Uniform deployment of second-order agents on a line segment

Sergey E. Parsegov and Anton V. Proskurnikov

***Abstract*—In this paper we consider a special problem of formation control, that is, the problem of uniform spread (or *deployment*) of several identical agents on a line segment. This problem was exhaustively studied for first-order agents since the pioneering paper by I.A. Wagner and A.M. Bruckstein who considered it under the name of "row straightening algorithm". We extend it to the more realistic case of second-order agents.**

***Keywords*—Formation control, multi-agent systems, decentralized control, deployment, mobile agents**

I. Introduction

Recently it has been realized that functioning of many complex systems, arising in physics, biology, economics, sociology, computer science etc. is based on common principles, examined in the frameworks of *multi-agent systems* theory, theory of complex networks and *interconnected systems*. Numerous examples that include, but are not limited to, oscillator networks, smart power grids, robotic and sensor networks, distributed estimation and optimization, models of economical and social interaction, intellectual behavior of biologic populations may be found in recent monographs and reviews [1], [3], [5], [13], [21], [22] and references therein.

The cornerstone principles of multi-agent design are full or partial *autonomy* of the simpler parts of the systems (*agents*), *local interactions* between them without use of global information about the whole system, and *decentralization*, i.e. absence of the central controller or decision making unit. Such benefits of these principles as robustness, adaptivity, flexibility, and overall cheapness of decentralized solutions as compared to classical centralized designs motivated wide use of multi-agent systems in engineering and technology and rapid development of correspondent mathematical theory.

One of the most important areas in multi-agent systems theory is *formation control*, that is, rendering agents to form a desired static or dynamic geometric pattern such as some regular shape, flock or swarm. The problems of *dynamic* formation control [3], [11], [15], [21], [29] are primarily concerned with control of mobile agents such as wheeled robots, unmanned aerial or underwater vehicles, or spacecrafts, often being inspired by the motion of biological formations such as flocks, herds and schools [23]. Distributed algorithms providing agents to form a *static* formation have applications e.g. to such crucial problems of mobile sensor network theory as *deployment* of agents on some area or manifold [12], [14], [24], *optimal coverage* [4] and *partition of the area* [18].

Probably, one of the simplest formation control algorithm, providing deployment of the agents on a line segment, was proposed in [27], under the name "row straightening". The idea lying in the heart of the control strategy is *averaging*: each agent moves towards the middle of the segment connecting its adjacent neighbors, using only relative position measurements. Similar in flavor algorithms were proposed for multi-agent deployment of discrete-time "ant-like" agents on a ring graph [7], [8]. Another iterative procedure leading to uniform deployment of agents over ellipse is known as the "van Loan scheme" [6], [25]. Besides formation control, the averaging model was used in [27] to describe the process of pulse propagation through distributed resistor–capacitor circuits. In the recent paper [17] this procedure was extended to the case of disturbed agents, a nonlinear algorithm proposed in [16], [17] provides uniform deployment in *fixed* time independent of the initial conditions.

A serious drawback of control algorithms considered in [7], [17], [25], [27] is the assumption that agent has no self-dynamics, obeying the first-order integrator model. In the present paper we extend the averaging algorithm to more realistic second-order agents. We show that in presence of velocity damping in each agent the algorithm from [27] remains applicable. If the friction in the agents is negligible (that is, the agents obey a double integrator model), one may introduce velocity damping term in the protocol from [10]. However, this approach assumes that each agent has access to its absolute velocity, which is quite restrictive. We show, however, that uniform deployment may be provided by means of the protocol, based on relative measurements only: unlike the algorithms from [27], each agent measures not only its position relative to adjacent teammates but also relative velocities; which is a common idea for control of second-order agents [20]–[22]. Moreover, we show that the direct observation of the relative velocities may be replaced by low-pass differentiation of the relative positions [30] at the cost of slowing the convergence down. Therefore, for second-order integrators uniform deployment without velocity measurements is possible as well.

