Social Choice with a Poverty Line

E. Maskin

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  - ordinality (O)

Arrow Theorem: if f satisfies U, P, IIA, and O, then there exists i_* such that, for all $(u_1,\ldots,u_n)$ and all x, y, if $u_{i_*}(x) > u_{i_*}(y)$ then $W(x) > W(y)$

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– i_* is a dictator

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- shouldn't we take account of such comparisons?

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  - allows us to compare both utility *levels* and utility *differences* across individuals

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- optimal population is unbounded

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  - below this line, person’s welfare *subtracts* from social welfare

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  - reference level means can't transform utilities by adding β anymore and still get invariance

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  Decreasing Social Returns to Utility: social indifference curves are *convex*

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    - need to give increasingly more to 1 to compensate for 2's loss

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- where $a_i(v_i) = \begin{cases} 1, \text{ if } v_i > 0 \\ a(\geq 1), \text{ if } vi < 0 \end{cases}$

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- then there exist $(a_1(\cdot),\ldots,a_n(\cdot))$ such that for all $(u_1,\ldots,u_n), x, y$
- $W(x)R(y) \Leftrightarrow \sum a_i(u_i(x))u_i(x) \geq \sum a_i(u_i(y))u_i(y),$
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focus on n=2

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  - indifference curve is $u_1^{1/3} + u_2^{1/3} = c$
  - so slope of indifference curve is $\frac{du_2}{du_1} = \infty$ at $u_1 = 0$
  - violates DSR in second quadrant

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  – DSR implies can't have $a < 1$