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Simulation of the Particle Dynamics in the Two-Dimensional Poiseuille Flow with Low Reynolds Number

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Abstract—A particle moving in the Poiseuille flow in a channel experiences the influence of the velocity gradient and the channel walls. Several effects happen with the particle – velocity retardation, rotation, and migration transversal to the main flow direction. We simulate dynamics of a hard particle in a narrow two-dimensional channel and estimate the variation with the channel width of the particle velocity, frequency of rotation, and time to reach the steady state. We provide a comparison of our results with the existing analytical calculations. Simulations were done using the Lattice Boltzmann Method for the fluid flow and Immersed Boundary Method for the particle movement.

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1. INTRODUCTION

The motion of particles in a channel currently has an enormous interest, connected with the intensive research of the particle dynamics in microfluidic devices [1–3], with the red blood passage through vessels [4], and tumor cells spreading [5]. The common feature of the subject is the low Reynolds number of the flow and the finite-width geometry of the channel. A particle in a narrow channel experiences the influence of the surrounding viscous fluid and the proximity to the channel walls. So, one can expect that a particle in the channel to move axially in the flow direction and to move transversally due to the velocity gradient of the Poiseuille flow and the proximity to the channel walls. In addition to the translational motion of the particle as the whole, the particle will rotate due to the velocity difference between the sides closer to and further away from the walls.

The paper presents a simulation of Poiseuille flow in the two-dimensional channel of width W, with an immersed hard particle (a disk) with neutral buoyancy and a radius R in the viscous fluid. The Poiseuille flow is axisymmetric and non-homogeneous in the direction transversal to the axis. It is known from early experiments [6–8] that particles can migrate axially in the Poiseuille flow to the steady state, which can be out of the axis.

Analytical considerations suggest that a particle in the unbounded flow experiences the lifting force, which is a function of the shear gradient, particle radius, viscosity, density, and relative velocity [9, 10]. Interestingly, particle in the Poiseuille flow moves slower than the surrounding fluid due to the finite geometry. This effect was analytically treated by Simha [11], who suggests that the relative velocity of the cylinder in the channel is proportional to the square of the confinement ratio $\gamma = 2R/W$ multiplied

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by the value of the Poiseuille flow at the axis, $u_{rel} = 2/3\gamma^2 u_{max}$. In turn, the relative velocity produces the lifting force [9, 10].

In simulations of the disk movement in the two-dimensional Poiseuille flow, the fluid from the left side fills the channel with the length L and width W. The pressure difference, applied at the left and right sides of the channel produces the parabolic velocity profile, the Poiseuille flow,

$$u_{x=0} = u_{x_L} = u_{max}\left(1 - \left(\frac{2y}{W}\right)^2\right),$$

where y is the distance from the axis, $y \in (-W/2, W/2)$.

The initial velocity v of the disk immersed in the fluid is equal to the fluid velocity at the disk's center. The disk's center is placed at some distance y_0 from the axis.

We measure in simulations the relative velocity u_{rel} of the disks, the rotation frequency, and the time to reach the steady-state position as a function of the viscosity and confinement ratio. In the case of the small Reynolds numbers of our computer experiments, $Re \approx 1$ and neutral particle buoyancy, we found that the only parameter that selects the lifting force's direction is the confinement ratio γ. We also check that the relative velocity and frequency of rotation are the functions of the confinement ratio, as predicted analytically by Simha [11].

2. SIMULATION MODEL

The fluid simulated using lattice Boltzmann method (LBM) [12]. The distribution function $f_i(\vec{x}, t)$ represents the density of particles with velocities $\vec{c_i}$ $(i = 0, 1, \ldots, 8)$ at the grid position $\vec{x}$ at time t. We use the D2Q9 scheme with nine velocities $\vec{c_i}$ pointing towards the four corners of the square around the grid position $\vec{x}$, to the four sides of the square, and to the center (zero velocity).

Distribution function changes using the conventional LBM equations with the relaxation term proportional to the difference between the current $f_i(\vec{x}, t)$ and equilibrium distributions $f_i^{eq}(\vec{x}, t)$ [13]

$$f_i(\vec{x} + \vec{c_i}\Delta x, t + \Delta t) = f_i(\vec{x}, t) - \frac{\Delta t}{\tau}\left[f_i(\vec{x}, t) - f_i^{eq}(\vec{x}, t)\right]$$

with the discrete equilibrium distribution function

$$f_i^{eq}(\vec{x}, t) = w_i \rho\left(1 + \frac{\vec{u}\cdot\vec{c_i}}{c_s^2} + \frac{(\vec{u}\cdot\vec{c_i})^2}{c_s^4} - \frac{\vec{u}\cdot\vec{u}}{2c_s^2}\right),$$

where weights w_i correspond to a discrete set of velocities and c_s is the sound velocity. The density is calculated as $\rho(\vec{x}, t) = \sum_i f_i(\vec{x}, t)$ followed by the velocity calculation $\vec{u}(\vec{x}, t) = \sum_i \vec{c_i} f_i(\vec{x}, t)/\rho(\vec{x}, t)$.

