Syllabus for the course "Iwasawa Theory"

Area of Specialisation: 01.06.01 Mathematics and Mechanics

Doctoral programs in
01.01.03 Mathematical Physics
01.01.04 Geometry and Topology
01.01.05 Probability Theory and Mathematical Statistics
01.01.06 Mathematical Logics, Algebra and Theory of Numbers

Approved by the Academic Council of the Doctoral School in Mathematics
on 9th October, 2015

Moscow - 2015

Syllabus

1. Course Description
    a. Title of a Course: Iwasawa Theory
    b. Pre-requisites: Basic knowledge of Algebraic Number Theory
    c. Course Type: optional
    d. Abstract

Let p be an odd prime. In the course of his work on Fermat's Last Theorem, Kummer discovered a connection between arithmetic of fields generated over $\mathbb{Q}$ by p-th roots of unity, and the values of Riemann zeta function at odd negative integers. Besides that, Kummer established congruences between these values that lead to a definition of a p-adic analogue of the Riemann zeta function by interpolation. Almost a century later Iwasawa made equally important discovery that p-adic analogue of the Riemann zeta function is related to arithmetic of fields generated over $\mathbb{Q}$ by *all* p-primary roots of unity.

Iwasawa theory comes out of the following idea of him: in some infinite towers of fields it is often easier to describe all Galois modules simultaneously, not just one of them.

Iwasawa studied the example of p-Sylow subgroups I_n of the ideal class group of the field K_n in the tower of fields $K_1 \subset K_2 \subset K_3 \subset \ldots \subset$ such that $\mathrm{Gal}(K_n|K_1) \cong \mathbb{Z}/p^n\mathbb{Z}$ for a prime p. (In the case of fields $K_n = \mathbb{Q}(\mu_{p^n})$ generated over $\mathbb{Q}$ by p^n-th roots of unity, the group I_1 was known already to Kummer as a principal obstruction to a direct proof of Fermat's Last Theorem.) In a natural way, I_n's are $\mathbb{Z}_p$-modules, and also modules over $G_n = \mathrm{Gal}(K_n|K_1)$. It is not, however, convenient to work with the group ring $\mathbb{Z}_p[G_n]$ since, e.g., it is not a domain.

There are norm maps of $\mathbb{Z}_p[G_m]$-modules $I_m \to I_n$ for $m > n$, so that the inverse limit $\varprojlim I_n$ is a module over the inverse limit $\Lambda := \varprojlim \mathbb{Z}_p[G_n]$. It is much easier to understand the structure of the ring Λ: it is a complete 2-dimensional regular local ring, (non-canonically) isomorphic to the ring of power series $\mathbb{Z}_p[[T]]$. There is a theorem describing the structure of modules over Λ. Here is a simplest example of consequence of this theorem: p divides the class number of one of K_n's of and only if it divides the class number of all K_n's.

There is a profound connection of this theory with special values of L-functions. Iwasawa's idea, called Iwasawa's Main Conjecture, states that the "characteristic ideal" of the module I_n in the ring Λ admits a generator, which is, in a sense, p-adic L-function. This is shown, in particular, for all totally real number fields K_0.

2. Learning Objectives

The goals are proofs of Iwasawa's Main Conjecture in some cases and, in particular, the following formula:

$$\zeta(1-2k) = (-1)^k \prod_p \frac{|H^1_{\text{ét}}(\mathbb{Z}[1/p], \mu_{p^N}^{\otimes 2k})|}{|H^0_{\text{ét}}(\mathbb{Z}[1/p], \mu_{p^N}^{\otimes 2k})|} \quad \text{for all integer } k \geq 1 \text{ and sufficiently big } N.$$

3. Learning Outcomes

Students will learn the proof of Iwasawa's Main Conjecture

4. Course Plan

- Review of class field theory
- Iwasawa Algebras and p-adic (pseudo)-measures
- The algebra Zp[[T]], structure of Zp[[T]]-modules, isomorphism onto Iwasawa algebra
- Iwasawa's theory of Zp-extensions
- Complex L-Functions, functional equations, values at negative integers
- p-adic L-Functions
- Euler Systems
- Main Conjecture

5. Reading List

   a. Required

Elliptic curves and number theory, lecture notes by R. Sujatha

https://www.ias.ac.in/public/Resources/Events/Mid_Year_Meetings/16_sramdorai.pdf

   b. Optional

Iwasawa theory: a climb up the tower, by Romyar Sharif

http://math.ucla.edu/~sharifi/iwanot.pdf

6. Grading System

There will be (i) a problem list, based on [6], (ii) a list of available talks. The maximal grade will correspond to general understanding of the subject and to the most of the problems solved. A part of the required problems can be replaced by a talk.

7. Guidelines for Knowledge Assessment

Sample problems:

Let $L/\mathbb{Q}$ be a Galois extension of degree p and E be an elliptic curve defined over $\mathbb{Q}$. Let p be a fixed prime (of good ordinary reduction if required). We use L_∞, $\mathbb{Q}_\infty$ as the notation for the cyclotomic $\mathbb{Z}_p$-extension. Let $w|p$ be primes in L. We have $L_{\infty,w}/\mathbb{Q}_{\infty,p}$, a Galois extension and say the Galois group is denoted by G.

How does one one compute the Galois cohomology $H^i(G, \oplus_{w|p} H^1(L_{\infty,w}, E_{p^\infty}))$ for $i = 1, 2$.

Let E, E' be elliptic curves over $\mathbb{Q}$ and also suppose they are p-isogenous. How are the Euler systems corresponding to the two isogenous elliptic curves related, if at all?

8. Methods of Instruction

Lectures, problem sessions and self-study.

9. Special Equipment and Software Support (if required)

No requirements.

Competences: UK-1, 2, 5, PK-1, OPK-1,2 (according to *01.06.01 Mathematics and Mechanics* Educational Standard).