A Homothetic Test for the Matrix Updating Methods and Twin RAS Method

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Matrix updating methods are used for constructing the target matrix with the prescribed row and column marginal totals that demonstrates the highest possible level of its structural similarity (or resemblance, likeness, closeness, etc.) to initial matrix given. Various definitions of a matrix similarity measure generate a great manifold of different methods and techniques for matrix updating.

A notion of structural similarity between initial and target matrices has a vague framework that can be slightly refined in an axiomatic manner. To this end, one can consider a particular case of strict proportionality between row and column marginal totals for target and initial matrices with the same scalar multiplier. Here the question arises: can we accept the initial matrix homothety as optimal solution for proportionality case of matrix updating problem?

At first sight, this solution can be appreciated as rather logical and, moreover, it allows preserving in target matrix the same location of zeros as in the initial matrix. However, it is to be emphasized that the above question indeed seems neither simple nor evident, and its proposition cannot be proved formally. Nevertheless, in most practical situations an affirmative answer to this question is almost obvious.

In the paper, it is shown that well-known and widely used RAS and Kuroda's method (and the Kullback – Leibler divergence approach, moreover) for matrix updating serve as an additional instrumental confirmation to such an answer because they easily pass through the homothetic test proposed. Thus, all matrices from the initial matrix homothetic family demonstrate an excellent structural similarity between each other. This conclusion can be helpful for refining a collection of matrix updating methods based on constrained minimization of the distance functions.

A quite common way to define a measure for the structural similarity between initial and target matrices is to use some matrix norm for their difference to be minimized subject to linear constraints. From the viewpoint of the homothetic test introduced in the paper, now one can set a goal to dispose the target matrix as close as possible not to initial matrix, but to its homothetic family. As a result, the solution of matrix updating problem cannot be "worse" (in terms of the matrix norm chosen) than the original one.

The RAS method is fairly associated with a more general notion of conditionally minimizing non-negative function called the Kullback – Leibler divergence that can be used in the information theory for comparing "true" and "test" probability distributions (it is not a distance function really, because the symmetry and triangle inequality conditions do not hold for it). The RAS method corresponds to not-so-evident choice of the initial matrix as "test" data array and the target matrix as "true" data array. More natural opposite order of arguments in Kullback – Leibler divergence function generates a new method of matrix updating that can be called "twin RAS method". The iterative algorithm for estimating the target matrix by this method is developed in the paper.

As an instance of a failure in the homothetic testing, the GRAS method for updating the matrices and tables with some negative entries is analyzed. Illustrative numerical examples are given.

Keywords: matrix updating methods, matrix homothety, RAS and Kuroda's methods, Kullback – Leibler divergence, homothetic test, twin RAS method

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1. An introduction to the matrix updating problems

A general problem of updating rectangular (or square) matrices can be formulated as follows. Let $\mathbf{A}$ be an initial matrix of dimension $N \times M$ with row and column marginal totals $\mathbf{u}_{\mathbf{A}} = \mathbf{A}\mathbf{e}_M$, $\mathbf{v}'_{\mathbf{A}} = \mathbf{e}'_N \mathbf{A}$ where $\mathbf{e}_N$ and $\mathbf{e}_M$ are $N \times 1$ and $M \times 1$ summation column vectors with unit elements.

Further, let $\mathbf{u} \neq \mathbf{u}_\mathbf{A}$ and $\mathbf{v} \neq \mathbf{v}_\mathbf{A}$ be exogenous column vectors of dimension N×1 and M×1, respectively. The problem is to estimate a target matrix **X** of dimension N×M at the highest possible level of its structural similarity (or resemblance, likeness, closeness, etc.) to initial matrix **A** subject to N+M equality constraints

$$\mathbf{X}\mathbf{e}_M = \mathbf{u}, \qquad \mathbf{e}'_N\mathbf{X} = \mathbf{v}', \tag{1}$$

and under the consistency condition

$$\mathbf{e}'_N\mathbf{u} = \mathbf{e}'_M\mathbf{v}. \tag{2}$$

It is assumed that initial matrix **A** does not include any zero rows or zero columns, does not have less than N+M nonzero elements, does not include any rows or columns with a unique nonzero element, and does not contain any pairs of rows and columns with four nonzero elements in the intersections. Otherwise, it is advisable to clean matrix **A** from those undesirable features before applying any matrix updating method in practice.

