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Fixed-time Consensus Algorithm for Multi-agent Systems with Integrator Dynamics

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Abstract: The paper addresses the problem of exact average-consensus reaching in a prespecified time. The communication topology is assumed to be defined by a weighted undirected graph and the agents are represented by integrators. A nonlinear control protocol which ensures a finite-time convergence is proposed. With the designed protocol, any a priori specified convergence time can be guaranteed regardless of the initial conditions.

Keywords: multi-agent systems, consensus, fixed-time convergence.

1. INTRODUCTION

The consensus or agreement problem is a key problem in decentralized cooperative control of multi-agent systems Olfati-Saber et al. (2007), Ren & Cao (2011). This is partly due to many important applications in control of spacecrafts, mobile robots, UAVs, sensor networks and other fields such as optimization. In the leaderless consensus problem, all nodes/agents are required to converge to a common value which is not prespecified in advance (in other words, the disagreement between the agents in the system is to be minimized to zero).

In Olfati-Saber & Murray (2004), a classical continuous linear consensus control protocol was studied for networks of integrators. It was shown that the second smallest eigenvalue of the Laplacian matrix of the interaction graph, called *algebraic connectivity*, determines the convergence rate of consensus algorithms. Evidently, the protocols that provide high convergence rate are more preferred in applications. In Xiao & Boyd (2004), Shafi et al. (2011), the problems of vertex/edge weight design were considered that provide the desired spectra of the graph Laplacians. It should be noted that, by maximizing the algebraic connectivity, a better performance of linear algorithms can be obtained; however, the agreement still can be reached just asymptotically. Moreover, the convergence time essentially depends on the initial conditions of the agents. On the other hand, in numerous practical applications, the transient processes are to be completed in a prescribed time.

The *main objective* of this paper is to design an average-consensus control protocol which provides the multi-agent system with the *finite-time* convergence property; on top of that, *any guaranteed settling time is to be specified in advance regardless of the initial positions of the agents.*

The theory of finite-time stability and stabilization problems has been the subject of intensive research in the recent years; e.g., see Haimo (1986), Bhat & Bernstein (2000), Moulay & Perruquetti (2006), Orlov (2009), Polyakov & Poznyak (2012). For use of finite-time control ideology in consensus problems, see Cortés (2006), Hui et al. (2010), Wang & Xiao (2010), Xiao et al. (2009). Finite-time stability analysis usually exploits the theory of non-smooth Lyapunov functions and involves such concepts as weak and strong stability, differential inclusions, generalized gradients and derivatives, Roxin (1966), Hui et al. (2010).

Obviously, there is a great need in finite-time consensus algorithms; moreover algorithms that guarantee any predefined convergence time *regardless of the initial conditions of the agents* are most desired. The corresponding modification of the *finite-time stability* was called *fixed-time stability* in Polyakov (2012). Fixed-time algorithms can also be found in Andrieu et al. (2008) and Cruz-Zavala et al. (2011). In Parsegov et al. (2012), a fixed-time control protocol for a specific formation control problem was designed.

Polynomial state feedback control system design has attracted considerable attention in nonlinear control, see Ebenbauer & Allgöwer (2006). This class of control systems appears in models of a wide range of applications such as chemical processes, electronic circuits and mechatronics, biological systems, etc.

This paper presents new consensus control protocols of the polynomial type which guarantee *fixed-time convergence* to a common value.

The paper is organized as follows. The next section presents the notations used in the paper. Section 3 discusses the statement of the problem and basic assumptions, while Section 4 introduces mathematical preliminaries needed for further exposition. The main theoretical

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result is formulated and proved in Section 5. Finally, the results of numerical simulations and conclusions are given.