We also consider hierarchical control scheme analogous to that presented in [26]. In this case, we consider several equally sized groups of agents. The groups have to rendezvous at the points that are uniformly spread over the fixed line segment. In principle, this problem may be subdivided into several problems of uniform deployment for independent teams of agents. However, as was demonstrated in [26] for the closely related problem of cyclic pursuit, the performance and

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convergence of the algorithm may be considerably improved by using a “two-layer” protocol. This approach, applied to the problem at hand, implies that we solve first the problem of uniform deployment for group centroids, which is the “lower” level of the algorithm. The agents in each group participate not only in the averaging with neighbors from the adjacent groups but also in cyclic pursuit with their group mates (the ”upper” level of the algorithm) which guarantees the convergence to the centroid and hence rendezvous at the uniformly spread target points. We confine ourselves to the hierarchical algorithms for first-order agents, whereas the second-order case may be considered in the same way.

The paper is organized as follows. Section II presents preliminary information and problem formulation. Our main results, namely three designed control algorithms are presented and discussed in Section III, which also contains proofs and numerical simulations. In the last Section IV a two-layer hierarchical control scheme is discussed.

II. Preliminaries and problem statement

Throughout the paper, we deal with a team of $N \geq 1$ *mobile* agents indexed 1 through N and two *static* agents to which we assign indices 0 and $N+1$. Denoting the position of the j-th agent with $x_j \in \mathbb{R}^d$ (where $j = 0, 1, \dots, N+1$), we are interested in the control policies (or *protocols*) which guarantee uniform allocation of the mobile agents along the line segment, connecting the constant points x_0 and x_{N+1}.

The problem just outlined may be formally reduced to the classical reachability problem if the agents have to arrive at the desired positions in finite time or problem of stabilization, if only asymptotic convergence is required. For each agent the desired position is to be calculated. Aiming, for instance, at the arrangement the agents in the increasing order of indices, the target point for the j-th agent is $x_j^0 := x_0 + x_{N+1}(j-1)/N$. After this initial step, each agent moves to its planned target point independently of the remaining teammates. Assuming that the agent has first-order dynamics

$$\dot{x}_j(t) = u_j(t) \in \mathbb{R}^d, \quad j \in 1:N, \tag{1}$$

the simplest algorithm for the allocation of the agents may be given by a proportional controller:

$$\dot{x}_j(t) = \gamma(x_j^0 - x_j), \quad x_j^0 := x_0 + x_{N+1}(j-1)/N. \tag{2}$$

The algorithm (2), in spite of its simplicity, relies on a very restrictive assumption, which may fail in practice and in fact makes controller (2) inapplicable for large-scale formations, that is, agent’s ability to measure its position relative to the target point. In particular, each agent has either to compute its terminal point (being thus aware of its own index, positions of both endpoint agents and the formation size) and measure its own position in the global reference frame, or to distinguish the target in some other way (by using unique transponders etc.) In both cases the mobile agents in the formation are not “equal”, using different control algorithms and being thus non-replacable. Moreover, if a few of the agents fail, regrouping of the formation requires full re-initialization of target values.

A. Decentralized protocol for uniform deployment of single integrators

Contrary to a straightforward “centralized” solution (2), a more attractive *decentralized* protocol for the first-order agents (1) was examined in [10], [25], [27] for the first-order agents (1) which provides uniform deployment of the agents by using only “local” interactions:

$$u_j(t) = \frac{1}{2}(x_{j-1}(t) - x_j(t)) + \frac{1}{2}(x_{j+1}(t) - x_j(t)),\ j \in 1:N. \tag{3}$$

Protocol (3) has many advantages over algorithm (2). The agents use only relative measurements without any knowledge of the whole formation. Moreover, each agent only has to know its “successor” and “predecessor” in the formation, being unaware of its own index. If the agent j fails, the only necessary “rewiring” in the system is to link the agents $j-1$ and $j+1$ to be adjacent, after which protocol (3) will allocate the remaining $N-1$ agents uniformly.

