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LINEAR SYSTEMS

Upper Bounds on Peaks in Discrete-Time Linear Systems

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Abstract—Trajectories of stable linear systems with nonzero initial conditions are known to deviate considerably from the zero equilibrium point at finite time instances. In the paper we analyze transients in discrete-time linear systems and provide upper bounds on deviations (peaks) via use of linear matrix inequalities. An approach to peak-minimizing feedback design is also proposed. An analysis of peak effects for norms of powers of Schur stable matrices is presented and a robust version of the problem is considered. The theory is illustrated by numerical examples.

Keywords: linear discrete-time systems, stability, nonzero initial conditions, transient behavior, upper bounds, linear matrix inequalities, robustness

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1. INTRODUCTION

The analysis of transients in dynamical systems has always been an important direction of research in the theory of automatic control [1] and related areas; also see [2, 3] and other numerous monographs and textbooks. Traditionally, transient is usually understood as a response of a stable system with zero initial conditions to "typical" exogenous input signals such as unit step function, harmonic disturbance, and the like. Possible deviations of the system output from the steady state is referred to as *overshoot*, and there exist numerous publications on this subject; besides the bibliography sources mentioned above, see, e.g. [4].

At the same time, much less attention in the literature has been paid to a closely related phenomenon known as *peak*, which is caused by nonzero initial conditions in the absence of exogenous input. Clearly, both overshoot and peak are to be avoided in the engineering practice in order to implement "smooth" or even monotone transient. We mention [5] as one of the recent publications on peak effects in *continuous-time systems*, where numerous bibliography references are given. Certain considerations on the links between peaks and overshoot are presented in [6] (see section 5.1); also see discussion on p. 90 of [7]. For instance, in [6, 7] it is shown that, having a large peak of trajectory of an input-free system $\dot{x} = Ax$, $x_0 \neq 0$, a vector b can be found such that the overshoot in the system $\dot{x} = Ax + bu$, $x_0 = 0$, with unit step input u will be large.

To the best of our knowledge, peak effects in *discrete-time systems* are very poorly explored, and the corresponding results cannot be directly derived from their continuous-time counterparts, since these phenomena differ in nature. Among a very few publications in this direction we mention [8, 9], where the main emphasis has been put to adaptive control of discrete-time SISO systems. Closed-form expressions for the magnitude of peak and/or lower bounds were obtained in [10] for classes of scalar difference equations with various initial conditions. Paper [11] also deserves to be mentioned, where attempts have been made to link peak effects with poor controllability of the system.

Results close to those presented in this paper are obtained in [12], where discrete-time systems with integral and phase constraints were analyzed and designed. In the present paper the main emphasis is put on finding upper bounds on peaks; i.e., we are aimed at estimating from above maximal deviations of trajectories from unit-norm initial conditions. Similar results for continuous-time systems were obtained in [6, 13–15]; also see [16], p. 65.

Estimation of peak of *trajectories* of discrete-time systems is closely related to the estimation of norms of powers of Schur stable *matrices*. The well-known monograph [17] provides numerous general-type results on this problem which appears in the implementation of numerical iterative processes. More subtle results can be found in [18, 19]. In particular, the so-called *Kreiss constant* [20] is shown to bound from below the spectral norm of powers of a matrix A, and it can be used to compute an upper bound. This constant is defined through the resolvent of the linear operator A; also, it is closely related to the *ε-pseudospectrum* of the matrix A. The bounds obtained with this approach may happen to be very conservative for generic matrices; moreover, this constant is rather hard to compute numerically.

Clearly, for a given system and fixed initial conditions, the magnitude of peak can in principle be found by straightforward exponentiation and finding the desired value numerically. However for high dimensions and powers of matrices, such an approach may turn out to be numerically unstable. On top of that, theoretical estimates are of apparent interest.

Next, finding worst-case initial conditions leading to the maximal peak of the trajectory is not easy. Moreover, estimation of peak values for *classes of matrices*, in particular, for matrices containing additive norm-bounded uncertainty is a complicated problem. Yet another consideration is related to the use of this sort of results in control theory, namely, when designing peak-minimizing controllers.