Clearly, the system of equations (1) is dependent at consistency condition (2) that provides an existence of target matrix **X**. However, it is easy to show that any N+M–1 among N+M constraints (1) are mutually independent. Furthermore, it is evident that any feasible solution of matrix updating problem **X** can be simply transformed into another one by letting, e.g.,

$$x_{nm}^{\text{new}} = x_{nm} + \varepsilon, \quad x_{nj}^{\text{new}} = x_{nj} - \varepsilon; \qquad x_{im}^{\text{new}} = x_{im} - \varepsilon, \quad x_{ij}^{\text{new}} = x_{ij} + \varepsilon$$

where ε is an arbitrary scalar, or

$$x_{nm}^{\text{new}} = x_{nm} + \varepsilon, \quad x_{nj}^{\text{new}} = x_{nj} - \varepsilon/2, \quad x_{nk}^{\text{new}} = x_{nk} - \varepsilon/2; \qquad x_{im}^{\text{new}} = x_{im} - \varepsilon, \quad x_{ij}^{\text{new}} = x_{ij} + \varepsilon/2, \quad x_{ik}^{\text{new}} = x_{ik} + \varepsilon/2,$$

and so on.

Thus, general problem of matrix updating significantly depends on a definition of the measure for structural similarity between initial and target matrices. Various definitions of this measure generate a great manifold of different methods and techniques for matrix updating. As Temurshoev et al. (2011, p. 92) rightly noted, "it is impossible to consider *all* updating methods, because theoretically their number is infinite".

2. A homothetic test for the matrix updating methods

A notion of structural similarity between initial and target matrices has a vague framework that can be slightly refined in an axiomatic manner. In this context let us consider a particular case of strict proportionality between row and column marginal totals $\mathbf{u} = k\mathbf{u}_\mathbf{A}$ and $\mathbf{v} = k\mathbf{v}_\mathbf{A}$ for target and initial matrices with the same scalar multiplier k. Here the main question arises: can we accept the matrix homothety $\mathbf{X} = k\mathbf{A}$ as optimal solution for proportionality case of a general matrix updating problem? At first sight this solution can be appreciated as rather logical and, moreover, it allows preserving in X the same location of zeros as in the initial matrix. However, it is to be emphasized that the above question indeed seems neither simple nor evident, and its proposition cannot be proved formally.

Nevertheless, in most practical situations an affirmative answer to this question is almost obvious. In particular, as it is shown below, the well-known and widely used RAS and Kuroda's methods for matrix updating serve as an additional instrumental confirmation to such an answer. In this connection, examining the property "if $\mathbf{u} = k\mathbf{u}_\mathbf{A}$ and $\mathbf{v} = k\mathbf{v}_\mathbf{A}$ then $\mathbf{X} = k\mathbf{A}$" we will call by a homothetic test for matrix updating method. It is natural to propose that a successful passing through homothetic test were to be appreciated as a positive feature of any matrix updating method.

3. Homothetic testing of RAS method

The key idea of the RAS method is triple factorization of target matrix

$$\mathbf{X} = \mathbf{RAS} = \langle\mathbf{r}\rangle\mathbf{A}\langle\mathbf{s}\rangle = \hat{\mathbf{r}}\mathbf{A}\hat{\mathbf{s}} \tag{3}$$

where $\mathbf{r}$ and $\mathbf{s}$ are unknown N×1 and M×1 column vectors. Here angled bracketing around a vector's symbol or putting a "hat" over it denotes a diagonal matrix, with the vector on its main diagonal and zeros elsewhere (see Miller and Blair, 2009, p. 697).

