Topics in tournament theory

Hannu Vartiainen

Higher School of Economics
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- Many problems in economics and decision theory can be reduced into a binary relation, reflecting "preference" or "dominance"
  - decision making
  - voting
  - coalition formation
  - information structures
  - rankings
  - networks
- The maximal element of the relation, or an element in **the core**, is a natural "optimal choice"

- But: the binary relation often not well behaved
  - nontransitive
  - incomplete
- => no maximal element

Example

Tennis players a, b, c where a beats b, b beats c and c beats a

Example

Majority voting with alternatives x, y, z and voters $1, 2, 3$ with preferences

1	2	3
x	y	z
y	z	x
z	x	y

Example

Decision maker with regret: y appears better when about to choose x, z appears better when about to choose y, and x appears better when about to choose z

- How to choose then?
- Many applications in voting and social choice, game theory, decision theory, computer science(!)

These lectures

- To express some of the key approaches to tournament theory
  - noncooperative game theoretic
  - decision teoretic
  - cooperative game theoretic
- The notion of covering: its interpretation under different approaches
- Novel interpretation

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Set up

- There is a finite set X finite set of outcomes or chosable alternatives
- T is a binary relation on X, i.e, a set of ordered pairs $T \subset X \times X$
- notation: xTy instead of $(x, y) \in T$, with the interpretation "x beats y"
- The upper set at x is denoted by $T(x) = \{y \in X : yTx\}$
- The lower set at x is denoted by $T^{-1}(x) = \{y \in X : xTy\}$

Definition

Binary relation T is a Tournament if

- Irreflexive: xTy implies $x \neq y$
- Total: either xTy or yTx for all $x \neq y$
- Asymmetric: if xTy then not yTx for all $x \neq y$

- Tournament allows cycles of arbitrary length and a tournament without cycles is an ordering
- $\mathcal{X}$ is the set of nonempty subsets of X

Solution

- A **tournament solution** is a nonempty correspondence $S : \mathcal{X} \rightarrow X$ that associates to each $A \subseteq X$ a set of outcomes $S(A) \subseteq A$ that are deemed "implementable" when the choice problem is A
- If there is an outcome x such that xTy for all $y \in A \backslash \{x\}$ – there cannot be two or more elements with the same property - then it is natural to demand that $\{x\} = S(T, A) =>$ **Condorcet consistency**
- However, such an x rarely exists $=>$ S reflect a reasonable justifiaction and unavoidably contains compromizes
- Which S to choose?

Impossibility

- To motivate out approach, we show that three natural conditions on S are not compatible $=>$ institutions matter
- S satisfies Chernoff's condition if $A \subseteq A'$ implies $S(A') \cap A \subseteq S(A)$

Theorem

Let $\#X \geq 3$. There is a tournament solution $S : \mathcal{X} \rightarrow X$ that satisfies Condorcet consistency and Chernoff if and only if T is an ordering.

Proof.

T is an ordering $=>$ axioms: obvious.
Axioms $=>$ T is an ordering: Suppose that T is not an ordering. Then there is a subset$A = \{x, y, z\}$ of X such that $xTyTzTx$. From Chernoff $S(\{x, y, x\}) \cap \{x, y\} \subseteq S(\{x, y\})$. From Condorcet, $S(\{x, y\}) = \{x\}$. Hence $y \notin S(\{x, y, z\})$. Similar argument implies that $z \notin S(\{x, y, z\})$ and $x \notin S(\{x, y, z\})$. This violates the assumption that S is nonempty. □

- The problem: *Chernoff*

Covering

- The notion of **covering** seeks to identify elements that are fundamentally **un**stable in sense that they could never be elected via any reasonale procudure

Definition

(cf. Fishburn 1977, Miller 1980, Duggan 2013) Outcome y **covers** x in $B \subseteq X$ if

$$\{y\} \cup T(y) \cap B \subseteq T(x) \cap B$$

- This version of covering does not require that $x \in B$, only that y and anything that dominates y in B is also in B and dominates x
- Intuition: it is safe to say that x makes y uniplementable since wherever we move from y is also dominates x

- The covering relation in $B \subseteq X$ is transitive
- Since X is a finite set, the set of maximal elements of the covering relation in B, denoted $uc(B)$ called the **uncovered** set of B, is nonempty
- $=>$ No element in $uc(B)$ is covered in B
- Two-step principle: if $x \in uc(B)$, then, for any $y \in B$ there is z such that $z \in \{y\} \cup T(y) \cap B \setminus T(x)$
  - Thus from any uncovered element it takes at most two dominance steps in B to get back to x
  - Note: does not say that the designated z should be in $uc(B)$!