Overall, research in this direction is seen to be pretty much important.

In the current paper we are aimed at finding estimates of the value of peak for the *trajectories* of discrete-time systems and *norms of powers* of Schur stable matrices. In Section 2 we present two simple examples demonstrating that peaks may take arbitrarily large values, and, moreover, its magnitude can be found in closed form. Section 3 is devoted to the construction of upper bounds using linear matrix inequalities and to the description of a peak-minimizing design procedure; the theory is accompanied by a discussion and illustrated by numerical examples. In Section 4 we formulate a robust version of the peak problem and provide its solution.

2. MOTIVATING EXAMPLES

We present two examples of estimating the norms of powers of matrices; they admit for a closed-form solution and show that the value of peak may be arbitrarily large.

Having a Schur stable matrix A, we are interested in finding

$$\eta(A) = \max_{k=1,2,\ldots} \|A^k\|.$$

If $\eta(A) > 1$ and the maximum is attained at $k > 1$, we say that the quantity $\|A^k\|$ experiences peak, and the peak instant

$$k^* = \arg\max_{k=1,2,\ldots} \|A^k\|$$

is also of our interest.

We start with a very simple example.

Example 1. Consider the companion-form Schur stable matrix $A \in \mathbb{R}^{2\times 2}$ with eigenvalues $\lambda_1 = \lambda_2 = \lambda$:

$$A = \begin{pmatrix} 0 & 1 \\ -\lambda^2 & 2\lambda \end{pmatrix}. \tag{1}$$

By induction we have

$$A^k = \begin{pmatrix} -(k-1)\lambda^k & k\lambda^{k-1} \\ -k\lambda^{k+1} & (k+1)\lambda^k \end{pmatrix}. \tag{2}$$

Let the matrix l_∞-norm $\|A\|_\infty = \max_i \sum_j |a_{ij}|$ be used; we then find

$$\eta(A) = \max_{k=1,2,\ldots} (k\lambda^{k+1} + (k+1)\lambda^k).$$

Taking the first difference of the quantity under the max sign, we obtain

$$k^* = \frac{2\lambda}{1-\lambda^2} \tag{3}$$

for the peak instant $k^* = \arg\max \|A^k\|$, or, more precisely,

$$k^* = \left\lfloor \frac{2\lambda}{1-\lambda^2} \right\rfloor,$$

where $\lfloor\cdot\rfloor$ denotes rounding towards minus infinity. Peak takes place only with $k^* > 1$; i.e., for $\lambda > (\sqrt{5}-1)/2 \approx 0.6180$. For the value of peak we have

$$\eta(A) \approx \lambda^{\frac{2\lambda}{1-\lambda^2}} \frac{1+\lambda}{1-\lambda} > \frac{1+\lambda}{\mathrm{e}(1-\lambda)}. \tag{4}$$

From now onwards e denotes the base of the natural logarithm. It is seen that, as $\lambda \to 1$, both the peak value and instant grow; for instance, with $\lambda = 0.99$, the peak is attained at $k = 99$ and its value is equal to 73.2120 (estimate (4) gives 73.2080).

Example 2. Consider the matrix

$$A = \begin{pmatrix} \lambda & 0 & \cdots & 1 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \lambda & 0 \\ 0 & \ldots & 0 & \lambda \end{pmatrix} \in \mathbb{R}^{n\times n}, \tag{5}$$

where $|\lambda| < 1$. For its spectral norm we have $\|A\|^2 = \lambda^2 + 0.5 + 0.5(1+4\lambda^2)^{1/2} > 1$.