Putting (3) into (1), we have the system of nonlinear equations

$$\hat{\mathbf{r}}\mathbf{A}\hat{\mathbf{s}}\mathbf{e}_M = \hat{\mathbf{r}}\mathbf{A}\mathbf{s} = \langle\mathbf{A}\mathbf{s}\rangle\mathbf{r} = \mathbf{u}\,, \qquad \mathbf{e}'_N\hat{\mathbf{r}}\mathbf{A}\hat{\mathbf{s}} = \mathbf{r}'\mathbf{A}\hat{\mathbf{s}} = \mathbf{s}'\langle\mathbf{A}'\mathbf{r}\rangle = \mathbf{v}'\,.$$

Proper transformations of this system lead to following pair of iterative processes:

$$\mathbf{r}_{(i)} = \left\langle \mathbf{A}\left\langle \mathbf{A}'\mathbf{r}_{(i-1)}\right\rangle^{-1}\mathbf{v}\right\rangle^{-1}\mathbf{u}\,, \quad i = 1 \div I; \qquad \mathbf{s}_{(I)} = \left\langle \mathbf{A}'\mathbf{r}_{(I)}\right\rangle^{-1}\mathbf{v}\,; \tag{4}$$

$$\mathbf{s}_{(j)} = \left\langle \mathbf{A}'\left\langle \mathbf{A}\mathbf{s}_{(j-1)}\right\rangle^{-1}\mathbf{u}\right\rangle^{-1}\mathbf{v}\,, \quad j = 1 \div J; \qquad \mathbf{r}_{(J)} = \left\langle \mathbf{A}\mathbf{s}_{(J)}\right\rangle^{-1}\mathbf{u} \tag{5}$$

where i and j are iteration numbers, and the character " ÷ " between the lower and upper bounds of index's changing range means that the index sequentially runs all integer values in the specified range.

As concerning a homothetic test for RAS method at $\mathbf{u} = k\mathbf{u}_\mathbf{A}$ and $\mathbf{v} = k\mathbf{v}_\mathbf{A}$, it can be easily shown that under starting condition $\mathbf{r}_{(0)} = \mathbf{e}_N$ or $\mathbf{s}_{(0)} = \mathbf{e}_M$ the RAS method iterative process (4) or (5) demonstrates one-step convergence to pair of vectors $\mathbf{r} = \mathbf{e}_N$, $\mathbf{s} = k\mathbf{e}_M$ or to $\mathbf{r} = k\mathbf{e}_N$, $\mathbf{s} = \mathbf{e}_M$, respectively. Hence, RAS algorithm's implementation gives $r_n s_m = k$ for any n and m, $n = 1 \div N$, $m = 1 \div M$, from which $\mathbf{X} = \mathbf{r}\mathbf{s}' \circ \mathbf{A} = k\mathbf{A}$ where the character " $\circ$ " denotes the Hadamard's (element-wise) product of two matrices with the same dimensions. Besides, it is easy to see that replacing the initial matrix $\mathbf{A}$ with its homothety $k\mathbf{A}$ leaves the RAS method iterations (4) and (5) invariant. Thus, the RAS method passes through a homothetic test successfully.

4. Homothetic testing of Kuroda's method

Kuroda (1988) proposed an original method for matrix updating that reduces to constrained minimization of the twofold-weighted quadratic objective function

$$\mathrm{f}_{\mathrm{K}}\left(\mathbf{x}_{\mathbf{u}}, \mathbf{x}_{\mathbf{v}}\right) = \frac{1}{2}\mathbf{x}_{\mathbf{u}}'\mathbf{W}_1\mathbf{x}_{\mathbf{u}} + \frac{1}{2}\mathbf{x}_{\mathbf{v}}'\mathbf{W}_2\mathbf{x}_{\mathbf{v}} \tag{6}$$

where $\mathbf{W}_1$ and $\mathbf{W}_2$ are the nonsingular diagonal matrices of order NM with the relative reliability or relative confidence factors (weights), $\mathbf{x}_{\mathbf{u}}$ and $\mathbf{x}_{\mathbf{v}}$ are NM-dimensional column vectors that are defined through applying the vectorization operator "vec", which transforms a matrix into a vector by stacking the columns of the matrix one underneath the other (see, e.g., Magnus and Neudecker, 2007), as follows:

$$\mathbf{x}_{\mathbf{u}} = \mathrm{vec}\left(\hat{\mathbf{u}}^{-1}\mathbf{X} - \hat{\mathbf{u}}_{\mathbf{A}}^{-1}\mathbf{A}\right), \qquad \mathbf{x}_{\mathbf{v}} = \mathrm{vec}\left(\mathbf{X}\hat{\mathbf{v}}^{-1} - \mathbf{A}\hat{\mathbf{v}}_{\mathbf{A}}^{-1}\right).$$