- The **ultimate uncovered set** UUC is defined recursively:
  - $uc^0 := X$
  - $uc^{k+1} = uc(uc^k)$, for all $k = 0, \ldots$
  - $UUC := uc^\infty$
- No element in UUC is covered in UUC

- We now specify an important concept of Laslier 1997 (weaker than the original one by Dutta 1988)

Definition

The set $D \subseteq X$ is a **covering set** if any outcome $x \notin D$ is covered in $D \cup \{x\}$

- There is a "fixed point" flavor in this concept; set D constitutes of elements that are not "unstable", given that the unstability of these elements is evaluead againts the assumption that precisely the other elements can be stable
- The existence of a covering set is immediate as X is a covering

Proposition

uc^k is a covering set for all $k = 0, \ldots$

Proof.

We proceed by induction. uc^0 is a covering set. Suppose that uc^k is a covering set. Then for any $x \notin uc^k$ there is $x' \in uc^k$ such that

$$\{x'\} \cup T(x') \cap (uc^k \cup \{x\}) \subseteq T(x) \cap (uc^k \cup \{x\}).$$

Suppose that $x' \notin uc^k \backslash uc^{k+1}$. Since the covering in uc^k -relation is transitive, the number of elements in X is finite, there is x'' that is maximal in the relation such that $x'' \in uc^{k+1}$ and such that

$$\{x''\} \cup T(x'') \cap uc^k \subseteq T(x') \cap uc^k$$

□

Proof.

[Proof cont.] By chaining the covering operations we get

$$\{x''\} \cup T(x'') \cap (uc^k \cup \{x\}) \subseteq T(x) \cap (uc^k \cup \{x\}).$$

Hence x'' covers x in $uc^k \cup \{x\}$, as desired. □

Corollary

UUC is a covering set

Tournament game

- We now study, *how* a decision from a toruanemtn could be made
- Let there be two parties, 1 and 2, competing over winning an election
- The election between the parties is made on the basis of their positions or "platforms" in X
- The voters preferences are represented by a majority relation T (tournament) on X
- The party competition Downsian: the parties only care about winning the election

- Parties 1, 2 choose their positions $x_1, x_2 \in X$, respectively, simultanously after which the majority election takes place

Definition

A **tournament game** is finite symmetric two-player zero-sum game (X, g) where X is the set of actions of bothe players and their payoffs $(u_1, u_2)(x_1, x_2)$ from each plaform combination $(x_1, x_2) \in X^2$ are defined by $u_1(x_1, x_2) = g(x_1, x_1) = -u_2(x_1, x_1)$ such that

$$g(x_1, x_1) = \begin{cases} 1, & \text{if } x_1 T x_2 \\ 0, & \text{if } x_1 = x_2 \\ \text{-1}, & \text{if } x_2 T x_1 \end{cases}$$

- The crux of the matter is strategic behavior: optimal plaform of party i depends on the choice of party j, and vice versa
- In a game (X, g), strategy $x_1 \in X$ is 1's *best response* to $x_2 \in X$ if

$$x_1 = \max_{y \in X} g(y, x_2)$$

and strategy $x_2 \in X$ is 2's best reponse to $x_2 \in X$ if

$$x_2 = \min_{y \in X} g(x_1, y)$$

Remark

If T supports a Condorcet winner x, then x is a best response of any player to against strategy of the other player, and x is the unique best response against itself

- But what if a Condorcet winner does not exist?
- An important way to solve interactive stratgic situations is the Nash equilibrium

Definition

A strategy pair (x_1, x_2) forms a (pure strategy) **Nash equilibrium** of (X, g) if x_1 is a best response to x_2, and vice versa

Proposition

A strategy pair (x_1, x_2) forms a (pure strategy) Nash equilibrium of (X, g) if and only if there is x such that $x = x_1 = x_2$ and such that x is the Condorcet winner associated to T

- Hence typically a pure strategy Nash equilibrium does not exist
- What does rationality of the agents, and common knowldge of it, imply on their behavior?
- A player is rational if he does not choose an action that is (weakly) dominated by another action

Definition

An action x is **weakly dominated** in B by x' if $g(x', y) \geq g(x, y)$, for all $y \in B$, with at least one inequality

- Denote actions that are (weakly) undominated in B by $ud(B)$ and the actions that are iteratively weakly dominated by $ud^0(X) = X$, $ud^{k+1}(X) = ud(ud^k(X))$, $IUD = ud^\infty(X)$

Lemma

$ud(B) = uc(B)$, for any $B \subseteq X$

Proof.