Next,

$$A^k = \begin{pmatrix} \lambda^k & 0 & \cdots & k\lambda^{k-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \lambda^k & 0 \\ 0 & \ldots & 0 & \lambda^k \end{pmatrix},$$

so that we arrive at

$$\|A^k\|^2 = \lambda^{2k-2}\left(\lambda^2 + \frac{k^2}{2} + \frac{k}{2}\sqrt{4\lambda^2 + k^2}\right);$$

i.e., $\|A^k\| \approx k\lambda^{k-1}$. By taking the first difference $[k\lambda^{k-1} - (k+1)\lambda^k]$, we obtain $k^* \approx \lambda/(1-\lambda)$ for the peak instant and

$$\eta(A) \approx \frac{1}{1-\lambda}\lambda^{\frac{\lambda}{1-\lambda}}$$

for its value; for λ close to unity we have

$$\eta(A) \approx \frac{1}{\mathrm{e}\lambda(1-\lambda)}.$$

Hence, likewise Example 1, the magnitude of peak of $\|A^k\|$ may take arbitrarily large values.

Similar conclusions are valid for matrices of the form (5), but having distinct eigenvalues, nonzero super-diagonals, etc., as well as for the matrix l_∞-norm.

For general-form matrices, accurate values of peak are not computable, and below we consider upper bounds.

3. UPPER BOUNDS ON DEVIATIONS

In this section we consider discrete-time linear systems of the form

$$x_{k+1} = Ax_k, \quad x_k \in \mathbb{R}^n, \quad k = 0, 1, \ldots, \tag{6}$$

with initial conditions x_0, $\|x_0\| \le 1$; from now on, the Euclidian vector norm and the spectral matrix norm $\|A\|$ is used. For Schur stable systems we obtain simple upper bounds on the peak of *trajectories*:

$$\max_{\|x_0\|\le 1}\ \max_{k=0,1,\ldots} \|x_k\|;$$

also, an approach to peak-minimizing state feedback design will be presented. The results are obtained via use of linear matrix inequality technique (e.g., see, [7, 16], where the foundations of the theory are presented and numerous control-related applications are discussed) and formulated as semidefinite programs (SDP).

3.1. Analysis

The following result holds.

Theorem 1. *Let γ be a solution of the semidefinite program*

$$\min \gamma \ \textit{subject to}\ APA^\top - P \prec 0, \quad I \preccurlyeq P \preccurlyeq \gamma I, \tag{7}$$

in the variables $\gamma \in \mathbb{R}$ and $P = P^\top \in \mathbb{R}^{n\times n}$.

Then the trajectories of system (6) *satisfy*

$$\max_{\|x_0\|\le 1}\ \max_{k=0,1,\ldots} \|x_k\| \le \eta_{\mathrm{upp}}(A) = \gamma^{1/2}.$$

Proof. Schur stability of system (6) is equivalent to the existence of the quadratic Lyapunov function $V(x_k) = x_k^\top P^{-1} x_k$ with matrix $P \succ 0$ satisfying the discrete-time Lyapunov inequality

$$APA^\top - P \prec 0. \tag{8}$$

Consider the ellipsoid

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n \colon x^\top P^{-1} x \le 1 \right\}. \tag{9}$$

By the definition of the Lyapunov function, the condition $x_0 \in \mathcal{E}$ implies $x_k \in \mathcal{E}$ for all $k > 0$. In particular, if the ellipsoid contains the unit ball $\mathcal{B} = \{x \in \mathbb{R}^n \colon \|x\| \le 1\}$, which can be written as $P \succcurlyeq I$, then $x_0 \in \mathcal{B}$ implies $x_k \in \mathcal{E}$ for all $k > 0$. This means $\|x_k\| \le \sqrt{\lambda_{\max}(P)}$ (i.e., the norm of the solutions does not exceed the square root of the length of the largest semiaxis of the ellipsoid) for all $k > 0$. Hence, to obtain the best upper bound on $\|x_k\|$, the quantity $\lambda_{\max}(P)$ is to be minimized over all positive-definite matrices P satisfying the Lyapunov inequality together with the condition $P \succcurlyeq I$.

Note that the Lyapunov inequality in (7) in strict; i.e., the optimization is performed over an open set. In the formulation of this theorem and the subsequent assertions we will omit subtleties related to use of strict/nonstrict inequalities; a detailed discussion of this issue can be found in [16]; also see [7, Subsection 2.1].