2. NOTATIONS

The following notation will be used throughout the paper:

- $\mathbb{R}$ is the set of real numbers; $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$;
- $D^*\varphi(t)$ denotes the upper right-hand Dini derivative of the function $\varphi : \mathbb{R} \to \mathbb{R}$ at the point $t \in \mathbb{R}$, i.e.
$$D^*\varphi(t) := \limsup_{h\to+0} \frac{\varphi(t+h)-\varphi(t)}{h};$$
- the following sign-preserving exponentiation operator is used:
$$s^{[k]} := \text{sign}(s)|s|^k, \tag{1}$$
where $s \in \mathbb{R}$ and $\text{sign}(s)$ is the sign function
$$\text{sign}(s) = \begin{cases} 1 & \text{if } s > 0, \\ -1 & \text{if } s < 0, \\ 0 & \text{if } s = 0. \end{cases}$$
For instance, $(-2)^{[2]} = -4$.
For the matrix variable $A \in \mathbb{R}^{n\times m}$, this operator $A^{[p]}$ is understood componentwise. For example,
$$A = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}, \quad A^{[2]} = \begin{bmatrix} 1 & -4 \\ 9 & 0 \end{bmatrix}.$$
- $\lambda_*(P)$ is the smallest *positive* eigenvalue of the symmetric matrix $P \in \mathbb{R}^{n\times n} : P = P^\top$;
- for any $p \geq 1$, the l_p-norm of the vector $z \in \mathbb{R}^n$ is defined by
$$\|z\|_p := \left(\sum_{i=1}^n |z_i|^p\right)^{\frac{1}{p}}; \tag{2}$$
- the unit vector is defined by $\mathbf{1}_n = [1, 1, \ldots, 1]^\top \in \mathbb{R}^n$;
- given $\nu = [\nu_1, ..., \nu_n]^\top \in \mathbb{R}^n$, the notation $\text{diag}(\nu)$ is used for the diagonal matrix with the elements $\nu_i \in \mathbb{R}, i = 1, 2, ..., n$.

3. PROBLEM STATEMENT

We consider a group of n numbered mobile agents. Let their positions at time $t \geq 0$ be denoted by $x_i(t) \in \mathbb{R}$, $i = 1, 2, ..., n$. The dynamic model of each agent is described by a simple integrator:
$$\dot{x}_i = u_i, \quad i = 1, 2, \ldots, n, \tag{3}$$
where $u_i \in \mathbb{R}$ is the state feedback, called control protocol, to be designed based on the information received by agent i from its neighbors, and $x = [x_1, x_2, \ldots, x_n]^\top$.

The objective in this paper is to design a feedback control protocol u_i which

- solves the average-consensus problem in a fixed time for all initial conditions, i.e.
$$\exists T_{\max} \in \mathbb{R}_+ \ : \ x_i(t) = x^*, \quad t > T_{\max},$$
where $i = 1, 2, ..., n$, and $x^* := (1/n) \sum_{i=1}^n x_i(0)$;
- exploits only the local information about the distances of the agent from its neighbors according to the communication topology, i.e.
$$u_i = \sum_{j=1}^n \phi_{ij}\Big(x_j - x_i\Big), \quad i = 1, 2, \ldots, n, \tag{4}$$
where ϕ_{ij} are continuous functions of distances for all i, j, and $\phi_{ij} = 0$ if the corresponding agents are not connected.

Below, we consider some helpful notions, definitions and auxiliary lemmas needed for further discussion.

4. PRELIMINARIES

4.1 Graphs and Linear Consensus Protocol

Algebraic graph theory plays an important role in the analysis of consensus problems. Each agent of a multi-agent system communicates with some of the agents according to the communication topology (structure of the system). Such a structure can be represented by a (generally speaking) directed graph. In this work, we consider undirected graphs which are connected and do not contain self-edges.

We use the weighted undirected graph $\mathcal{G}(A)$ to represent the communication topology, where $A = [a_{ij}]$, $i, j = 1, 2, \ldots, n$, is referred to as the weighted *adjacency matrix*. Each agent in the multi-agent system is associated with a vertex in the graph, and a_{ij} is thought of as a weight (capacity) of the information channel represented by the edge $\{i, j\}$. For $a_{ij} = 0$, there is no edge between the corresponding vertices/agents. The undirected topology simply means that the neighboring agents receive the same information about the distances between them.

Note that under the assumptions above, a well-known linear control protocol Olfati-Saber & Murray (2004), Chebotarev & Agaev (2009) of the form
$$u_i = \sum_{j=1}^n a_{ij}(x_j - x_i), \quad i = 1, 2, \ldots, n, \tag{5}$$
does solve the average-consensus problem; however the agents reach consensus asymptotically and the settling time depends on the initial conditions.