Hard disks of the radius $R\Delta x$ are immersed in the moving fluid (here Δx is the lattice spacing). The mutual interaction force $\vec{g}$ between fluid and hard disk calculated using the distribution function. The disk boundary is represented as a set of Lagrangian points [14] $\vec{X_k}(t + \Delta t)$ and $\vec{U_k}(t + \Delta t)$, $(k = 1, \ldots, N)$ are velocities at those points. The fluid velocity $\vec{u}'$ at the Lagrangian points $\vec{X_k}$ are calculated using interpolation (we use dimensionless variables following Inamuro's review paper [14]).

$$\vec{u}'(\vec{X_k}, t + \Delta t) = \sum_{\vec{x}} \vec{u}'(\vec{x}, t + \Delta t) W(\vec{x} - \vec{X_k})$$

with summation over all lattice nodes $\vec{x}$, and the weighting function W is defined [15] by

$$W(x, y, z) = w\left(\frac{x}{\Delta x}\right) w\left(\frac{y}{\Delta x}\right),$$

$$w(r) = \begin{cases} 1/8\left(3 - 2|r| + \sqrt{1 + 4|r| - 4r^2}\right), & |r| \le 1, \\ 1/8\left(5 - 2|r| - \sqrt{-7 + 12|r| - 4r^2}\right), & 1 \le |r| \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

The force $\vec{g}(\vec{x}, t + \Delta t)$ acting on the particle is calculated iteratively [14].

The hard disk motion is simulated with the classical mechanics equations of motion, with the linear velocity $\vec{U}$ evolution

$$M\frac{d\vec{U}(t)}{dt} = \vec{F}(t),$$

and angular velocity evolution

$$\frac{1}{2}MR^2\frac{d\omega}{dt} = T_z.$$

where M is the disk mass, $\vec{F}(t) = -\sum \vec{g}(t)$ is the total force acting on the disk, and T_z is the component of the force moment, $\vec{T} = -\sum \vec{r} \times \vec{g}$.

3. SIMULATION RESULTS

In the following, we will use dimensionless variables, and all LBM variables are normalized by the particle radius $R\Delta x$ and by the maximum velocity u_{max} of Poiseuille flow. The normalized variables are denoted with a tilde. Therefore, in our units, the particle radius $\tilde{R}$=1 and maximum velocity $\tilde{u}_{max}$=1.

Table 1 presents results of the simulation of buoyant disk movement in the channel with confinement ratio γ. The disk moves to the steady-state position, which is on the axis for the values of $\gamma = 1/2$ and 1/4, and some intermediate position between the axis and the wall for smaller values of γ. Disks rotate due to the velocity gradient with the period $\tilde{T}$ presented in the second column of Table 1—the corresponding angular velocity $\tilde{\Omega} =$ in the third column, and the relative velocity u_{rel} in the last column of the table.

The fit to the data of relative velocity gives $\tilde{u}_{rel} \approx 0.51\gamma^2\tilde{u}_{max}$, which is the good compared with the analytical estimation $u_{rel} = 2/3\gamma^2 u_{max}$ by Simha [11]. We have to note that the numerical value of coefficients can change because of the assumptions used in the analytical calculations.

The normalized frequency of rotation $\tilde{\omega} = 2\pi/\tilde{T}_1$ numerically coincides with the normalized angular velocity $\tilde{\Omega} = 2\pi/\tilde{T}_1\tilde{R}$ (since, in our units, $\tilde{R} = 1$), and the fit to the data in third column of Table 1 gives $\tilde{\omega} \approx 0.26\gamma$. The fit coincides well with the analytical estimates [11].

In our simulations, we checked the sensitivity of the fits to the initial position of the disk in the tube $\tilde{y}_0 = 1$ and for other values of Reynolds number. Tables 2–5 show that the relative speed of the disk is independent of the initial position of the disk. The period T_1 presented in the tables is the first circle of rotation which depends on the velocity gradient of the flow at the initial disk position. It is smaller with the disk's smaller shift $\tilde{y}_0$. This dependence coincides nicely with the functional dependence of the angular rotation estimated by Saffman [10] $\tilde{\omega} \propto \sqrt{\tilde{y}}/\tilde{W}$, which is the square root of the transversal velocity gradient.