A numerical solution of the SDP (7) can be found by using numerous computational tools; e.g. such as the MATLAB-compatible toolboxes LMI lab [21] (part of the Robust Control Toolbox) or `cvx` [22].

Theorem 1 provides an upper bound—over all initial conditions in the unit ball—on the peak of norms of solutions of a stable system. In other words, we have obtained an upper bound on the norms of matrix powers $\|A^k\|$; this immediately follows from the definition of the matrix spectral norm.

Accuracy of the estimates obtained along this way is an open issue; see discussion is Subsection 3.3.

3.2. Design

Consider now the design problem for the system

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots. \tag{10}$$

Here, the pair of matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $m < n$, is assumed to be controllable, and the matrix A is not necessarily Schur stable; $u_k \in \mathbb{R}^m$ is the control input.

We restrict our considerations by linear static state feedback

$$u_k = Kx_k, \tag{11}$$

so that the problem is to find a matrix $K \in \mathbb{R}^{m\times n}$ that leads to the minimal peak of trajectories of the closed-loop system

$$x_{k+1} = (A + BK)x_k$$

for all initial conditions $\|x_0\| \le 1$.

The only difference with the analysis problem is the presence of the additional variable K, and the first inequality in the SDP (7) is to be modified properly.

Let a stabilizing gain matrix K be already found; then, by Schur lemma [16], the Lyapunov inequality (8) for the closed-loop matrix $A + BK$ can be re-written in the equivalent form as

$$\begin{pmatrix} P & (A+BK)P \\ P(A+BK)^\top & P \end{pmatrix} \succ 0.$$

Introducing the auxiliary matrix variable $Y = KP$ this inequality writes

$$\begin{pmatrix} P & AP+BY \\ PA^\top + Y^\top B^\top & P \end{pmatrix} \succ 0, \tag{12}$$

which is a linear matrix inequality in P, Y.

Further considerations mimic the proof of Theorem 1, and we arrive at the following result.

Theorem 2. *Let γ, P, Y provide a solution to the SDP*

$$\min \gamma \ \textit{subject to} \ (12) \ \textit{and} \ I \preccurlyeq P \preccurlyeq \gamma I$$

in the variables $\gamma \in \mathbb{R}$, $P = P^\top \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{m\times n}$.

Then the control law (11) *with gain matrix $K = YP^{-1}$ stabilizes system* (10)*, and the trajectories of the closed-loop system with matrix $A_c = A + BK$ admit the following estimate:*

$$\max_{\|x_0\|\le 1} \ \max_{k=0,1,\dots} \|x_k\| \le \eta_{\mathrm{upp}}(A_c) = \gamma^{1/2}.$$

3.3. Discussion and Examples

The formulations of Theorems 1 and 2 are simple, but the accuracy of the resulting estimates is not addressed. We therefore discuss some of their properties and present the results of numerical simulations which testify to a reasonable performance.

Consider the analysis problem.

First, it is clear that trajectories of system (6) experience no peak for any initial conditions if and only if $\|A\| < 1$. This inequality is equivalent to $AA^\top \prec I$, i.e., to the existence of the Lyapunov function with $P = I$ (see the Lyapunov inequality in (7)), whence it follows that the estimate given by Theorem 1 is exact, and ellipsoid (9) is the unit ball.

The next issue relates to the choice of the worst-case initial conditions x_0 from the unit ball:

$$x_{\mathrm{wc}} = \arg \max_{\|x_0\|=1,\, k=1,2\dots} \|x_k\|.$$

This is an open problem; however, experiments testify to the fact that the point $x_0 = e_{\min}$, $Pe_{\min} = e_{\min}$ "often" happens to be close to the worst one. Here, $e_{\min}$ denotes the eigenvector of the matrix P (the solution given by Theorem 1) associated with its minimal eigenvalue equal to unity. It is these initial conditions which will be considered in all the experiments below.