If $g(x', y) \geq g(x, y)$, for all $y \in B$, with at least one inequality, then $x'Ty$ implies xTy, for all $y \in B$, and for some y, $x'TyTx$ or, if $y \in \{x, x'\}$, $x'Tx$. Since T is a tournament, this amounts to $\{x'\} \cup T(x') \cap B \subseteq T(x) \cap B$ □

Corollary

$IUD = UUC$

- Another interesting relation of the tournament game is to Shapley's 1964 saddle points
- In the current context: sets $S \subseteq X$ is a **weak generalized saddle** if for all $x \notin S$ there is $x' \in S$ such that $g(x', y) \geq g(x, y)$ for all $y \in S$, with at least one strict inequality
- Known properties of weak saddles
    - Exist in all normal form games
    - Contains a (support of) Nash equilibrium

Proposition

Dutta's covering set = weak generalized saddle

Proof.

If S is a weak generalized saddle, then $S = ud(S \cup \{x\})$ for all $x \notin S$. By the previous Lemma, $ud(S \cup \{x\}) = uc(S \cup \{x\})$, for all $x \in X$. Hence $S = uc(S \cup \{x\})$ for all $x \notin S$, implying that S is also a covering set. □

- A **mixed strategy** associated to the tournament game is a pair of probability distributions p_1,p_2 on X detemining the actions of both the players, and inducing the payoffs

$$u_1(p_1, p_2) = \sum_{x_1} \sum_{x2} p_1(x_1) p_2(x_1) g(x_1, x_1) = -u_2(p_1, p_2)$$

- Strategy (p_1,p_2) constitutes a Nash equilibrium if p_1 is a best response to p_2, and vice versa
- By the minmax-theorem - since the tournament game is zero-sum - there is a unique equilibrium payoff for both the players
- However, we can say more

Theorem

The tournament game supports a unique Nash equilibrium (p_1^*, p_2^*). This equlibrium is symmetric such that $p^* = p_1^* = p_2^*$.

- The support of p^* is called the **bipartisan set**

Proposition

The bipartisan set B is a subset of UUC

Proof.

Consider a tournament game restricted to the outcome set UUC, and identify the unique symmetric equilibrium point $\hat{p}$ associated to UUC. We argue that $\hat{p}$ is also an equilibrioum point at the larger game X. It suffices to show that any $x \notin UUC$ is not a better response to $\hat{p}$. Since UUC is a covering set, there is $x' \in UUC$ such that $g(x', y) \geq g(x, y)$ for all $y \in UUC$, with at least one strict inequality. Since the support of $\hat{p}$ is in UUC, also

$$\sum_y \hat{p}(y) g(x', y) > \sum_y \hat{p}(y) g(x, y).$$

□

[Proof cont.]Since any element x'' in the support of $\hat{p}$ is a best reponse against $\hat{p}$,

$$\sum_{x''}\sum_{y}\hat{p}(x'')\hat{p}(y)g(x'',y) \geq \sum_{y}\hat{p}(y)g(x',y).$$

Thus

$$\sum_{x''}\sum_{y}\hat{p}(x'')\hat{p}(y)g(x'',y) > \sum_{y}\hat{p}(y)g(x,y),$$

implying that $x \notin UUC$ cannot be a best reponse to $\hat{p}$.

Dynamic elections

- Bernheim and Slavov (2009) expand the idea of Condorcet dominance to a setting where political decisions are made repeatedly
- The set X of outcomes is now interpreted as the set of social **states** which may change in dates $t = 0, 1, \dots$.
- Policy making is now an ongoing process where the individuals gain benefits from the policy choices in each period t
- To be concrete, let there be a set $\{1, \dots, n\}$ of agents, each i endowed with a per period utility function $u_i : X \to \mathbb{R}$
- Define the majority relation $T \subset X^2$ over states by

$$xTy \quad \text{if} \quad |\{i : u_i(x) \geq u_i(y)\}| > \frac{n}{2}$$

- A **play path** is a sequence $\bar{x} = (x_0, ..., x_K)$ of outcomes originating from some intial outcome x_0
- Denote the set of paths, i.e. **histories**, by $H = \cup_{k=0}^{\infty} X^k$, with $\emptyset$ as the initial history
- A **policy programme** σ specifies a social action given the history of past actions $\sigma : H \to X$
- The interpretation of a policy programme is that if $\sigma(h, x) = y \in X$, then after a history h of states, the current state x is successfully challenged by a winning coalition (majority) with outcome y which then becomes the prevailing state in the next period
- Since σ is conditioned on all the histories, we may use juxtaposition $\sigma(h, x_t)$ for $\sigma(x_0, ..., x_t)$, where $h = (x_0, ..., x_{t-1})$

