$u_4(h_i)$	1	1	1	0	0
$u_5(h_i)$	1	1	1	0	0

Exchange Economies

Classical Exchange Economies

Proposition 5: The Walrasian allocation rule ψ is manipulable.

Example 2: $\ell = 2, N = 3, \{e_1, e_2, e_3\} = \{(10{,}10), (35{,}5), (15{,}15)\}$. $u_i = x_i y_i$ for all i.

$$u_1[\psi_1(\{1{,}2{,}3\}, e)] = 112.5 \qquad u_3[\psi_3(\{1{,}2{,}3\}, e)] = 253.125$$

If agent 3 leaves the scene or never appears at the first place

Agent 1's allocation at the reduced problem		Agent 3's endowment		Final resources of the manipulating coalition
$\psi_1(\{1{,}2\}) = (20{,}6.\bar{6})$	$+$	$e_3 = (15{,}15)$	$=$	$(35{,}21.\bar{6})$

Redistribution of total resources afterwards — Pareto improvement

$$z_1 = (15{,}7.\bar{6}), \quad z_3 = (20{,}14) \quad \rightarrow \quad u_1(z_1) = 115, \quad u_3(z_3) = 280$$

Fair Division

The General Setting

$\boldsymbol{N} \in \boldsymbol{\mathcal{N}}$: Set of agents

$\boldsymbol{\Omega} \in \boldsymbol{\mathcal{C}}$: Common endowment to be distributed

$\boldsymbol{\mathcal{C}}$: Consumption space

$\boldsymbol{R_i} \in \boldsymbol{\mathcal{R}_i}$: Preference of i over $\mathcal{C}$

$\boldsymbol{R} \in \boldsymbol{\mathcal{R}}$: Preference profile

$(\boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{R}})$: Fair division model

$(\boldsymbol{N}, \boldsymbol{\Omega}, \boldsymbol{R})$: Fair division problem

Fair Division

Remark on the manipulation: If K does not appear in the allocation process, it means they renounce their claims.

Then, $K = S$ is never a better outside option. Core has no bite here.

PM: (As Ω is fixed) No one is better off when an additional agent arrives.

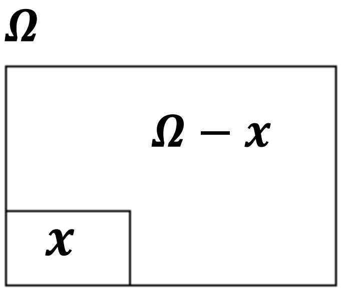
Theorem 1: Given a model $(\mathcal{N}, \mathcal{C}, \mathcal{R})$, if a PO allocation rule φ is PM, then it is also AP.

Fair Division

Proposition 8: Let everyone has strictly monotone preferences, φ be an allocation rule.

Initial problem

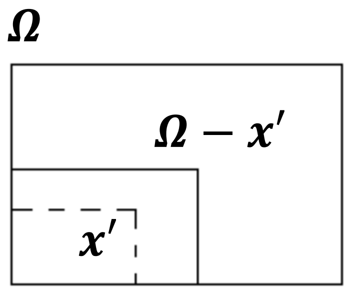
(N, Ω)



$\varphi_j(N) = x$

Additional set K of agents claim Ω

$(N \cup K, \Omega)$



$\varphi_j(N \cup K) = x' > x$

Then, everyone except j manipulates φ at problem $N \cup K$ by absence of K.

Fair Division

$\Omega \in \mathbb{R}_+^\ell$ & no money in the model

Two famous rules:

1. *Ω-egalitarian equivalent allocation*: Picks the unique efficient allocation x s.t. all agents are indifferent to the same bundle $\lambda.\Omega$ for $\lambda \in \mathbb{R}$.

2. *Competitive equilibrium with equal incomes (CEEI)*: First distribute Ω among agents equally. Agents start with (Ω/n). Then calculate the competitive allocation.

Corollary to Theorem: *Ω-egalitarian equivalent* allocation rule is AP.

Fair Division

Example 3: Let $\ell = 2$, $\Omega = (24{,}24)$, $|N| = 4$

$$u_1 = min\{2x + 8, y\}, \qquad u_i = min\{18x + 100{,}25y + 132\} \text{ for } i = 2{,}3{,}4.$$

$$CEEI_1(\{1{,}2{,}3\}) = (1{,}10) \qquad CEEI_1(N) = (2{,}12)$$

Corollary to Proposition 8: *CEEI* is not AP.

Remark 1: Competitive idea is less vulnerable to manipulation compared to exchange economies (See Example 4).

Fair Division

Example 4:

$N = \{1, \dots, 11\}$

2 divisible goods (beans and rice)

$u_1 = r_1 + 10b_1$, $\quad u_i = 10r_i + b_i$ for $i \in S = \{2, \dots, 11\}$.

Fair Division		**Exchange Economy**
$\Omega = (11,11)$		$e_i = (1,1)$ for all $i \in N$ (*private endowment*)
$CEEI\ (N, \Omega, R)$	=	$\psi(N, e, R)$ *(Walrasian allocation)*
No coalition can manipulate		S can manipulate the Walrasian rule by leaving any proper subset $K \subset S$ out of the market.

Fair Division

Ω is a single object & monetary transfers are available

Valuation: $a_i \geq 0$

Problem: (N, a)

Assignment: Only one agent receives the object

Transfers: A vector of balanced money transfers: $\sum_{i \in N} t_i = 0$

Allocation: Assignment & balanced transfers

Utility of *i*: $u_i(allocation) = a_i + t_i$ if i gets the object, $u_i = t_i$ otherwise

Fair Division

Assume *equal treatment of equals* (final utilities of agent with same valuations are the same)

WLOG order agents s.t. $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$

An *efficient solution* yields a unique final utility distribution U s.t.

$$\sum_{i \in N} U_i(N, a) = a_n \qquad \text{and} \qquad U_i \geq 0$$

Stand-alone cooperative game: $v(S, a^S) = max_{i \in S} a_i$

Corollary to Theorem: Shapley value and the Dutta-Ray egalitarian solution of the associated stand-alone cooperative game are AP.

Fair Division

$$0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$$

Proposition 9: A utility distribution satisfies condition ***C*** if and only if for any problem (N, a), when an additional agent j arrives, we have:

Cases	***C =AP***	***C =PM***
$a_{n-1} < a_n < a_j$	*Utility of only agent n can increase (up to $a_j - a_n$)*	*No one in N gains*
Otherwise	*No one in N gains*	*No one in N gains*

Envy-freeness: No one prefers another agent's allocation to her own allocation

Proposition 10: AP and EF are incompatible

Concluding Remarks

- Our manipulation idea is a generalization of the secession idea in core stability, and AP solutions are core selections.

- AP is too demanding in exchange economies

- Thomson (2012) introduced a weaker axiom "withdrawal-proofness" in Exchange Economies and Fair Division problems. Our negative results coincide with him. However, we have positive results for a stronger concept.

- AP imposes core-like participation constraints in Fair Division problems where core stability has no bites.

Concluding Remarks

- PM has been “mainly” considered as a solidarity property in TU Games and Fair Divison problems. We show that it also has a strong stability aspect.

- AP and PM have close formal implications. It is not easy to find sensible solutions that are not PM but AP.