The dynamics of the multi-agent system (3) under control protocol (5) can be rewritten in the vector form
$$\dot{x} = -\mathcal{L}x, \tag{6}$$
where the $n \times n$ symmetric matrix $\mathcal{L}$ defined by $\mathcal{L} = \text{diag}(A\mathbf{1}_n) - A$ is referred to as the *Laplacian of the graph* $\mathcal{G}(A)$.

The following properties of the matrix $\mathcal{L}$ of an undirected graph are important for the analysis of consensus problems (see Olfati-Saber & Murray (2004), Chebotarev & Agaev (2009), Ren & Cao (2011)):

- $\mathcal{L}$ has at least one zero eigenvalue with the associated eigenvector $\mathbf{1}_n$;
- $\mathcal{L}$ has a simple zero eigenvalue if and only if the corresponding graph is connected;
- the quadratic form $x^\top \mathcal{L} x = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x_j - x_i)^2$ is positive semidefinite, i.e., all nonzero eigenvalues of $\mathcal{L}$ are positive;
- for a connected graph, the second smallest eigenvalue $\lambda_*(\mathcal{L}) = \lambda_2(\mathcal{L})$ is called the *algebraic connectivity* of $\mathcal{L}$; it quantifies the rate of convergence of consensus algorithms;
- for the algebraic connectivity we have

$$\min_{\|x\|_2 \neq 0,\ \mathbf{1}_n^\top x=0} \frac{x^\top \mathcal{L} x}{\|x\|_2^2} = \lambda_*(\mathcal{L}) > 0; \quad (7)$$

therefore, $\sum_{i=1}^{n} x_i = 0$ implies $x^\top \mathcal{L} x \geq \lambda_*(\mathcal{L}) x^\top x$.

4.2 Fixed-time Convergence

Let us consider the following system:

$$\dot{z} = g(t, z), \quad z(0) = z_0, \quad (8)$$

where $z \in \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a possibly discontinuous nonlinear function. In this case, the solutions of (8) are understood in the sense of Filippov (1988). Assume that the origin is an equilibrium point of system (8).

Definition 1. (Bhat & Bernstein (2000)). The origin is said to be a *globally finite-time stable* equilibrium point for system (8) if it is globally asymptotically stable and any solution $z(t, z_0)$ of (8) attains it in finite time, i.e., $z(t, z_0) = 0\ \forall\, t \geq T(z_0)$, where $T\colon \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is the *settling-time function.*

For example, any solution of the system $\dot{z} = -z^{\frac{1}{3}}, z \in \mathbb{R}$, converges to the origin in the finite time $T(z_0) := \frac{2}{3}\sqrt[3]{|z_0|^2}$.

Definition 2. (Polyakov (2012)). The origin is said to be a *fixed-time stable* equilibrium point of system (8) if it is globally finite-time stable and the settling-time function $T(z_0)$ is bounded, i.e. there exists $T_{\max} > 0\colon\ T(z_0) \leq T_{\max}$ $\forall\, z_0 \in \mathbb{R}^n$.

For example, the origin is a fixed-time stable equilibrium point of the system

$$\dot{z} = -z^{\left[\frac{1}{2}\right]} - z^{[2]}, \quad z \in \mathbb{R}, \quad z(0) = z_0,$$

since its solution has the form

$$z(t,z_0) = \begin{cases} \operatorname{sign}(z_0) \tan^2\left(\dfrac{T(z_0) - t}{2}\right), & 0 \leq t \leq T(z_0), \\ 0, & t > T(z_0), \end{cases}$$

where $T(z_0) = 2\arctan(\sqrt{|z_0|}) \leq \pi$.