Within a homothetic test for Kuroda's method, the row and column marginal totals for target matrix are $\mathbf{u} = k\mathbf{u}_{\mathbf{A}}$ and $\mathbf{v} = k\mathbf{v}_{\mathbf{A}}$, hence

$$\hat{\mathbf{u}}^{-1}\mathbf{X} - \hat{\mathbf{u}}_{\mathbf{A}}^{-1}\mathbf{A} = \hat{\mathbf{u}}_{\mathbf{A}}^{-1}\left(\frac{1}{k}\mathbf{X} - \mathbf{A}\right), \qquad \mathbf{X}\hat{\mathbf{v}}^{-1} - \mathbf{A}\hat{\mathbf{v}}_{\mathbf{A}}^{-1} = \left(\frac{1}{k}\mathbf{X} - \mathbf{A}\right)\hat{\mathbf{v}}_{\mathbf{A}}^{-1}.$$

Therefore, at $\mathbf{X} = k\mathbf{A}$ the vectors $\mathbf{x}_{\mathbf{u}}$ and $\mathbf{x}_{\mathbf{v}}$ vanish both, and the quadratic function (6) reaches its absolute minimum value equal to zero. It means that from viewpoint of Kuroda's method, the matrix homothety $\mathbf{X} = k\mathbf{A}$ provides the optimal solution for general problem of matrix updating in a case of strict proportionality between row and column marginal totals for target and initial matrices. Thus, Kuroda's method passes through a homothetic test successfully as well as RAS method.

5. Applying homothetic test to Kullback – Leibler divergence

The RAS method is associated with a more general notion of conditional minimizing non-negative function called the Kullback – Leibler divergence that can be used for comparing "true" and "test" probability distributions (see Kullback and Leibler, 1951). Letting $a = \mathbf{e}_N'\mathbf{A}\mathbf{e}_M$ and $x = \mathbf{e}_N'\mathbf{X}\mathbf{e}_M$, we have the first distribution as $\mathbf{A}/a$, and the second one – as $\mathbf{X}/x$, or possibly vice versa, but with much more vague interpretation. So all elements of $\mathbf{A}$ and $\mathbf{X}$ are implied to be non-negative.

In these denotations the Kullback – Leibler divergence (sometimes called "information gain") has genuine representation as

$$\mathrm{f}_{\mathrm{KL}}(\mathbf{A}/a;\mathbf{X}/x)=\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{a_{nm}}{a}\ln\left(\frac{x}{a}\frac{a_{nm}}{x_{nm}}\right)=\frac{1}{a}\mathrm{f}_{\mathrm{KL}}(\mathbf{A};\mathbf{X})+\ln\frac{x}{a} \tag{7}$$

and inverse representation, with an opposite order of its arguments, as

$$\mathrm{f}_{\mathrm{KL}}(\mathbf{X}/x;\mathbf{A}/a)=\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{x_{nm}}{x}\ln\left(\frac{a}{x}\frac{x_{nm}}{a_{nm}}\right)=\frac{1}{x}\mathrm{f}_{\mathrm{KL}}(\mathbf{X};\mathbf{A})+\ln\frac{a}{x} \tag{8}$$

where $\mathrm{f}_{\mathrm{KL}}(\mathbf{A};\mathbf{X})$ and $\mathrm{f}_{\mathrm{KL}}(\mathbf{X};\mathbf{A})$ are corresponding Kullback – Leibler functions for non-normalized data.

Thus, the approach based on the Kullback – Leibler divergence comes to minimization of objective function (7) or (8) subject to linear constraints (1) under the consistency condition (2). It is easy to see that in homothetic testing with $\mathbf{X}=k\mathbf{A}$ the non-negative functions (7) and (8) reach their absolute minimum (zero) values since

$$\mathrm{f}_{\mathrm{KL}}(\mathbf{A}/a;k\mathbf{A}/ka)=\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{a_{nm}}{a}\ln\left(\frac{ka}{a}\frac{a_{nm}}{ka_{nm}}\right)=\frac{a}{a}\ln(1)=0\,,$$

$$\mathrm{f}_{\mathrm{KL}}(k\mathbf{A}/ka;\mathbf{A}/a)=\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{ka_{nm}}{ka}\ln\left(\frac{a}{ka}\frac{ka_{nm}}{a_{nm}}\right)=\frac{ka}{ka}\ln(1)=0\,.$$

It means that from viewpoint of the Kullback – Leibler divergence approach the matrix homothety $\mathbf{X}=k\mathbf{A}$ can be considered as optimal solution for proportionality case of general problem of matrix updating. Thus, the Kullback – Leibler divergence approach passes through a homothetic test successfully as well as RAS and Kuroda's methods.