Next, the experiments were performed with matrices A in companion form; this case is of interest because it represents the vector form of the scalar difference equation (the coefficients of this equation are the elements of the last row of A). In a number of situations, the upper bounds obtained are "precise." For example, for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -0.9801 & 1.4001 \end{pmatrix} \tag{13}$$

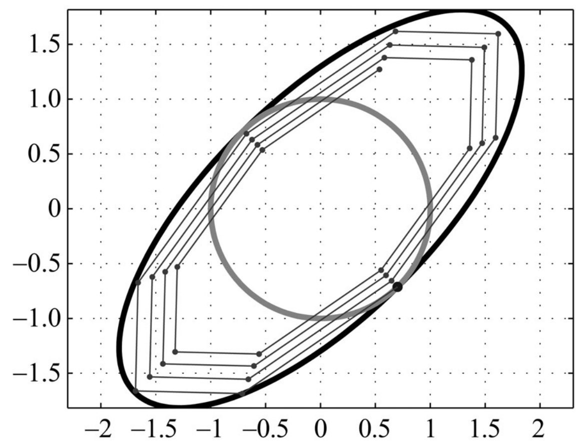

Fig. 1. The trajectory of the system with $\lambda_{1,2} = 0.99\,\mathrm{e}^{\pm j\pi/4}$, and the bounding ellipse.

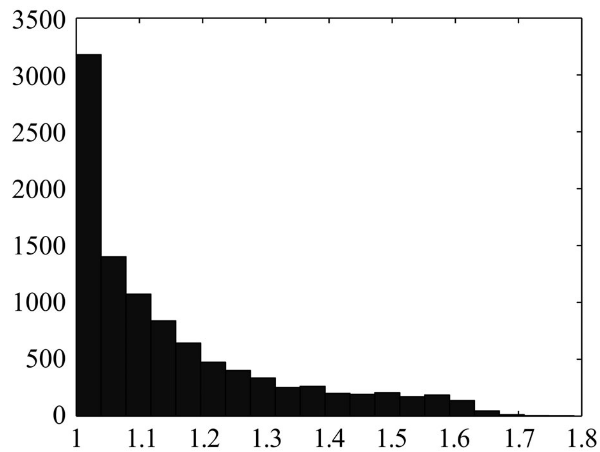

Fig. 2. Accuracy of the upper bound from Theorem 1; a histogram from 10 000 randomly generated Schur stable matrices in $\mathbb{R}^{3\times 3}$.

with eigenvalues $\lambda_{1,2} = 0.99\,\mathrm{e}^{\pm j\varphi}$, $\varphi = \pi/4$, Theorem 1 gives $\eta_{\mathrm{upp}}(A) = 2.3803$, which exceeds the actual magnitude of peak (calculated by straightforward exponentiating $\|A^k\|$, $k = 1, 2, \ldots$) by less than 0.6 %. Figure 1 depicts the trajectory of the system, the bounding ellipse, and the unit circle of initial conditions.

On the other hand, for $\varphi = 0$ (i.e., for $\lambda_1 = \lambda_2 = 0.99$), the trajectory experiences a large peak equal to 73.2120 (see Example 1), and the estimate $\eta_{\mathrm{upp}}(A) = 99.5025$ is much worse. For systems with companion-form matrix, the magnitude of peak (and, probably, the conservatism) is high if the eigenvalues are multiple and/or located close to the stability boundary. Results of this sort for scalar systems can be found in [10].

As said, accuracy of the estimates for matrices of general form is an open question. We performed the following experiment. Schur stable matrices A were generated randomly; for each of them, the estimate $\eta_{\mathrm{upp}}(A)$ was found by Theorem 1, and the quantity $\eta(A, x_0) \doteq \max_k \|x_k\|$ (with x_0 mentioned above) was obtained by direct computations. The accuracy of the estimate $\eta_{\mathrm{upp}}(A)$ was characterized by the quantity $\mathrm{acc} = \eta_{\mathrm{upp}}(A)/\eta(A, x_0)$. The corresponding histogram is depicted in Fig. 2; the quality of the estimate provided by Theorem 1 is seen to be reasonable.