Lemma 3. (Polyakov (2012)). If there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ such that

(1) $V(z) = 0 \Leftrightarrow z = 0$;
(2) any solution $z(t)$ of (8) satisfies the inequality $D^* V(z(t)) \leq -(\alpha V^p(z(t)) + \beta V^q(z(t)))^k$ for some $\alpha, \beta, p, q, k > 0$: $pk < 1$, $qk > 1$,

then the origin is globally fixed-time stable for system (8) and the following estimate of the settling time holds:

$$T(z_0) \leq \frac{1}{\alpha^k(1 - pk)} + \frac{1}{\beta^k(qk - 1)}, \quad \forall\, z_0 \in \mathbb{R}^n.$$

This lemma together with its refinement given below are the cornerstones for the design of the nonlinear fixed-time average-consensus control protocol.

Consider the special case where the constants p and q are of the form $p = 1 - \frac{1}{2\gamma}$ and $q = 1 + \frac{1}{2\gamma}$, $\gamma > 1$.

Lemma 4. (Parsegov et al. (2012)). If there exists a continuous radially unbounded function $V\colon \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ such that

(1) $V(z) = 0 \Leftrightarrow z = 0$;
(2) any solution $z(t)$ of (8) satisfies the inequality $D^* V(z(t)) \leq -\alpha V^p(z(t)) - \beta V^q(z(t))$ for some $\alpha, \beta > 0$, $p = 1 - \frac{1}{2\gamma}$, $q = 1 + \frac{1}{2\gamma}$, $\gamma > 1$,

then the origin is globally fixed-time stable for system (8) and the following estimate of the settling time holds:

$$T(z_0) \leq T_{\max} := \frac{\pi\gamma}{\sqrt{\alpha\beta}} \quad \forall\, z_0 \in \mathbb{R}^n.$$

The lemma above provides a more precise estimate of the settling time (see Parsegov et al. (2012) for the details).

In the next section we propose a new fixed-time consensus control protocol and study its rate of convergence using the results of Lemma 3 and Lemma 4.

5. FIXED-TIME CONSENSUS CONTROL PROTOCOL

Consider now the situation where every edge $\{i, j\}$ is associated with some function ϕ_{ij} that satisfy

$$\phi_{ij}\Big(x_j - x_i\Big) = -\phi_{ji}\Big(x_i - x_j\Big) \quad (9)$$

for any two neighboring agents i, j, and $\phi_{ij} \equiv 0$ if there is no edge between them. In Olfati-Saber & Murray (2004), the functions ϕ_{ij} are referred to as *action functions.*

Recall that $s^{[k]} := \operatorname{sign}(s)|s|^k$ (see Section 2) and choose the following action functions that meet condition (9):

$$\phi_{ij} = \alpha(a_{ij}(x_j - x_i))^{[\mu]} + \beta(a_{ij}(x_j - x_i))^{[\nu]}. \quad (10)$$

Here $\alpha, \beta \in \mathbb{R}_+$, $\mu \in (0, 1)$, $\nu > 1$, are the control parameters and $a_{ij} = a_{ji} \geq 0$, $i, j = 1, 2, \ldots, n$, are the elements of the adjacency matrix A.

Theorem 5. Consider system (3) with connected communication topology, i.e. the graph $\mathcal{G}(A)$ is connected. Then, under the control protocol (4) with action functions (10), system (3) solves the average-consensus problem in finite time which is globally bounded by $T^1_{\max}$:

$$T^1_{\max} = \frac{2}{\bar{\alpha}(1 - \mu)} + \frac{2}{\bar{\beta}(\nu - 1)}, \quad \forall\, x_0 \in \mathbb{R}^n, \quad (11)$$

where

$$\bar{\alpha} = \alpha 2^{\mu} \left(\lambda_*(\mathcal{L}_\mu)\right)^{\frac{\mu+1}{2}}, \quad \bar{\beta} = \beta 2^{\nu} n^{\frac{1-\nu}{2}} \left(\lambda_*(\mathcal{L}_\nu)\right)^{\frac{\nu+1}{2}}.$$

Here, $\alpha, \beta \in \mathbb{R}_+$, $\mu \in (0, 1)$ and $\nu > 1$ are the control parameters, $\mathcal{L}_\mu$ and $\mathcal{L}_\nu$ are the Laplacians of the graphs $\mathcal{G}\left(A^{\left[\frac{2\mu}{\mu+1}\right]}\right)$ and $\mathcal{G}\left(A^{\left[\frac{2\nu}{\nu+1}\right]}\right)$, respectively.