Notice, finally, that the function $\mathrm{f}_{\mathrm{KL}}(\mathbf{X};\mathbf{A})$ for non-normalized data from the inverse representation of Kullback – Leibler divergence (8), despite a certain shortcoming in its interpretation, serves as an objective function in mathematical programming formulation of the RAS method, e.g., in Appendix 7.1 "RAS as a Solution to the Constrained Minimum Information Distance Problem" to Miller and Blair (2009). However, it is to be emphasized that, strictly speaking, the Kullback – Leibler divergence is not a distance function really because the symmetry and triangle inequality conditions do not hold for it.

6. The twin RAS method

Recall that in RAS method the initial matrix **A** is used as a "test" probability distribution whereas the target matrix **X** forms "true" probability distribution. From viewpoint of information theory, it would be rather logical to accept an opposite order of matrix choice. Then the following mathematical programming problem arises: to minimize objective function $\mathrm{f}_{\mathrm{KL}}(\mathbf{A};\mathbf{X})$ from formula (7) subject to linear constraints (1) under the consistency condition (2).

The Lagrangean function for this problem is

$$\mathrm{L}_{\mathbf{AX}}(\mathbf{X};\boldsymbol{\lambda},\boldsymbol{\mu})=\sum_{n=1}^{N}\sum_{m=1}^{M}a_{nm}\ln\left(\frac{a_{nm}}{x_{nm}}\right)-\sum_{n=1}^{N}\lambda_n\left(\sum_{m=1}^{M}x_{nm}-u_n\right)-\sum_{m=1}^{M}\mu_m\left(\sum_{n=1}^{N}x_{nm}-v_m\right)$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are column vectors of Lagrange multipliers with dimensions $N\times1$ and $M\times1$. The appropriate first partial derivatives of Lagrangean function with respect to x_{nm} are

$$\frac{\partial\,\mathrm{L}_{\mathbf{AX}}}{\partial x_{nm}}=-\frac{a_{nm}}{x_{nm}}-\lambda_n-\mu_m=0\,,\quad n=1\div N,\ \ m=1\div M.$$

These equations can be resolved with respect to x_{nm} as

$$x_{nm}=\frac{-a_{nm}}{\lambda_n+\mu_m}\,,\quad n=1\div N,\ \ m=1\div M. \tag{9}$$

Inserting (9) into two constraints (1) gives

$$-\sum_{m=1}^{M}\frac{a_{nm}}{\lambda_n+\mu_m}=u_n\,,\ n=1\div N,\qquad -\sum_{n=1}^{N}\frac{a_{nm}}{\lambda_n+\mu_m}=v_m\,,\ \ m=1\div M,$$

from which we have the following pair of iterative processes:

$$\lambda_n^{(i)}=-\frac{1}{u_n}\sum_{m=1}^{M}\frac{a_{nm}}{1+\mu_m^{(i-1)}\big/\lambda_n^{(i-1)}}\,,\ n=1\div N,\qquad \mu_m^{(i)}=-\frac{1}{v_m}\sum_{n=1}^{N}\frac{a_{nm}}{1+\lambda_n^{(i)}\big/\mu_m^{(i-1)}}\,,\ m=1\div M; \tag{10}$$

$$\mu_m^{(j)}=-\frac{1}{v_m}\sum_{n=1}^{N}\frac{a_{nm}}{1+\lambda_n^{(j-1)}\big/\mu_m^{(j-1)}}\,,\ m=1\div M,\qquad \lambda_n^{(j)}=-\frac{1}{u_n}\sum_{m=1}^{M}\frac{a_{nm}}{1+\mu_m^{(j)}\big/\lambda_n^{(j-1)}}\,,\ n=1\div N, \tag{11}$$

with starting conditions $\boldsymbol{\lambda}=\mathbf{e}_N$ and $\boldsymbol{\mu}=\mathbf{e}_M$, where i and j are iteration numbers.