Since protocol (2) is coordinate-wise decoupled, one may assume without loss of generality that $d = 1$: $x_j(t) \in \mathbb{R}$. Let $x = [x_1, x_2, \dots, x_N]^\top$ be the state vector of the multi-agent system; then the dynamics of the overall system can be written in compact form as

$$\dot{x} = Ax + b, \tag{4}$$

where the matrix A and the vector b are of the form

$$A := \begin{bmatrix} -1 & 0.5 & 0 & \dots & 0 \\ 0.5 & -1 & 0.5 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0.5 & -1 \end{bmatrix} \in \mathbb{R}^{N\times N}, \tag{5}$$

$$b := [0.5x_0, 0, \dots, 0, 0.5x_{N+1}]^\top \in \mathbb{R}^N. \tag{6}$$

The matrix A is a tridiagonal matrix with eigenvalues [2]

$$\lambda_k = -2\sin^2\frac{k\pi}{2(N+1)}, \quad k \in 1:N. \tag{7}$$

Since $\lambda_k < 0, k \in 1:N$, the matrix is Hurwitz and hence system (4) has a unique and exponentially stable equilibrium

$$x_* := -A^{-1}b = x_0[1,\dots,1]^\top + \frac{x_{N+1} - x_0}{N+1}[1,2,\dots,N]^\top \in \mathbb{R}^N. \tag{8}$$

In other words, protocol (3) establishes uniform allocation of the agents on the segment with endpoints x_0 and x_{N+1} independently of the initial conditions. From formula (7) the following estimate for the convergence rate is immediate:

$$\|x(t) - x_*\| \leq e^{-\hat{\lambda}t}\|x(0) - x^*\| \tag{9}$$

for any solution of system (4), where $x(0)$ is the vector of the initial positions and the convergence rate $\hat{\lambda}$ is given by

$$\hat{\lambda} = \min_k |\lambda_k| = 2\sin^2\frac{\pi}{2(N+1)}. \tag{10}$$

It should be noticed that a discrete-time counterpart of system (1),(3) may be examined in the same way [27].

In the present paper, we are going to consider the problem of uniform deployment of the agents with more realistic *second-order* dynamics.

$$\ddot{x}_j + a\dot{x}_j = u_j, \quad j \in 1:N. \tag{11}$$

Here $a \geq 0$ stands for the constant friction coefficient, in the case where $a = 0$, model (11) is the double integrator model. The problems of multi-agent consensus and formation control for second-order agents have recently attracted intensive interest, basically motivated by concerns of multi-agent mobile robotics, see e.g. [3], [21], [22], [28].

III. Main Results

In this section our main results are given, which offer distributed protocols for uniform deployment of second-order agents (11) along the line segment with fixed endpoints.

To start with, we examine applicability of algorithm (3) for the second-order agents (11). We introduce some notations. For two numbers $p, q \in \mathbb{R}$ let $h_1(p,q), h_2(p,q) \in \mathbb{C}$ be the two roots (real or complex) of the equation $h^2 + hp + q = 0$ and $H(p,q) := \max(Re\, h_1(p,q), Re\, h_2(p,q))$. In other words,

$$H(p,q) = \begin{cases} -p/2, & p^2 - 4q < 0 \\ \frac{-p+\sqrt{p^2-4q}}{2}, & p^2 - 4q \geq 0. \end{cases} \tag{12}$$

The following theorem shows that in presence of the velocity damping ($a > 0$) the protocol turns out to be applicable to such agents and estimates the convergence rate.

Theorem 1: Let $a > 0$. Then protocol (3) establishes uniform allocation of the agents (11) along the line segment with endpoints x_0 and x_{N+1}, that is, $x(t) \to x_*$ and $\dot{x}(t) \to 0$ as $t \to +\infty$. The convergence is exponential:

$$\|x(t) - x_*\| + \|\dot{x}(t)\| \leq Ce^{-\mu t}, \tag{13}$$

where $C = C(x(0), \dot{x}(0))$ and $\mu := -H(a, \hat{\lambda}) > 0$.