Needless to say, the results obtained along this way depend on the method of random generation of matrices, their dimension, etc., though overall, they testify to rather meaningful quality of the estimates.

Theorem 1 gives an upper bound on $\|A^k\|$. Estimates based on the Kreiss constant [20]

$$\mathcal{K}(A) = \sup_{z\in\mathbb{C}:|z|>1} (|z|-1)\|(zI-A)^{-1}\|$$

mentioned above turn out to be much worse. In particular, the quantity $K_{\text{upp}}(A) \doteq \mathrm{e}\, n\, \mathcal{K}(A)$ can be shown to bound $\|A^k\|$ from above, see [18, 19]. For the test matrix

$$A = \begin{pmatrix} \frac{20}{21} & 2 & 0 & 0 \\ 0 & \frac{40}{41} & 2 & 0 \\ 0 & 0 & \frac{60}{61} & 2 \\ 0 & 0 & 0 & \frac{80}{81} \end{pmatrix} \tag{14}$$

analyzed in [19] (cf. (5)), the true value of peak of its norm is equal to $\eta(A) \approx 1.4721 \times 10^5$, and it is observed at step $k = 142$. We point out that $\|A\| = 2.8072$, i.e., during iterations, the norm increases more than 50 000 times, before start decreasing! The Kreiss upper bound equals $K_{\text{upp}}(A) \approx 7.88 \times 10^5$, which does not make much sense. Using Theorem 1, we obtain $\eta_{\text{upp}}(A) \approx 1.6823 \times 10^5$, i.e., the accuracy of this estimate is about 14%. We also note that the matrix P computed in the optimization problem in Theorem 1 has a huge condition number (about 10^{10}). This forced us to properly tune the numerical parameters of the LMI solvers (such as the maximal number of iterations, accuracy, etc.).

We now turn to the peak-minimizing control design.

First, note an obvious peculiarity of the approach in the discrete time case. It is well-known that for controllable systems, the trajectory can be settled to zero in a finite number of steps, by choosing zero poles for the closed-loop system. For systems in the canonical controllable form [3], the matrix of the closed-loop system has companion form with the zero last row; this guarantees the absence of peak. In continuous-time systems, such a complete mitigation of peak is impossible (e.g., see discussion in [6]). For matrices of general form, setting eigenvalues to zero may cause significant peak.

We illustrate the efficiency of the peak-attenuation procedure of Theorem 2 by applying it to system (10) with Schur stable matrix A of the form (14), and $B = (0,\ 0,\ 0,\ 1)^\top$, i.e., having scalar control input (11). Theorem 2 gives the matrix A_c of the closed-loop system, and the upper bound for the peak was found to be $\eta_{\text{upp}}(A_c) = 16.1031$. The actual peak computed numerically is $\eta(A_c) = 15.6778$, i.e., our estimate is quite accurate. Notably, the value of peak is 10 000 times smaller than that of the open-loop system.

3.4. Peak Effects and Superstability

Yet another direction of the analysis of peak effects relates to the notion of superstability; e.g., see [23, 24] for numerous useful properties of such systems.

Recall that a matrix A (and the associated system (6)) is said to be *superstable* if $\|A\|_\infty < 1$. Superstability is a sufficient condition for stability; importantly, such systems do not exhibit peaks, since the ℓ_∞-norm of solutions decreases monotonically.

Note that the value of peak in stable systems is not invariant to change of basis; i.e., the transients of two systems with matrices having the same spectra may differ dramatically. Indeed, if a matrix is diagonalizable, then in the new basis there is obviously no peak, whereas in the old basis it may happen to be large.

Superstability is also not invariant to coordinate transformations, since it is formulated in terms of conditions on its entries, not eigenvalues. Therefore, change of basis may gain superstability and the absence of peak; see [23–25] for more details.