The velocity gradient will produce the lifting force, moving the disk from the initial position to the axis or from the axis. In the case of the small Reynolds numbers of our computer experiments, $Re \approx 1$ and neutral particle buoyancy, the only parameter that selects the lifting force's direction is the confinement ratio γ.

Table 1. The confinement ratio γ, period $\tilde{T}_1$ of the first particle rotation, angular velocity $\tilde{\Omega}$, and relative velocity of the particle $\tilde{u}_{rel}$. Reynolds number $Re = 1/4$. Initial position of the disk is $\tilde{y}_0 = 1$.

γ	$\tilde{T}_1$	$\tilde{\Omega}$	$\tilde{u}_{rel}$
1/2	33.96	0.1850	−0.1352
1/4	97.44	0.0645	−0.0333
1/6	215.40	0.0292	−0.0128
1/8	382.80	0.0164	−0.0101
1/10	599.52	0.0105	−0.0054

Table 2. The initial position of the disk $\tilde{y}_0$, period $\tilde{T}_1$ of the first particle rotation, corresponding frequency $\tilde{\omega}$, and relative speed of the particle $\tilde{u}_{rel}$. $Re = 0.250$, $\gamma = 1/2$

$\tilde{y}_0$	$\tilde{T}_1$	$\tilde{\omega}$	$\tilde{u}_{rel}$
1	33.96	0.1850	−0.1352
1/2	57.24	0.1098	−0.1380
1/4	125.40	0.0501	−0.1372

Table 3. The initial position of the disk $\tilde{y}_0$, period $\tilde{T}_1$ of the first particle rotation, corresponding frequency $\tilde{\omega}$, and relative speed of the particle $\tilde{u}_{rel}$. $Re = 0.375$, $\gamma = 1/2$

$\tilde{y}_0$	$\tilde{T}_1$	$\tilde{\omega}$	$\tilde{u}_{rel}$
1	30.84	0.2037	−0.1505
1/2	55.80	0.1126	−0.1452
1/4	124.80	0.0503	−0.1445

Table 4. The initial position of the disk $\tilde{y}_0$, period $\tilde{T}_1$ of the first particle rotation, corresponding frequency $\tilde{\omega}$, and relative speed of the particle $\tilde{u}_{rel}$. $Re = 0.250$, $\gamma = 1/6$

$\tilde{y}_0$	$\tilde{T}_1$	$\tilde{\omega}$	$\tilde{u}_{rel}$
1	200.19	0.0314	−0.0128
1/2	432.36	0.0145	−0.0324
1/4	850.68	0.0074	−0.0297

Table 5. The initial position of the disk $\tilde{y}_0$, period $\tilde{T}_1$ of the first particle rotation, corresponding frequency $\tilde{\omega}$, and relative speed of the particle $\tilde{u}_{rel}$. $Re = 0.375$, $\gamma = 1/6$

$\tilde{y}_0$	$\tilde{T}_1$	$\tilde{\omega}$	$\tilde{u}_{rel}$
1	30.84	0.2037	−0.1505
1/2	55.80	0.1126	−0.1452
1/4	1145.16	0.0055	−0.0053

Table 6. The confinement ratio $\gamma = 2\tilde{R}/\tilde{W}$ and steady state level $\beta = 2\tilde{y}_{st}/\tilde{W}$. $Re = 1$.

γ	β
1/2	0
1/6	0.267
1/8	0.311
1/10	0.335

The confinement ratio also selects the level $\beta = 2\tilde{y}_{st}/\tilde{W}$ of the steady flow position. After the transient time, depending on the initial position of a particle in the channel, the particle reaches the equilibrium position at a distance y_{st} from the axis. In the experiment of particle suspension flowing in

Poiseuille flow of viscous fluid was found that particles concentrated around some finite distance both from the axis and wall [6]. This effect was supported in the experiments of other groups [7, 8] and there are some analytical estimations of the position [9, 10]. This effect is also obtained in the simulations. See for extended discussion the recent paper [16].

Table 6 shows the estimation of the steady position of the particle in the channel with $Re = 1$, which demonstrates the variation of the steady state level β with confinement ratio.

4. DISCUSSION

We simulate a disk motion in the two-dimensional Poiseuille flow and find a good agreement of our simulation results for the relative speed flow and disk rotation frequency with the known analytical estimates for a low Reynolds number. The particle's steady state position in the channel depends on the confinement ratio, qualitatively in line with the analytical estimations. The combination of the Lattice Boltzmann Method and Immersed Boundary Method can accurately simulate the shear flow in two-dimensional geometry.

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