In the absence of friction ($a = 0$) protocol (3) obviously does not lead to uniform allocation of the agents as the system (4) is only Lyapunov stable but not exponentially stable, admitting e.g. solutions $x(t) = x_* + Re[v_k e^{i\omega_k}]$ where v_k is an eigenvector of A, matching to the eigenvalue λ_k, and $\omega_k := \sqrt{|\lambda_k|}$. Nevertheless, Theorem 1 suggests the following modification of protocol (3), applicable not only for double integrator but even for the case of unstable agent (11) ($a < 0$).

Corollary 1: The following control algorithm

$$u_j(t) = -\varkappa \dot{x}_j(t) + \frac{1}{2}(x_{j-1}(t) - x_j(t)) + \frac{1}{2}(x_{j+1}(t) - x_j(t)) \tag{14}$$

provides establishes uniform allocation on the line segment with endpoints x_0 and x_{N+1} whenever $a + \varkappa > 0$. The protocol provides exponential convergence with rate (13) with $\mu := -H(a + \varkappa, \hat{\lambda}) > 0$.

Proof: Protocol (14), applied to the agents (11), leads to the same closed-loop system as one can obtain by applying the original algorithm (14) to the agents with modified velocity damping gain $a \mapsto a + \varkappa$. The claim of Corollary is now obvious from Theorem 1. ■

Unlike protocol (3), algorithm (14) employs not only relative measurements but also *absolute* velocity of the agent. There are some classes of applications where the velocity may be available even though agents cannot measure their absolute positions. For instance, marine vehicles may be equipped with electromagnetic or Doppler log, which measure speed over water or over ground, giving however very imprecise absolute position measurements. For practical implementation, however, it is desirable to have an algorithm for uniform deployment of double integrator agents ($a = 0$) which is based only on relative measurements. We consider the following control algorithm

$$\begin{aligned} u_j(t) = \frac{1}{2}(x_{j-1}(t) - x_j(t)) + \frac{1}{2}(x_{j+1}(t) - x_j(t)) + \\ + \frac{p}{2}(\dot{x}_{j-1}(t) - \dot{x}_j(t)) + \frac{p}{2}(\dot{x}_{j+1}(t) - \dot{x}_j(t)),\ j \in 1:N. \end{aligned} \tag{15}$$

Here $p > 0$ is a fixed coefficient. The following result shows that protocol (15) uniformly allocates the agents over the line segment and provides exponential convergence.

Theorem 2: Let $a = 0$ and $p > 0$. Then protocol (15) establishes uniform allocation of the agents (11) along the line segment with endpoints x_0 and x_{N+1}, that is, $x(t) \to x_*$ and $\dot{x}(t) \to 0$ as $t \to +\infty$, moreover, (13) holds with

$$\mu = -\max_k H(-p\lambda_k, -\lambda_k) > 0, \text{ where } \lambda_k \text{ are from (7).} \tag{16}$$

Protocol (15) allows to deploy agents uniformly using only relative velocity measurements. A closer analysis reveals, however, that in fact the direct velocity measurement may be avoided at the cost of slower convergence. By denoting $w_j(t) := (x_{j-1}(t) + x_{j+1}(t))/2 - x_j(t)$, algorithm (15) may be rewritten as $u_j(t) = w_j(t) + p\dot{w}_j(t)$. The idea, borrowed from [30], is to replace the derivative $\dot{w}_j(t)$ by the output of a low-pass differentiator $\dot{w}_j(t) \approx \dot{y}_j(t)$, where

$$\dot{y}_j(t) = -\gamma y_j(t) + w_j(t),\ \gamma > 0.$$

By doing so, algorithm (15) shapes into the following

$$\begin{aligned} & u_j(t) = w_j(t) + p\dot{y}_j(t) = (1+p)w_j(t) - p\gamma y_j(t), \\ & \dot{y}_j(t) = -\gamma y_j(t) + w_j(t) \\ & w_j(t) = \frac{1}{2}(x_{j-1}(t) - x_j(t)) + \frac{1}{2}(x_{j+1}(t) - x_j(t)). \end{aligned} \tag{17}$$

Protocol (17) also provides uniform deployment with exponential convergence, as shown by the following theorem.