- We will now focus on policy programmes that are **absorbing** in the sense that after all histories, the policy path converges in finite time to an absorbing state in which it stays permanently: for any history h there is an integer T_h such that $\sigma^t(h) = x$, for all $t > T_h$
- The absorbing state of the policy programme σ that starts from history h is then well defined for all h, and denoted by $\alpha[\sigma(h)]$.
- We assume that there is no discounting - the intertemporal payoffs are evaluated by the *limit-of-the-means criterion*
- For an absorbing policy programme it holds then that the intertemporal payoff if agent i from policy σ at history $h \in H$ is given by $u_i(\alpha[\bar{\sigma}(h)])$

Definition

(Bernheim and Slavov 2009) An absorbing policy programme σ is a **dynamic Condorcet winner** (DCW) if

$$\alpha[\bar{\sigma}(h,x)]T\alpha[\bar{\sigma}(h,x,y)], \quad \text{for all } (h,x) \in H, \text{ for all } y \in X. \quad (1)$$

Theorem

If there is a Condorcet cycle $w^1, w^2, ..., w^K$ *such that* $w^k T w^{k+1}$ *for all* $k = 1, ..., K-1$ *and* $w^K T w^1$, *then each state* w^k *can be supported as an absorbing state of an absorbing DCW*

- This prolematic for two reasons
  1. The model exploits heavily the property that a deviation triggers a long punishement - a devitation from x to yTx is just a meaningless transitory phase towards z such that xTz where the play stays forever
  2. The model does not give any light on how one-shot decisions should be made
- As a consequence of point 1, for a DCW it may be the case that $xT\alpha[\bar{\sigma}(h,x)]$ which implies, among other things, that the Condorcet winner may not always be implemented (when such exists)
- We shall now argue, that both the above problems can be solved via a simple modification of the problem

Endogenous agenda formation

- We focus one-shot decision making of a society whose social preferences are characterized by the tournament (majority) relation T
- The decision process is endogenous in a sense that any majority can amend the status quo in any way it wants - until no more amenedments are made (the model is based on Vartiainen 2014)
- Decision making procedure: at stage $t = 0, 1, \ldots$:
  - Status quo x_t may be implemented ($=$STOP) or replaced by a majority with outcome, say y, which then becomes the status quo $x_{t+1} = y$ at stage $t + 1$
  - If x_t is not replaced, then x_t is implemented
- Denote by $H = \cup_{k=0}^{\infty} X^k$ the set of all possible finite histories of status quos, in which no status quo has been implemented during the play (typical elements h or (h, x) of H)

- The policy programme is now a function $\sigma : H \to X \cup \{\text{STOP}\}$ with the interpretation that at $(x_0, ..., x_t) \in H$, if $\sigma(x_0, ..., x_t) =$STOP, then x_t is implemented and if $\sigma(x_0, ..., x_k) = y \in X$, then the a majority will challenge x_t with y and the next status quo will be y
- Thus, a policy programme σ now specifies how the decision process, i.e. the sequence of tentative status quos, evolves and which outcome - if any - eventually becomes implemented
- Let us focus on policy programmes that terminate in finite time after all histories

- Let $\bar{\sigma}(h)$ denote the sequence of status quos in X that is induced by the programme σ from the history h onwards

$$\bar{\sigma}(h) = (\sigma^0(h), \sigma^1(h), \ldots)$$

- Since σ is terminating, at any h there is t_h such that $\sigma^{t_h}(h) = \text{STOP}$
- Denote by $\mu[\bar{\sigma}(h)]$ the final status quo of the path $\bar{\sigma}(h)$ (if σ is terminating, then $\mu[\bar{\sigma}(h)]$ is well defined, for all h)
- If a policy action $a \in X \cup \{\text{STOP}\}$ is chosen at history $(h, x) \in H$, then

$$\mu[\bar{\sigma}(h, x, a)] = \begin{cases} \mu[\bar{\sigma}(h, x, y)], & \text{if } a = y \in X \\ x, & \text{if } a = \text{STOP} \end{cases}$$

