Hierarchical Cyclic Pursuit: Algebraic Curves Containing the Laplacian Spectra

Sergei E. Parsegov, Pavel Y. Chebotarev, Pavel S. Shcherbakov, and Federico Martín Ibáñez

***Abstract*—This article addresses the problem of multiagent communication in networks with a regular directed ring structure. These can be viewed as hierarchical extensions of the classical cyclic pursuit topology. We show that the spectra of the corresponding Laplacian matrices allow exact localization on the complex plane. Furthermore, we derive a general form of the characteristic polynomial of such matrices, analyze the algebraic curves its roots belong to, and propose a way to obtain their closed-form equations. In combination with frequency-domain consensus criteria for high-order single-input single-output linear agents, these curves enable one to analyze the feasibility of consensus in networks with a varying number of agents.**

***Index Terms*—Algebraic curves, cyclic pursuit, hierarchy, Laplacian spectra of digraphs.**

I. Introduction

THE Laplacian spectra of graphs play an important role in solving distributed optimization and control problems since they mainly determine the stability and the convergence rate of the corresponding dynamical systems [1], [2], [3]. For a fixed graph, finding the spectrum does not cause any difficulties, but if we consider graphs with a scalable structure (i.e., those constructed by the repetition of the same component), the problem of exact calculation or localization of the spectrum turns out to be nontrivial. A huge amount of literature is devoted to the derivation of formulas for the Laplacian spectra of *undirected topologies*, including various lattices such as rectangular grids, honeycombs [4], hierarchical small-world networks [5], products and coronas of graphs [6], and many others.

However, when analyzing the dynamics of network systems, *directed* communication topologies are of major interest. Say, it can be observed that a group of high-order agents may converge to consensus under an undirected interaction topology, but it fails to do so under the corresponding unidirected one, even though this topology contains a spanning converging tree. A precise localization of the Laplacian spectra of digraphs serves as the basis for the analysis of consensus problems in such situations.

In this article, we study several generalizations of the *cyclic pursuit* multiagent strategy. Its history can be traced back to 1878, when Darboux [7] published his elegant work, where he studied a geometric averaging procedure and proved its convergence to consensus. Basically, cyclic pursuit is a strategy where agent i pursues its neighbor $i-1$ modulo N, where N is the number of agents. Evidently, such a communication structure is an unidirected ring or a "predecessor–follower" topology, i.e., a Hamiltonian cycle.

Cyclic pursuit strategies attracted the attention of different scientific communities (e.g., see [8], [9], [10], [11], [12], and [13]) due to a wide range of applications including but not limited to numerous formation control tasks, such as patrolling, boundary mapping, etc. Their extensions to hierarchical structures were considered in [14], [15], [16], and [17]. The work in [18] and [19] addressed the case of heterogeneous agents; the effect of communication delays was analyzed in [20]. Geometrical problems related to cyclic pursuitlike algorithms were studied in [21]. Some pursuit algorithms use the rotation operator in order to follow desired trajectories [22]. The work in [23] shows the connection of discrete-time weighted cyclic pursuit with the general DeGroot model. Another group of strategies (protocols) is based on bidirectional topologies [24], that is, each agent i has relative information about its neighbors $i-1$ and $i+1$ (modulo N). The row straightening problems studied in [25] and [26] also imply symmetric communications except for fixed "anchors" (the endpoints of a segment). The problems of *vehicle platooning* with cyclic communications (e.g., see [27], [28], [29], and [30]) are also closely related to the problems of cyclic pursuit. In this case, the network system also has inputs including the desired intervehicular distances and communication disturbances. The analysis of the closed-loop stability of such systems is reduced to the study of state matrices close or identical to those studied in cyclic pursuit.

Regular ring structures model symmetric hierarchical interaction between agents. In some cases, these structures allow for

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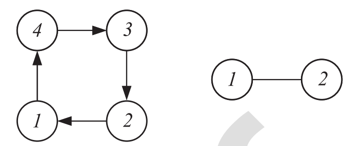
closed-form expressions for the spectra of the corresponding Laplacian matrices, which helps to analyze the control protocols these matrices are involved in. While cyclic pursuit can be treated as a special case of consensus seeking, the properties of the underlying interaction topology are closely related to classical mathematical considerations including the study of algebraic curves. For the basic cyclic pursuit topology, the eigenvalues of the corresponding Laplacian matrix are roots of unity [14]. No matter how many agents/nodes constitute the network, the spectrum lies on the unit circle. This fact prompted us to study hierarchical and other generalized ring topologies, which led to higher-order curves that contain their Laplacian spectra.