The twin RAS method implies the iterative calculations using (10) or (11) and the consequent finding the elements of target matrix **X** by formula (9). Practical computations testify that the twin RAS method converges together with RAS method. It is clear from Section 5 that the twin RAS method passes through a homothetic test successfully.

7. Improving the matrix updating methods within a homothetic paradigm

Acceptance of the matrix $\mathbf{X}=k\mathbf{A}$ as optimal solution for proportionality case of a general matrix updating problem leads to establishing the fact that the matrices from homothetic family $k\mathbf{A}$ demonstrate an excellent structural similarity between each other. This homothetic paradigm can be helpful for improving a collection of matrix updating methods based on constrained minimization of the distance (or quasi-distance) functions.

A quite common approach to define a measure for the structural similarity between initial and target matrices is to use some matrix norm for the difference $\mathbf{X}-\mathbf{A}$ to be minimized subject to linear constraints (1) under the consistency condition (one can find the proper reviews, e.g., in Miller and Blair, 2009 and Temurshoev et al., 2011), so that the optimal solution is

$\mathbf{X}^* = \arg\min_{\mathbf{X}} \|\mathbf{X} - \mathbf{A}\|$.

However, within homothetic paradigm we can set a goal to dispose the target matrix as close as possible not to initial matrix **A**, but to its homothetic family $k\mathbf{A}$. As a result, the optimal solution becomes

$$(\mathbf{X}^*, k^*) = \arg\min_{\mathbf{X},k} \|\mathbf{X} - k\mathbf{A}\|$$

and, clearly, it cannot be "worse" (in terms of the certain matrix norm chosen) than the original one.

The problem to minimize the distance between target matrix **X** and uniparametrical family $k\mathbf{A}$ is presented above in preliminary formulation. The further handling of this problem becomes more operational with a vectorization of matrices **A** and **X** by transforming them into *NM*-element column vectors, respectively, $\mathbf{a} = \text{vec}\mathbf{A}$ and $\mathbf{x} = \text{vec}\mathbf{X}$. It is fruitful to express the latter vector in form of multiplicative pattern $\mathbf{x} = \hat{\mathbf{a}}\mathbf{q}$ where **q** is *NM*-dimensional column vector of unknown relative coefficients. Note that diagonal matrix $\hat{\mathbf{a}}$ is singular if the initial matrix contains at least one zero element.

In vector notation the transition from $\|\mathbf{x} - \mathbf{a}\|$ to $\|\mathbf{x} - k\mathbf{a}\|$ leads to an idea of orthogonal projecting an unknown target vector **x** onto the homothetic ray $k\mathbf{a}$ in *NM*-dimensional vector space with scalar product operation. To make this vector norm minimization problem independent on scale of initial data, it is expedient to introduce into consideration the relative distance function $\|\mathbf{q} - k\mathbf{e}_{NM}\|$ instead of $\|\mathbf{x} - k\mathbf{a}\|$, i.e., to consider the orthogonal projection of an unknown target vector **q** onto the relative homothetic ray $k\mathbf{e}_{NM}$.

8. Numerical examples and concluding remarks

Consider the Eurostat input–output data set given in "Box 14.2: RAS procedure" (see Eurostat, 2008, p. 452) for compiling some numerical examples. The 3×4-dimensional initial matrix **A** combines the entries in intersections of the columns "Agriculture", "Industry", "Services", "Final d." with the rows "Agriculture", "Industry", "Services" in "Table 1: Input-output data for year 0". Note that all the elements of this matrix are nonzero. The row marginal total vector **u** of dimension 3×1 is the proper part of the column "Output" in "Table 2: Input-output data for year 1", and the column marginal total vector $\mathbf{v}'$ of dimension 1×4 involves the proper entries of the row "Total" in the near-mentioned data source.

Initial matrix **A** and marginal totals **u**, $\mathbf{v}'$ are presented in Table 1. The first numerical example is to handle the data set available by RAS method with iterative processes (4) or (5) and by twin RAS method with iterative processes (10) or (11). The computation results are grouped in the left half of Table 2 for RAS method and in the right half of Table 2 for twin RAS method; they seem to be very similar among themselves.