- In particular, $\mu[\bar{\sigma}(h, \sigma(h))] = \mu[\bar{\sigma}(h)]$

Definition

(Vartiainen 2013) A history dependent terminating policy programme σ satisfies the **one-deviation property** if

$$\text{not } \mu[\bar{\sigma}(h,a)]\, T\, \mu[\bar{\sigma}(h)], \quad \text{for all } a \in X \cup \{\textsc{stop}\}, \text{ for all } h \in H$$

- That is, after each history, a majority will not benefit from changing the prescribed action given the consequences
- An immediate implication of the one-deviation property is that

$$\text{not } xT\mu[\bar{\sigma}(h,x)], \quad \text{for all } (h,x) \in H$$

- Which outcomes are implementable via a policy programme σ with the one-deviation property?

Definition

A nonempty set $C \subseteq X$ is a **consistent choice set** if, for any $x \in C$ and for any $y \in X$, there is $z \in C$ such that $z \in (\{y\} \cup T(y)) \setminus T(x)$

- In other words, any x in C is not covered in $C \cup \{y\}$ by *any* y in X
- Recall the definition of the covering set D: any outcome $x \notin D$ is covered in $D \cup \{x\}$
- The set Y of alternatives is **implementable** via a dynamic policy programme σ if

$$Y = \{x \in X : \sigma(h, x) = \text{STOP}, \text{ for some } h \in H\}$$

Lemma

Let a terminating policy programme σ satisfy the one-deviation property. Then the set Y of outcomes that are implementable via σ is a consistent choice set.

Proof.

Take any $(h, x) \in H$ such that $\sigma(h, x) = \text{STOP}$. Then $\mu[\bar{\sigma}(h, x)] = x \in Y$. Take any $y \in X$, and let $z = \mu[\bar{\sigma}(h, x, y)] \in Y$. Since it cannot be the case that $yT\mu[\bar{\sigma}(h, x, y)]$, we have $z \in \{y\} \cup T(y)$. Since σ satisfies the one-deviation property, not zTx, or $z \notin T(x)$, as desired. □

- If z is a Condorcet winner, then z is the only outcome that a consistent choice can contain, and hence the only one that is implementable via a policy programme σ that meets the one-deviation property

- To see the converse, let C be a consistent choice set
- We construct a terminating policy programme $\sigma^C : H \to X$ in such a way that σ^C meets the one-deviation property and implements the outcomes in C
- Construct a function $z : X \times X \to X$ such that, for any $x \in C$ and for any $y \in X$,

$$z(x, y) \in C \cap \{y\} \cup T(y) \setminus T(x)$$

- Interpret C as an index set and construct recursively a partitioning $\{H_x\}_{x \in C}$ of H as follows: choose $x_0 \in C$ and $\emptyset \in H_{x_0}$ and then, if $h \in H_x$, for any $y \in X$,

$$(h, y) \in \begin{cases} H_y, & \text{if } y \in C \setminus P(x) \\ H_{z(x,y)}, & \text{if } y \notin C \setminus P(x) \end{cases}$$

- An element of the partition $\{H_x\}_{x \in C}$ is called a *phase*
- Given the function z and the collection phases $\{H_x\}_{x \in C}$, choose a policy programme σ^C such that, for all $h \in H_x$,

$$\sigma^C(h, y) = \begin{cases} \text{STOP}, & \text{if } y \in C \backslash P(x) \\ z(x, y), & \text{if } y \notin C \backslash P(x) \end{cases} \quad (2)$$

- Intuition: σ^C it implements any element in C such that any deviating majority coalition will become punished by implementing an outcome in C that the deviating coalition does not prefer relative to the outcome that was originally to become implemented
- The role of a phase in the construction is to remind which majority is to be punished and the z-function specifies how the punishement is conducted
- Transition between phases determines when and how the majority that is to be punished should be changed
- The circularity in punishments makes the programme robust against profitable majority deviations in all phases, *i.e.*, after all histories

Lemma

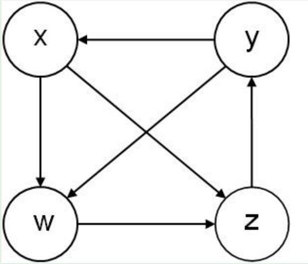
Policy programme σ^C is terminating, satisfies the one-deviation property, and implements elements in C

Theorem

Set Y of alternatives is implementable via a terminating policy programme that satisfies the one-deviation property if and only if Y is a consistent choice set.