Theorem 3: Let $a = 0$ and $p, \gamma > 0$. Then protocol (17) establishes uniform allocation of the agents (11) along the line segment with endpoints x_0 and x_{N+1}, that is, $x(t) \to x_*$ and $\dot{x}(t) \to 0$ as $t \to +\infty$, moreover, (13) holds, where

$$\mu = -\max\{Re\, z : z^3 + \gamma z^2 - (p+1)\lambda_k z - \gamma\lambda_k = 0\} > 0. \tag{18}$$

A. Proofs: stability analysis

To prove stability of the closed-loop systems, resulting from applying the protocols (3),(15),(17) to the agents (11), we employ the following stability criterion, elaborated by B.T. Polyak and Y.Z. Tsypkin [19] and reformulated later by S. Hara [9] using the concept of "generalized frequency variable".

Suppose that we are interested in stability of the following high-order linear system

$$\phi\left(\frac{d}{dt}\right)x(t) = Ax(t), \tag{19}$$

where $\phi(s)$ is a scalar polynomial and A is a constant $N\times N$-matrix. Denoting the characteristic polynomial of A by $D(s) := \det(sI - A)$, one easily show that system (19) is stable if and only if the polynomial $G(s) := D(\phi(s))$ is Hurwitz. More generally, let $\phi(s) = \psi(s)/\rho(s)$ be a rational function, which is analytic in the closed right half-plane $\bar{\mathbb{C}}_+ := \{s : Re\, s \geq 0\}$ (that is, ρ is Hurwitz polynomial). System (19) may be interpreted in this case as follows:

$$\psi\left(\frac{d}{dt}\right)x(t) = \rho\left(\frac{d}{dt}\right)Ax(t),$$

being stable if and only if the rational function $G(s) = D(\phi(s))$ has no zeros in $\bar{\mathbb{C}}_+$. Although this property may be verified in a straightforward way, disregarding the structure of $G(s)$, this procedure is computationally expensive for polynomials of high degree. Instead, one may use the concept of Ω-*domain*.

Definition 1: [19] The Ω-domain of the function $\phi(s)$ is defined to be the set of points λ on the complex plane, for which the function $\phi(s)-\lambda$ has no right zeros:

$$\Omega = \{\lambda \in \mathbb{C} : \phi(s) - \lambda \neq 0, Re\, s \geq 0\}$$

The Polyak-Tsypkin criterion [19] reduces the problem of stability of system (19) to a couple of simpler problems, which can be effectively solved, that is, computation of eigenvalue of A and computation of Ω-domain for $\phi(s)$:

Theorem 4: The characteristic function $G(s) = D(\phi(s))$ has no zeros in $\bar{\mathbb{C}}_+$ if and only if all of the zeros of $D(s)$ (e.g. eigenvalues of A) belong to Ω-domain of the function $\phi(s)$.

Moreover, since the set of roots of $G(s)$ is constituted by the roots of N equations $\phi(s) = \lambda_k$, where λ_k ($k \in 1:N$) are the eigenvalues of A. Hence the solutions of (19) have the following asymptotic behavior at infinity.

Lemma 1: For any solution of (19) one has

$$|x(t)| \leq Ce^{\alpha t}, \quad \alpha := \max\{Re\, s : \phi(s) = \lambda_k \text{ for some } k \in 1:N\}.$$

For our goals the precise computation of Ω-domain will be unnecessary (the details of the corresponding algorithm may be found in [19]). Our protocols lead in fact to system (19) with Hurwitz matrix A of the form (5), and Ω-domain in fact contains all negative real numbers.

We are now ready to prove our main results.