Table 1 Initial matrix **A,** Eurostat (2008), p. 452

	$\mathbf{A}$				$\mathbf{u}_A$	$\mathbf{u}$
	20.00	34.00	10.00	36.00	100.00	94.78
	20.00	152.00	40.00	188.00	400.00	412.86
	10.00	72.00	20.00	98.00	200.00	212.68
$\mathbf{v}'_A$	50.00	258.00	70.00	322.00	700.00	
$\mathbf{v}'$	47.28	268.02	73.58	331.44		720.32

Table 2 RAS and twin RAS results for updating of data set from Table 1

RAS	$\mathbf{X}$				$\mathbf{u}_X$	$\mathbf{u}$
	17.94	32.77	9.76	34.31	94.78	94.78
	19.36	158.08	42.12	193.30	412.86	412.86
	9.98	77.17	21.70	103.84	212.68	212.68
$\mathbf{v}'_X$	47.28	268.02	73.58	331.44	720.32	
$\mathbf{v}'$	47.28	268.02	73.58	331.44		720.32

twin RAS	$\mathbf{X}$				$\mathbf{u}_X$	$\mathbf{u}$
	18,00	32,74	9,74	34,30	94,78	94.78
	19,34	158,09	42,12	193,31	412,86	412.86
	9,94	77,19	21,72	103,83	212,68	212.68
$\mathbf{v}'_X$	47,28	268,02	73,58	331,44	720,32	
$\mathbf{v}'$	47.28	268.02	73.58	331.44		720.32

As it is well-known, "… RAS can only handle non-negative matrices, which limits its application to SUTs that often contain negative entries…" – see Temurshoev et al. (2011, p. 92). For the initial matrices with some negative entries, the generalized RAS (GRAS) method has been proposed by Junius and Oosterhaven (2003) and later redeveloped by Lenzen et al. (2007). (Notice that in the absence of negative entries GRAS method coincides with RAS method.) The next numerical example is assigned to verify GRAS method's response to homothetic testing.

Let us disturb three elements of data set from Table 1, say (1, 3), (3, 1) and (3, 3), by reversing their sign for year 0. After proper recalculation of the initial marginal totals $\mathbf{u}_A, \mathbf{v}_A$ we obtain the data set located in Table 3.

Table 3 Initial matrix **A** with some negative entries

	$\mathbf{A}$				$\mathbf{u}_A$
	20,00	34,00	**-10,00**	36,00	80,00
	20,00	152,00	40,00	188,00	400,00
	-10,00	72,00	**-20,00**	98,00	140,00
$\mathbf{v}'_A$	30,00	258,00	10,00	322,00	620,00

For homothetic testing let at first $k = 2$ and then $k = 3$. The results of computations by GRAS method are grouped in the both halves of Table 4 respectively. They demonstrate large deviations of the elements of first target matrix calculated from the doubled matrix 2**A**, and at $k = 3$ these deviations become noticeably larger.

Table 4 GRAS results for updating the data set from Table 3 (doubled and tripled)

GRAS	$\mathbf{X}$				$\mathbf{u}_\mathbf{X}$	$2\mathbf{u}_\mathbf{A}$
	32,11	66,37	**-8,93**	70,44	160,00	160,00
	34,65	320,14	48,32	396,89	800,00	800,00
	-6,76	129,49	**-19,39**	176,66	280,00	280,00
$\mathbf{v}'_\mathbf{X}$	60,00	516,00	20,00	644,00	1240,00	
$2\mathbf{v}'_\mathbf{A}$	60,00	516,00	20,00	644,00		1240,00

GRAS	$\mathbf{X}$				$\mathbf{u}_\mathbf{X}$	$3\mathbf{u}_\mathbf{A}$
	45,00	98,43	**-7,97**	104,54	240,00	240,00
	49,95	488,41	55,72	605,93	1200,00	1200,00
	-4,95	187,17	**-17,75**	255,53	420,00	420,00
$\mathbf{v}'_\mathbf{X}$	90,00	774,00	30,00	966,00	**1860,00**	
$3\mathbf{v}'_\mathbf{A}$	90,00	774,00	30,00	966,00		1860,00

Thus, it is important to emphasize that some matrix updating methods do not pass through a homothetic test, inter alia, the GRAS method can serve as an example of negative response to homothetic testing.

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