Example

Let $X = \{x, y, z, w\}$ and yTx, zTy, wTz, xTw, xTz, and yTw (Fig , where $x \longrightarrow y$ means xTy, etc.).Then the unique consistent choice set is $\{x, y, z\}$, and hence w cannot be implemented via a terminating policy programme that meets the one-deviation property.



- We now prove the existence of the solution
- Recall that UUC has the property that no element x in UUC is covered in UUC, and every element y outside UUC is covered in $UUC \cup \{y\}$

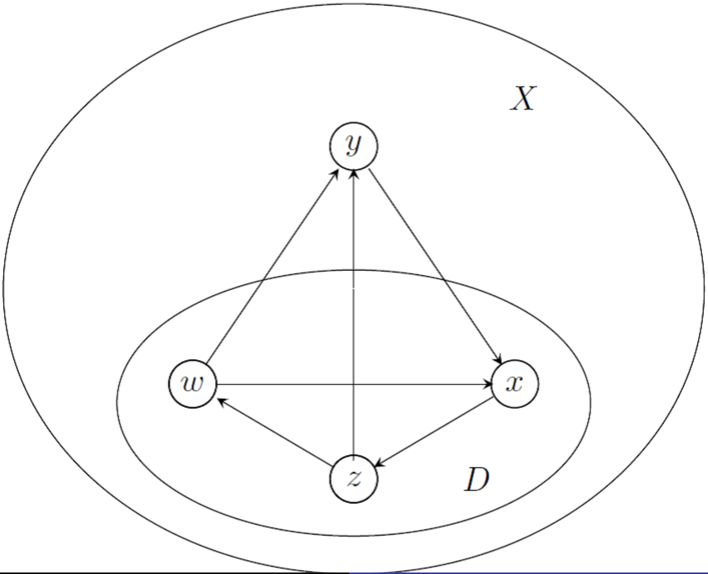
Lemma

Any $x \in UUC$ is not covered in $UUC \cup \{y\}$ by any $y \in UUC$

Proof.

Suppose that $x \in UUC$ is covered in $UUC \cup \{y\}$ by $y \in UUC$, i.e. $(\{y\} \cup T(y)) \cap (UUC \cup \{y\}) \subseteq T(x) \cap (UUC \cup \{y\})$. By the definition of UUC, $y \notin UUC$, and there is $w \in UUC$ that covers y in UUC, i.e. $\{w\} \cup T(w) \cap UUC \subseteq T(y) \cap UUC$. By the asymmetry of T, $y \in T(x)$ implies $x \notin T(y)$. Hence also $x \notin \{w\} \cup T(w)$. By the totality of T, $w \in T(x)$. Since x is uncovered in UUC, there exists $z \in UUC$ such that $z \in \{y\} \cup T(w) \setminus T(x)$. Since w covers y in UUC, it follows that $z \in T(y)$. Hence $z \in \{y\} \cup T(y) \setminus T(x)$, contrardicting that y covers x in $UUC \cup \{y\}$. □

Proof.



- Whence, since the *UUC* exists:

Theorem

A consistent choice set exists. Hence, a policy programme having the one-deviation property exists.

Theorem

UUC is the maximal consistent choice set.

Corollary

There is a terminating policy programme meeting the one-deviation property that implements outcomes in UUC. Moreover, UUC contains all outcomes that can be implemented via any terminating policy programme meeting the one-deviation property.

Dynamic stable set

- Von Neumann-Morgenstern (vNM) stable set is a solution with pedigree
- Applicable also when no maximal element exists (the Core is empty)
- However, in many contexts does not function well
  - Farsighteness of the agents
  - Existence (in patricular tournament!)

- Let X be the set of outcomes and P a relation on X, reflecting dominance
- Then (X, P) is an *abstract decision problem* (Lucas 1999)
- **Stable set** $V \subseteq X$ is defined by:
  1. (External stability) If $x \notin V$, then there is $y \in V$ such that yPx
  2. (Internal stability) If $x \in V$, then there is **no** $y \in V$ such that yPx

- Unlike the Core, the stable set accounts some of the underlying dynamics; it tests alternatives **only** against those that are deemed stable
- This feature of the solution reflects neatly strategic sophistication but it also guarantees the existence in many circumstances where the core is empty
- Moroever, in a tournament context, the stable set exists *only* when there is a Condorcet winner
    - Internal stability: stable set can only contain a single element
    - External stability: this element must dominate the others

- The problem: what happens after a blocking should also be accounted for
- But what happens after a blocking may depend what has happened before the blocking
- We now develop a solution that allows **history dependent** blockings (whence "dynamic" stable set)
- Apply to the tournament context, which is the most demanding in terms of the existence of the solution, and to to make the model comparable with our previous discussion