1) Proof of Theorem 1: By denoting $x = [x_1, x_2, \ldots, x_N]^\top$, the closed-loop system (11), (3) may be rewritten as

$$\phi\left(\frac{d}{dt}\right)x(t) = Ax(t) + b, \tag{20}$$

where A and b are given respectively by (5),(6) and $\phi(s) := s^2 + as$. The stability of the equilibrium solution $x_* = -A^{-1}b$ is equivalent then to the stability of autonomous system (19). Since $a > 0$ by assumption, the equation $\phi(s) = \lambda$ has no unstable roots whenever $\lambda < 0$ (the polynomial $\phi(s) - \lambda$ is Hurwitz), from where stability is immediate since all eigenvalues of A are real and negative, given by (7). In accordance with Lemma 1, the convergence rate μ in (13) is $\mu = \max_k H(a, -\lambda_k)$ which entails that $\mu = H(a, \hat{\lambda})$ since $H(a,\cdot)$ is non-decreasing.■

2) Proof of Theorem 2: Let $a = 0$. The closed-loop system (11), (15) shapes into

$$s^2 x = (ps+1)(Ax+b), \quad s := \frac{d}{dt},$$

which may be treated as system (20) with a rational function $\phi(s) := s^2/(ps+1)$. The stability of the equilibrium solution $x_* = -A^{-1}b$ is equivalent to the stability of autonomous system (19) which is entailed by Theorem 4 since the equation $\phi(s) = \lambda$ has no unstable roots whenever $\lambda < 0$ (the polynomial $s^2 - p\lambda s - \lambda$ is Hurwitz if $p > 0$ and $\lambda < 0$). In particular, Ω-domain contains all the eigenvalues λ_k. The formula for convergence rate is obvious from Lemma 1. ■

3) Proof of Theorem 2: Retracing arguments from two previous proofs, system (11), (15) boils down into

$$s^2 x = q(s)(Ax+b), \quad s := \frac{d}{dt},\ q(s) := 1 + \frac{ps}{s+\gamma},$$

which is equivalent to (20) with $\phi(s) = s^2(s+\gamma)/(s(p+1)+\gamma)$. To prove stability, it suffices to verify that the function $\phi(s) - \lambda$ has no unstable roots as $\lambda < 0$. This is implied by the Routh-Hurwitz criterion, stating that a polynomial $s^3 + as^2 + bs + c$ is Hurwitz if and only if $a, b, c > 0$ and $ab > c$. In particular, the polynomial $s^2(s+\gamma) - \lambda(p+1)s - \lambda\gamma$ is Hurwitz whenever $\gamma, p > 0$ and $\lambda < 0$. Therefore, the equilibrium solution is exponentially stable. The formula for convergence rate is immediate from Lemma 1. ■

B. Numerical Examples

To demonstrate the efficiency of the proposed control protocols for deployments we consider the problem of uniform deployment for agents on the plane:

$$\ddot{\xi}_j + a\dot{\xi}_j = u_j,\ \xi_j = [x_j, y_j]^\top \in \mathbb{R}^2. \tag{21}$$

As was mentioned in the foregoing, all our results are applicable for the space of arbitrary dimension since the protocols are coordinate-wise decoupled. All the tests deal with a team of $N = 5$ agents (21) to be deployed uniformly on the line segment with endpoints $\xi_0 = [-2, 3.1]^\top, \xi_6 = [2, 2.1]^\top$.

Our first numerical test illustrates the motion of agents (21) with $a = 2$, coupled via the protocol (3). Fig. 1 illustrates that the protocol uniformly allocates the agents on the line segment in accordance with Theorem 1.

Our next numerical tests illustrate the applicability of algorithms (15) and (17) for agents without velocity damping: $a = 0$. For simulations, we take $p = 10$ and $\gamma = 1.4$ (the latter parameter is employed only by algorithm (17)). Fig. 2 illustrates the performance under protocol (15) which uses relative velocities, and Fig. 3 shows the dynamics under protocol (17). Both protocols provide uniform deployment as claimed respectively by Theorems 2 and 3. It should be noticed, however, that the algorithm (15) provides more "smooth" trajectories and faster convergence. Such a behavior is expectable since the low-pass filter in fact introduces a delay in the velocity measurement.