Proof. The proof uses some ideas introduced in Olfati-Saber & Murray (2004) for nonlinear action functions φ_{ij} of the general form.

Namely, denote

$$x^* := (1/n) \sum_{i=1}^{n} x_i(0)$$

and introduce the vector $\delta = [\delta_1, \delta_2, \ldots, \delta_n]^\top$ called disagreement, $x_i(t) = x^* + \delta_i(t)$. It is easy to see that $\dot{\delta}_i(t) = \dot{x}_i(t)$ and $x_j - x_i = \delta_j - \delta_i$ for all $i, j = 1, 2, \ldots, n$.

Since equality (9) holds, we have

$$\sum_{i=1}^{n} \dot{x}_i(t) = 0. \quad (12)$$

Hence, $(1/n)\sum_{i=1}^n x_i(t) = (1/n)\sum_{i=1}^n x_i(0) = x^*$ and

$$\sum_{i=1}^n \delta_i(t) = 0, \quad \forall\, t > 0. \tag{13}$$

Therefore, average consensus will follow from the stability of the system

$$\dot\delta_i = \sum_{j=1}^n \phi_{ij}\Big(\delta_j - \delta_i\Big), \quad i = 1, 2, \ldots, n. \tag{14}$$

To prove stability, introduce now the following Lyapunov function candidate:

$$V(\delta) = \tfrac{1}{2}\delta^\top\delta = \tfrac{1}{2}\sum_{i=1}^n \delta_i^2. \tag{15}$$

Its total derivative is computed as

$$\dot V(\delta) = \sum_{i=1}^n \delta_i\dot\delta_i = \sum_{i=1}^n \delta_i \sum_{j=1}^n \phi_{ij}\Big(\delta_j - \delta_i\Big).$$

Taking property (9) into account, we obtain

$$\begin{aligned}
\dot V(\delta) &= \sum_{i=1}^n\sum_{j=1}^n \delta_i\phi_{ij}\Big(\delta_j - \delta_i\Big)\\
&= \tfrac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \delta_i\phi_{ij}\Big(\delta_j\text{-}\delta_i\Big) + \tfrac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \delta_j\phi_{ji}\Big(\delta_i\text{-}\delta_j\Big)\\
&= \tfrac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \delta_i\phi_{ij}\Big(\delta_j\text{-}\delta_i\Big) - \tfrac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \delta_j\phi_{ij}\Big(\delta_j\text{-}\delta_i\Big)\\
&= -\tfrac{1}{2}\sum_{i=1}^n\sum_{j=1}^n (\delta_j - \delta_i)\phi_{ij}\Big(\delta_j - \delta_i\Big).
\end{aligned} \tag{16}$$

Since $s = \mathrm{sign}(s)|s|$, relations (10) and (16) imply

$$\begin{aligned}
\dot V(\delta) &= \tfrac{1}{2}\sum_{i,j=1}^n (\delta_i - \delta_j)\left(\alpha a_{ij}^\mu(\delta_j - \delta_i)^{[\mu]} + \beta a_{ij}^\nu(\delta_j - \delta_i)^{[\nu]}\right)\\
&= -\tfrac{1}{2}\sum_{i,j=1}^n \left(\alpha a_{ij}^\mu|\delta_j - \delta_i|^{\mu+1} + \beta a_{ij}^\nu|\delta_j - \delta_i|^{\nu+1}\right) \le 0.
\end{aligned}$$

Since $1 < \mu+1 < 2$, $\nu+1 > 2$, we use the norm equivalence property:

$$\|z\|_l \le \|z\|_r \le n^{2(\frac{1}{r} - \frac{1}{l})}\|z\|_l$$

for any $z \in \mathbb{R}^{n^2}$ and $l > r > 1$ (where $\|\cdot\|_p$ is defined by (2)). Evidently, $\|z\|_2 \le n^{1-\frac{2}{\nu+1}}\|z\|_{\nu+1}$ and $\|z\|_2 \le \|z\|_{\mu+1}$. With this in mind, we evaluate $\dot V(\delta)$ as follows:

$$\begin{aligned}
\dot V(\delta) \le& -\tfrac{1}{2}\alpha\left(\sum_{i,j=1}^n a_{ij}^{\frac{2\mu}{\mu+1}}(\delta_j - \delta_i)^2\right)^{\frac{\mu+1}{2}}\\
& -\tfrac{1}{2}\beta n^{1-\nu}\left(\sum_{i,j=1}^n a_{ij}^{\frac{2\nu}{\nu+1}}(\delta_j - \delta_i)^2\right)^{\frac{\nu+1}{2}}.
\end{aligned}$$