- Blocking procedure:
  - An initial status quo outcome $x_0 \in X$
  - At stage $t = 0, 1, \ldots$:
    - Status quo x_t may be blocked with outcome y which then becomes the status quo at stage $t + 1$
    - If x_t is not blocked, then x_t is implemented
- Sequence of nonimplemented status quos constitute a **history of blockings**
- Aain, denote by $H = \cup_{k=0}^{\infty} X^k$ the set of finite histories, with typical elements h or (h, x)

- Novelty here: whether status quo x is blocked may depend on the past history h
- At which histories (h, x), the status quo is not blocked?
- History $(h, x) \in H$ is **directly dominated** by history $(h, x, y) \in H$ if xTy
- Stable set can now be defined on the set of histories

- The set $V \subset H$ is now **directly dynamically stable** if:
  1. (External direct stability) If $h \notin V$, then there is an element in V that **directly** dominates h
  2. (Internal direct stability) If $h \in V$, then there is **no** element in V that **directly** dominates h

- Harsanyi's (1974) criticism: once an internally stable history is deviated to a history outside the stable set, the new history will also be deviated to, and this time to a history inside the stable set
- Indirect and profitable deviation of this sort is not restricted by internal stability
- Harsanyi's remedy: use **in**direct dominance as the dominance criterion: $(h, x) \in H$ is **indirectly dominated** by $(h, x_0, ..., x_K) \in H$ if $x = x_0$ and $x_K P x_k$, for all $k = 0, ..., K-1$
- Note: the farsighted stable set, appealing to the indirect dominion has received much attention recently (e.g. Ray and Vohra 2015)

- The set $V \subseteq H$ is a **indirectly dynamically stable** if:
  1. (External indirect stability) If $h \notin V$, then there is an element in V that **in**directly dominates h
  2. (Internal indirect stability) If $h \in V$, then there is **no** element in V that **in**directly dominates h

- But Xue (1998) points out that indirect dominance is not a satisfactory criterion either: an indirect dominance path may not be credible since nothing in the notion of indirect dominance guarantees that at the interim stage of the deviation path the decision maker(s) should not deviate to some *other* path.
- Thus both the dynamic versions of the stable set - direct and indirect - are vulnerable to criticism
- Thus, an obvious remedy to the problem would be to demand that the solution satifies **both** direct and indirect stability criteria at the same time

Strong dynamic stable set

- The following concept is analyzed by Vartiainen 2015, Salonen and Vartiainen 2015

Definition

A **strong dynamic stable set** $V \subset H$ is defined by:

1. (External direct stability) If $h \notin V$, then there **is** an element in V that **directly** dominates h
2. (Internal indirect stability) If $h \in V$, then there **is no** element in V that **indirectly** dominates h

Remark

A set $V \subseteq H$ is directly **and** *indirectly dynamically stable if and only if V is a strong dynamic stable set*

- Free from the Harsanyi and Xue critiques

- Our main interest is in outcomes that can be implemented via a dynamic stable set
- Denote by

$$\mu(V) := \{x : (h, x) \in V, \text{ for some } h \in H\},$$

  the set of final elements of the histories in V, i.e. outcomes that are implementable via histories in V
- The set Y of outcomes is **implementable** via the dynamic stable set V if $Y = \mu(V)$
- A characterization of strong dynamic stable sets directly in terms of sets of implementable outcomes

Characterization

- Recall:

Definition

A set $C \subseteq X$ is a **consistent choice set** if any y in C is not covered in $C \cup \{x\}$ by any x in X

Lemma

V is a strong dynamic stable set only if $\mu(V)$ is a consistent choice set

Proof.

Let V be a strong dynamic stable set. We show that $x \in \mu(V)$ is not covered in $\mu(V) \cup \{y\}$ by y. Suppose, on the contrary, that y covers x in $\mu(V) \cup \{y\}$. Then necessarily $y \in T(x)$. Identify $h \in H$ such that $(h, x) \in V$. Since $y \in T(x)$, (h, x, y) directly dominates (h, x). By internal stability, $(h, x, y) \notin V$. By external stability, there is $(h, x, y, z) \in V$ such that (h, x, y, z) directly dominates (h, x, y). Thus $z \in \mu(V)$ and $z \in T(y)$. By internal stability, however, (h, x, y, z) cannot indirectly dominate (h, x), implying - since it does directly dominate (h, x, y) - that $z \notin T(x)$. Thus $z \in \mu(V) \cap T(y) \setminus T(x)$ which implies that y does not cover x in $\mu(V) \cup \{y\}$. □