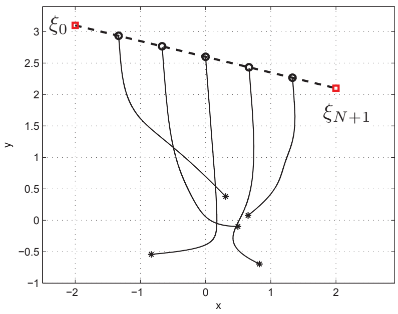

Fig. 1. The protocol (3) for agents with velocity damping ($a = 2$)

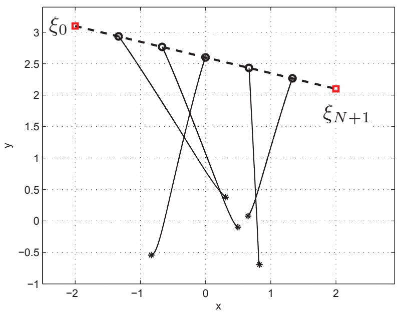

Fig. 2. Frictionless agents ($a = 0$) under protocol (15)

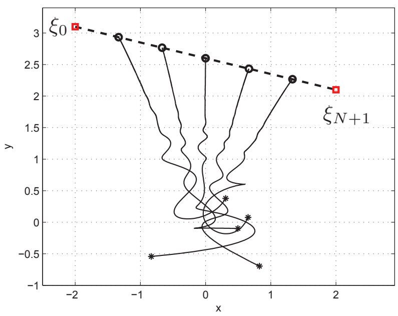

Fig. 3. Frictionless agents ($a = 0$) under protocol (17)

IV. Extension: Hierarchical Pursuit-Based Algorithms

In this section we consider a group of uniform allocation for several equal groups of agents. Suppose that the team of N agents divided into n_g groups, each containing n agents so that $N = n \times n_g$. The agents of each group has to rendezvous at the prescribed point, which points are to be uniformly spread over the line segment with endpoints x_0 and x_{n_g+1}. Let $x_{k,j}$ stands for the position of the j-th agent of the k-th group. For simplicity, we confine ourselves to the first-order agents, whereas the second-order case may be considered likewise.

To solve the rendezvous problem, one may subdivide the team of agents into n_g independent sub-teams, each consisting of the agents with the same index, taken from different groups $\{x_{k,j}\}_{k=1}^{n_g}$ (here $j \in 1:n$). Each sub-team may perform the task of uniform deployment independently by using protocol (3). Instead, one may consider more sophisticated "two-layer scheme" which provides, firstly, convergence of the centroids of the groups to the desired rendezvous points (the "lower layer") and, secondly, spiral motion of the group-mates around their centroid using the circular pursuit in each group. Such an approach allows to increase the converge rate and improve overall performance of the closed-loop system, as discussed in [26] on the example of the problem of circular formation stabilization. As an additional benefit, one obtains that agents *surround* the desired rendezvous point and the approach it with the same speed, which performance can not be achieved without interactions inside the group.

A. Two-Layer Hierarchical Scheme for Single Integrators

As was mentioned, we consider first-order agents

$$\dot{x}_{k,j} = u_{k,j}, \quad k \in 1:n_g, j \in 1:n, \tag{22}$$

coupled via the control protocol as follows:

$$u_{k,j} = x_{k,j+1} - x_{k,j} + \frac{x_{k+1,j} + x_{k-1,j}}{2} - x_{k,j}. \tag{23}$$

Here $x_{0,j} = x_0$, $x_{n_g+1,j} = x_{n_g+1}$ for any j, and if $j = n$, we put by definition $j+1 = 1$. The protocol (23) makes the the j-th agent of the k-th group tends to locate itself in the middle of the segment joining the respective agents from the adjacent groups, pursuing also the $(j+1)$-th teammate. The overall dynamics of N agents has the form

$$\dot{x} = \bar{A}x + \bar{b}, \tag{24}$$

where the matrix of the system has the form $\bar{A} = I_{n_g} \otimes C + D \otimes I_n$. The matrix C is a circulant matrix for each group of agents

$$C = \begin{bmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & \dots & 0 & -1 \end{bmatrix} \in \mathbb{R}^{n \times n}, \tag{25}$$

the matrix $D \in \mathbb{R}^{n_g \times n_g}$ has the form (5) and $\bar{b} = b \otimes \mathbf{1}$, where $\mathbf{1} = [1, 1, \dots, 1]^\top \in \mathbb{R}^n$, where $b \in \mathbb{R}^{n_g}$ is defined similarly to (6). The matrices I_n, I_{n_g} are identity matrices, and $\otimes$ stands for the Kronecker product of two matrices [2].