Certain classes of matrices admit simple conditions on gaining superstability; one of them is the class of Schur stable diagonalizable matrices with real spectra, which are superstable in the new basis and experience no peak. A more interesting example is the class of matrices with distinct eigenvalues λ_i that satisfy the condition

$$|\mathrm{Re}\,\lambda_i| + |\mathrm{Im}\,\lambda_i| < 1. \tag{15}$$

By a nonsingular transformation, such a matrix can be converted to the real block-diagonal form, and by (15) it becomes superstable [23], hence, no peak. More subtle results can be found in [25]; they relate to a characterization of diagonal transformations which make a matrix superstable; finding such transformations reduces to solving a linear program.

In the case of the spectral matrix norm, description of the respective transformations is more complicated, though it can be performed for certain classes of matrices. For instance, if the above-mentioned diagonal transformations with matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ are considered, then in the new coordinates for matrix (5) we obtain

$$\tilde{A} = D^{-1}AD = \begin{pmatrix} \lambda & 0 & \cdots & \dfrac{d_n}{d_1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \lambda & 0 \\ 0 & \ldots & 0 & \lambda \end{pmatrix}$$

and taking the quantity $\frac{d_n}{d_1}$ small enough, we obtain $\|\tilde{A}\| < 1$.

The notion of superstability is also handy in the design of peak-annihilating controllers. Finding such a controller reduces to solving a linear program, and if a solution exists, the closed-loop system exhibits no peak. Note however that the class of superstabilizable systems is relatively small; for instance, a system in the canonical controllable form cannot be superstabilized.

4. UPPER BOUNDS: A ROBUST VERSION

The results of Theorems 1 and 2 can be modified to cope with uncertainties in the system matrix.

Consider the analysis problem. Let the matrix A of the system

$$x_{k+1} = Ax_k, \quad x_k \in \mathbb{R}^n, \quad k = 0, 1, \ldots, \tag{16}$$

contains uncertainty of the form

$$A = A_0 + F\Delta H, \tag{17}$$

where the nominal matrix $A_0 \in \mathbb{R}^{n\times n}$ is Schur stable, $F \in \mathbb{R}^{n\times p}$, $H \in \mathbb{R}^{q\times n}$ are fixed and known, and the matrix perturbation $\Delta \in \mathbb{R}^{p\times q}$ is bounded in the spectral norm:

$$\|\Delta\| \leqslant \delta. \tag{18}$$

Structured uncertaint of this form is typical to numerous control problems (e.g., see [7]); we are interested in certifying stability of the whole family and the goal is to evaluate the maximal possible peak.

As above, the initial conditions x_0 in (16) are assumed to be bounded $\|x_0\| \le 1$. Let us estimate from above the maximum value of $\|x_k\|$ in (16), (17) for all such initial conditions and all admissible uncertainties (18).

The mandatory requirement is that the system be stable; hence, we first evaluate the maximum magnitude δ of the span of Δ that retains the stability of the system for all admissible values of Δ. More specifically, using the concept of robust quadratic stability, we find the *quadratic stability margin*, which is defined as the maximal value of δ that admits the existence of the common quadratic Lyapunov function for the uncertain system.

A straightforward modification of Theorem 2.3.3 [7] on the robust quadratic stability of the family (17), (18) leads to the following result, which was obtained in [7] for the case $\delta = 1$. The so-called Petersen's lemma [26] is a key tool for the derivation of the proposition below (various modifications and reformulations of this lemma and can be found in [7]).

Proposition. *The feasibility of the matrix inequality*

$$\begin{pmatrix} P - FF^\top & A_0 P & \mathbf{0} \\ PA_0^\top & P & PH^\top \\ \mathbf{0} & HP & \delta^{-2} I \end{pmatrix} \succ 0$$

in the matrix variable $P \succ 0$ is equivalent to the quadratic stability of system (16)–(18) *for all $\|\Delta\| \leqslant \delta$. Any solution P defines the common quadratic Lyapunov function $V(x) = x^\top P^{-1} x$.*

Here, $\mathbf{0}$ denotes the zero matrix of compatible dimensions.