Next, from (7) and (13) we have

$$\sum_{i,j=1}^n a_{ij}^{\frac{2\mu}{\mu+1}}(\delta_j - \delta_i)^2 = 2\delta^\top\mathcal{L}_\mu\delta \ge 4\lambda_*(\mathcal{L}_\mu)V(\delta) \tag{17}$$

and

$$\sum_{i,j=1}^n a_{ij}^{\frac{2\nu}{\nu+1}}(\delta_j - \delta_i)^2 = 2\delta^\top\mathcal{L}_\nu\delta \ge 4\lambda_*(\mathcal{L}_\nu)V(\delta). \tag{18}$$

Taking (17), (18), and (13) into account we arrive at

$$\begin{aligned}
\dot V(\delta(t)) \le& -2^\mu\alpha(\lambda_*(\mathcal{L}_\mu))^{\frac{\mu+1}{2}}V^{\frac{\mu+1}{2}}\\
& -2^\nu\beta n^{1-\nu}(\lambda_*(\mathcal{L}_\nu))^{\frac{\nu+1}{2}}V^{\frac{\nu+1}{2}}.
\end{aligned}$$

Introduce now the following quantities: $p = \frac{\mu+1}{2}$, $q = \frac{\nu+1}{2}$, $\bar\alpha = 2^{2p-1}\alpha(\lambda_*(\mathcal{L}_\mu))^p$, $\bar\beta = 2^{2q-1}\beta n^{2(1-q)}(\lambda_*(\mathcal{L}_\nu))^q$.

Then the total derivative of the Lyapunov function computed along the trajectories of (14) satisfies the following inequality

$$\begin{aligned}
&\dot V(\delta(t)) \le -\bar\alpha V^p(\delta(t)) - \bar\beta V^q(\delta(t)),\\
&\bar\alpha, \bar\beta > 0, \quad 0 < p < 1, \quad q > 1.
\end{aligned} \tag{19}$$

By Lemma 3, this immediately implies that the origin is a fixed-time stable equilibrium point of the auxiliary system (14), and estimate (11) for the settling time holds. The proof is complete.

The theorem presents quite a conservative settling time estimate, since its proof is based on the results of Lemma 3. A more accurate estimate can be derived with the use of Lemma 4 as formulated in the corollary below.

Corollary 6. If, under the conditions of Theorem 5, the parameters μ and ν of protocol (4), (10) are chosen as $\mu = 1 - \frac{1}{\gamma}$, $\nu = 1 + \frac{1}{\gamma}$ for some $\gamma > 1$, then the settling time can be estimated by the following value

$$T_{\max}^2 := \frac{\pi\gamma n^{\frac{1}{2\gamma}}}{2\sqrt{\alpha\beta}\big(\lambda_*(\mathcal{L}_\mu)\big)^{\frac{1}{2}-\frac{1}{4\gamma}}\big(\lambda_*(\mathcal{L}_\nu)\big)^{\frac{1}{2}+\frac{1}{4\gamma}}}. \tag{20}$$

The proof immediately follows from inequality (19) and Lemma 4.

We therefore have designed the fixed-time average-consensus control protocol and obtained the two estimates of the settling time. We conclude this section with a brief discussion on the practical aspects of the algorithms of this sort.

Fixed-time control protocols possess higher theoretical rates of convergence as compared to both finite-time and linear protocols. Moreover, this is also the case in real-life applications, where so-called “practical stability concept” is of primary interest. Indeed, under practical realization, control protocols provide only convergence to a neighborhood of the origin. This is explained by the presence of various unavoidable “impurities” such as system uncertainties, exogenous disturbances, delays, measurement noises, etc.