Using the fact that the matrix $\bar{A}$ is the Kronecker sum [2] of the matrices C and D: $\bar{A} = C \oplus D$, we arrive at the following.

Theorem 5: System (24) is stable and the set of eigenvalues of the matrix $\bar{A}$ is the direct sum of the sets of eigenvalues of the matrices C and D. Moreover, there exists a final position of agents $x_{k,j} \to x_0 + \frac{k}{n_g+1}(x_{n_g+1} - x_0)$, $t \to \infty, k = 1, \dots, n_g$, i.e. for every initial condition, each agent converges to the centroid of the group and the centroids tend to allocate uniformly on a segment with fixed endpoints x_0 and x_{n_g+1}. The rate of convergence estimate for $n_g \to \infty$ is $\hat{\lambda} \approx -\frac{\pi^2}{2n_g^2}$.

Proof: As it was shown before the eigenvalues of the matrix D have the form $\lambda_k = -2\sin^2 \frac{k\pi}{2(n_g+1)}$, $k = 1, \dots, n_g$, and the eigenvalues of C are also known and can be written as $\bar{\lambda}_j = e^{2\pi\imath j/n} - 1$, $j = 1, \dots, n$. The eigenvalues of the matrix $\bar{A} = I_{n_g} \otimes C + D \otimes I_n = C \oplus D$ can be easily found by the properties of the Kronecker sum. The matrix C has one zero eigenvalue and the nonzero eigenvalues are left, the eigenvalues of the matrix D are negative. Thereby, the eigenvalues of the matrix $\bar{A}$ are left and the system (24) is stable. It is evident that the rate of convergence estimate of the system (24) for $n_g \to \infty$ has the form $\hat{\lambda} \approx -\frac{\pi^2}{2n_g^2}$. ■

B. Numerical simulation

To illustrate the performance of the algorithm, we made a simulation for the case of $N = 9$ agents on the plane $\mathbb{R}^2$, subdivided into three groups of three agents each ($n = n_g = 3$). The trajectories of agents are displayed on Fig. 4. The simulation confirms the convergence of each group to its target point, which points are uniformly spread over the line segment.

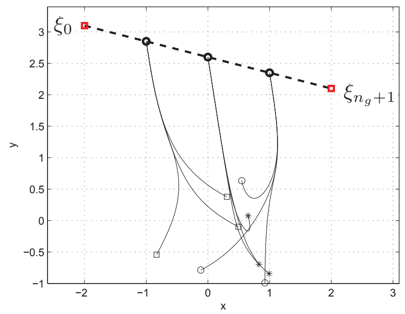

Fig. 4. First-order agents coupled via hierarchic control algorithm

V. Conclusions

In this paper we consider distributed control algorithms (protocols) for uniform deployment of second-order agents. We show that the well-known first-order averaging algorithm proposed in [27] is applicable if the agents model involves velocity damping however its convergence deteriorates as the damping becomes negligible. To cope with this problem, one may introduce damping velocity term in the protocol, which assumes, however, access to the absolute velocity. To discard this assumption, we develop a novel algorithm for uniform deployment based on relative velocity measurements. We show also this algorithm remains feasible, replacing relative velocity with its estimate from a low-pass differentiator. Therefore, it is possible to solve uniform deployment problem for second-order agents even without velocity measurements. The applicability of the algorithms is confirmed by numerical simulations. We also consider a two-layer hierarchical protocol for uniform deployments of several equal groups of agents.

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