- Let C be a consistent choice set
- Construct partitioning $\{H_x\}_{x \in C}$ of H as follows
  - Initial step: choose $x_0 \in C$ and $\emptyset \in H_{x_0}$
  - Inductive step: If $h \in H_x$, then, for any $y \in X$,

$$(h, y) \in \begin{cases} H_y, & \text{if } y \in C \backslash P(x) \\ H_x, & \text{if } y \notin C \backslash P(x) \end{cases}$$

- A **phase** H_x summarizes all the relevant information contained in the history
- Construct a set $V^C \subseteq H$ such that

$$V^C = \{(h, y) : h \in H_x \text{ and } y \in C \backslash P(x)\}$$

- Then $\mu(V^C) = C$

Lemma

V^C is a strong dynamic stable set

Theorem

Set B of alternatives is implementable via a strong dynamic stable set if and only if B is a consistent choice set.

- Recall:

Theorem

UUC is a consistent choice set and it contains any consistent choice set

Corollary

A strong dynamic stable set exists. Moreover, any strong dynamic stable set implements outcomes in UUC

Axiomatic characterization

- Interpreting the tournament relation as a reflection of decision maker's preferences, the one-deviation restriction or the dynamic stable set could viewed as generalized models of individual decision making
  - Reduces to the standard model when the relation is an ordering (Condorcet winner always chosen)
  - Allows modeling of regret, habit formation etc. often regarded conceptually difficult
- But what are observational and behavioral implications of the theories (i.e. choice from the UUC)?
- Can choice from the UUC be tested?

- The problem is that do not know T, we only observe choices
- Let an (observed) **choice function** f specify, for each decision problem $B \subseteq X$, which outcomes the decision maker chooses - or can choose - from B
- Conditions on f (see also Moulin 1991, Lombardi 2009):

A1 (Binary dominance consistency) For all $A \subseteq X$, and for all $x \in A$, if $x \in \cap_{y \in A} f(\{x, y\})$, then $x \in f(A)$

- Worst outcome cannot be chosen

A2 (Stability) For all $A \subseteq X$ and for all $x \in A$,
$f(A) = f(f(A) \cup \{x\})$

- This condition implies idempotence ($f(A) = f(f(A))$) and a degree of independence of irrelevant alternatives

A3 (Weakened Chernoff) For all $A \subseteq X$, if there is $y \in A$ such that $x \notin \cup_{z \in A} f(\{x, y, z\})$, then $x \notin f(A)$

By choosing $y = x$, this condition also implies that $x \notin f(A)$ whenever $x \notin \cup_{z \in A} f(\{x, z\})$

A4 (Asymmetry) For all $\{x, y\} \subseteq X$, either
$f(\{x, y\}) \in \{\{x\}, \{y\}\}$.

- Denote the corresponding ultimate uncovered set by $UUC_P(A)$, obtained by employing binary relation P in the set $A \subseteq X$

Lemma

If f satisfies A1-A4, then there is a tournament T such that $f(A)$ is a consistent choice set for all $A \subseteq X$

Theorem

The ultimate uncovered set choice function $UUC_T(\cdot)$ satisfies A1-A4, for any tournament relation T. Moreover, if f satisfies A1-A4, then there is a tournament T such that $f(A) \subseteq UUC_P(A)$, for all $A \subseteq X$.

Comments

- The notions of ultimate uncovered set and covering set have connections to different branches of literature, relying on tournament structure
  - Iteratively weakly undominated strategies in the tournament game = the ultimate uncovered set => Dutta's covering set => consistent choice set
  - Laslier's covering set = Shapley's weak generalized saddle => Dutta's covering set
  - Bipartisan set a subset of of the ultimate uncovered set
- Hence:
  - Outcomes that are implementale via a policy process that has the one-deviation property or via a dynamic stable set contained in the ultimate uncovered set
  - Bipartisan set = consistent choice set

- All these models and most of the observations can be immediately extended to weak tournaments and some of them to incomplete relations
- The key observation: history dependence, which in principle is a complicated object, can be natrually and conveniently be be characterized by covering operations $=>$ many usefule insights and applications elsewhere in dame theory
- Generalization of this model to Chwe's 1988 framework is an ongoing work by Salonen and Vartiainen 2015, based on Vartiainen 2011
- Existence and characterization, application to matching markets and hedonic games

Thank you!