Hence, in practice both finite-time and linear protocols expose similar qualitative behavior; namely, they provide finite-time convergence *to the neighborhood of the origin.* Moreover, in practical applications finite-time algorithms are only locally faster than the linear ones. In contrast, the *fixed-time* control guarantees convergence to the same region in a fixed time which can be pre-specified *a priori* regardless of the initial conditions. The rate of convergence is higher than that of finite-time and linear protocols globally and locally. A more detailed analysis of the practical fixed-time stability is given in Polyakov (2012).

5.1 An Illustrative Example

To demonstrate the efficiency of the proposed fixed-time consensus protocol, we consider the multi-agent system which consists of six agents with integrator dynamics (3) and the interaction topology defined by the weighted undirected graph depicted in Fig. 1.

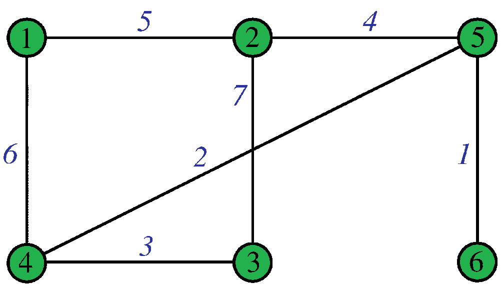

Fig. 1. Structure of the system.

For the same initial conditions

$$x(0) = x_0 := [350, 100, 200, 250, 400, 500]^\top,$$

Fig. 1 presents the results of simulations for the linear and the proposed nonlinear control protocols. The values of the parameters of the fixed-time control law (4), (10) were chosen as

$$\alpha = \beta = 1,$$

and

$$\gamma = 1.1,$$

leading to

$$\mu \approx 0.091, \qquad \nu \approx 1.909.$$

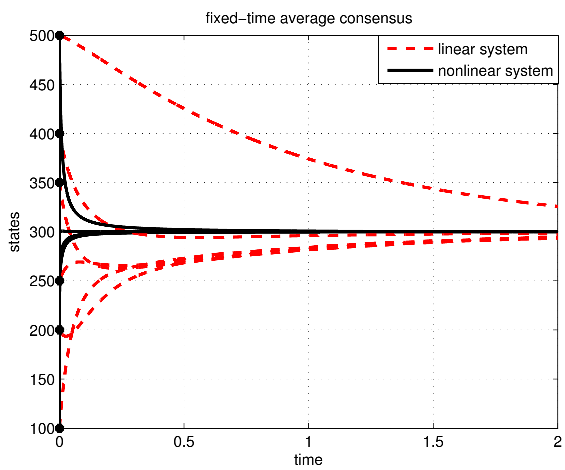

Fig. 2. The trajectories of the system under the linear and the proposed nonlinear protocols.

The results obtained confirm the theoretical conclusions of Theorem 5 showing fixed-time convergence to the average consensus in the nonlinear case.

The actual convergence time was approximately computed as $T(x_0) \approx 1.647$, while the estimates given by Theorem 5 and Corollary 6 were found as

$$T^1_{\max} \approx 4.885, \quad T^2_{\max} \approx 3.829. \qquad (21)$$

It is seen that the second estimate is much more accurate.

6. CONCLUSIONS

The contribution of the paper is the following:

- a nonlinear control protocol to solve an average-consensus problem is developed;
- it is proved that the guaranteed settling time for the system can be specified in advance regardless of the initial conditions of the agents (fixed-time convergence);
- the two different estimates for the settling time are obtained;
- as compared to the finite-time and linear consensus control protocols known from the literature, the fixed-time protocol proposed in this paper is guaranteed to have better convergence both globally and locally.

Clearly, the proposed algorithm is not free of drawbacks. For instance, it assumes the exact measurements of disagreements (though this undesirable property is somewhat inherited from the finite-time case). Next, we considered only undirected topologies, while directed graphs might represent more adequate models of interaction. These problems will be the subject of future research.

The theoretical results were successfully tested through several numerical experiments. The fixed-time stability framework applied in the paper looks promising in various multi-agent problems.

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