The maximization of δ over the solutions of the linear matrix inequality above is equivalent to the minimization of the quantity δ^{-2}; hence, introducing the new variable $\mu \doteq \delta^{-2}$, we immediately arrive at the following result.

Theorem 3. *Let $\tilde{\mu}$ be a solution of the following semidefinite program:*

$$\mu \longrightarrow \min$$

subject to the constraint

$$\begin{pmatrix} P - FF^\top & A_0 P & \mathbf{0} \\ PA_0^\top & P & PH^\top \\ \mathbf{0} & HP & \mu I \end{pmatrix} \succ 0 \tag{19}$$

in the matrix variable $P \succ 0$ and the scalar variable μ. Then the quadratic stability margin of the family (17), (18) *is given by $r = \tilde{\mu}^{-1/2}$.*

We now evaluate the magnitude of peaks of trajectories of the uncertain system (17), (18). To guarantee the quadratic stability of the family, we assume $\|\Delta\| \le \delta < r$; then, for $\mu = \delta^{-2}$, Ineq. (19) is feasible in P. This inequality is a robust counterpart of the Lyapunov inequality $APA^\top - P \prec 0$; hence, mimicking the logic of Theorem 1, we obtain its robust version.

Theorem 4. *Let $\tilde{\gamma}$ be a solution of the following semidefinite program:*

$$\gamma \longrightarrow \min$$

subject to the constraints

$$\begin{pmatrix} P - FF^\top & A_0 P & \mathbf{0} \\ PA_0^\top & P & PH^\top \\ \mathbf{0} & HP & \delta^{-2} I \end{pmatrix} \succ 0, \qquad I \preccurlyeq P \preccurlyeq \gamma I,$$

in the matrix variable $P = P^\top$ *and the scalar variable* γ. *Then, for the trajectories of system* (16)–(18) *we have*

$$\max_{\|\Delta\| \le \delta} \max_{\|x_0\| \le 1} \max_{k=0,1,\ldots} \|x_k\| \le \gamma^{1/2}.$$

Example 3. Consider system (16), (17) with matrix A_0 of the form (13) and $F = H = I$. Theorem 1 provides the upper bound $\eta_{\text{upp}}(A_0) = 2.3803$ on the magnitude of peak. Use of Theorem 3 leads to $r = 0.0071$ for the quadratic stability margin, a small value, since A_0 is close to instability. Letting $\delta = 0.9\,r$ and using Theorem 4, we obtain an upper bound on the magnitude of "robust" peak, equal to 15.3720, which is much greater than the estimate $\eta_{\text{upp}}(A_0) = 2.3803$ obtained for the uncertainty-free nominal matrix. This is explained both by the very formulation of the problem and, more importantly, by using the approach based on common Lyapunov functions, which may introduce additional high conservatism.

A robust version of the design problem can be formulated in a similar way, by combining the results of Theorems 2 and 4. We do not present a rigorous result, since the formulation and computations become more bulky, whereas its substantive richness is mild; indeed, likewise the analysis problem, the accuracy of the bound is hard to estimate.

5. CONCLUSIONS AND DIRECTIONS FOR FUTURE RESEARCH

In the paper, numerical routines were proposed for the computation of upper bounds on the deviations of trajectories of discrete-time systems (and norms of powers of Schur stable matrices) in the state space, having nonzero initial conditions from the unit ball. The peak-minimizing controller design problem was also addressed, and a robust modification of the analysis problem was considered.

The following research directions seem to be promising.

- Description of "good" classes of matrices which admit non-conservative upper bounds from Theorems 1 and 2; similarly, for "bad" matrices.
- Since peak effect is not invariant to change of basis, it would be interesting to characterize classes of coordinate transformations that reduce the value of peak or attenuate it completely.
- Further research related to application of the superstability tools looks natural. In this case, evaluation of peak and/or controller design assumes use of linear programming rather than the apparatus of linear matrix inequalities as in the theorems presented above.
- Finding the worst-case initial conditions (yielding maximal peak) in the unit ball, and the worst-case uncertainty in the robust formulation of the problem.

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