In this article, we study ring digraphs with a hierarchical "necklace" structure. It is convenient to explore the Laplacian spectra of such graphs with regularly interleaved directed and undirected arcs using the concept of hierarchy. Namely, we introduce a *macro-vertex*, which is a sequence of directed and undirected arcs (the lower level of the hierarchy) and a *directed ring of macro-vertices* (the upper level of the hierarchy). The topologies constructed in this way occupy an intermediate position between directed and undirected rings, which have been widely studied in relation to cyclic pursuit and control of homogeneous vehicular platoons running on a ring (e.g., see the nearest neighbor ring topologies presented in [28, Fig. 2(h) and (i)]).

A useful classification of consensus problems based on the notion of *complexity space* was proposed in [31, Fig. 1.1]. In accordance with it, three independent dimensions of complexity can be identified in which the simplest first-order consensus model can be generalized, namely: 1) the complexity of the agent model; 2) topological complexity (complexity of the structure of interactions); and 3) the complexity of couplings between agents. The contribution of our article to the general study of consensus in network systems can be attributed to the first two directions: The analysis and localization of the Laplacian spectra of special ring topologies to 2) and complex high-order models of agents to 1). Specifically, we prove that the Laplacian spectra of the studied digraphs lie on certain high-order algebraic curves irrespective of the number of macro-vertices forming the network. Along with this, we present an algorithm for obtaining equations of these curves. Based on this localization, we propose a geometric consensus condition in the frequency domain applicable to any number of interacting agents.

The rest of this article is organized as follows. Section II introduces some mathematical preliminaries needed for the subsequent analysis and discusses the statement of the problem. The main results that describe the Laplacian spectra of ring digraphs are presented in Section III. We prove that, regardless of the number of macro-vertices in such a digraph, its Laplacian spectrum lies on a certain algebraic curve and provide an algorithm to derive an implicit equation (of the form $p(x,y)=0$) of this curve in $\mathbb{R}^2$. In Section IV, we study consensus problems for a group of high-order linear SISO agents interacting through the discussed ring topologies, that is, performing *hierarchical* cyclic pursuit. We apply the frequency domain criterion [32], [33], [34] to derive a necessary and sufficient consensus condition, which does not depend on the number of agents in the network. The theoretical results are accompanied by numerical illustrations and. Finally, Section V concludes this article.

Throughout the article, $j := \sqrt{-1}$ denotes the imaginary unit while letters i and k are used for indexing purposes.



(a) (b)

Fig. 1. Hamiltonian cycle corresponding to the cyclic pursuit strategy with (a) four agents and (b) a macro-vertex. (a) Hamiltonian cycle on 4 nodes. (b) Macro-vertex on 2 nodes.

II. Preliminaries and Problem Statement

In this article, we study network systems that have a hierarchical ring structure. After defining the basic terminology, we formulate the problem.

Throughout the article, we consider finite digraphs allowing in some cases multiple arcs and loops. A digraph is denoted by $\mathcal{G}_N = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, N\}$ stands for the node set and $\mathcal{E}$ for the multiset[1] of arcs.

The formal definitions of the adjacency and Laplacian matrices of an unweighted digraph $\mathcal{G}_N$ are given below.

Definition 1: The *adjacency matrix* associated with a digraph $\mathcal{G}_N = (\mathcal{V}, \mathcal{E})$ is the matrix $\mathcal{A}_N = (a_{ik}) \in \mathbb{R}^{N\times N}$, where each entry a_{ik} is the number of arcs of the form (i,k) in $\mathcal{E}$.

Definition 2: The *Laplacian matrix* $\mathcal{L}_N \in \mathbb{R}^{N\times N}$ of $\mathcal{G}_N$ is the matrix with entries $l_{ii} = \sum_{k\neq i} a_{ik}$ and $l_{ik} = -a_{ik}$ for $i \neq k$, where $(a_{ik}) = \mathcal{A}_N$ is the adjacency matrix of $\mathcal{G}_N$.

For example, consider a graph that represents communications within the conventional cyclic pursuit strategy for four agents [see Fig. 1(a)]. Here, an arc from i to k shows that agent i pursues agent k.

The corresponding Laplacian matrix for the general case of N agents can be defined through the counterclockwise principal circulant permutation matrix [37] $\mathcal{P}_N$ as follows:

$$\mathcal{L}_N = I_N - \mathcal{P}_N$$

where $I_N \in \mathbb{R}^{N\times N}$ is the identity matrix

$$\mathcal{P}_N = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \quad (1)$$

[1] A multiset, unlike a set, allows multiple occurrences of each element. We need this in one particular case in which we assume the presence of multiple arcs in a digraph [see Fig. 4(b)].

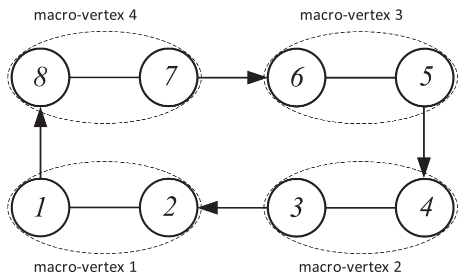


Fig. 2. Ring digraph with four macro-vertices.

and

$$\mathcal{L}_N = \begin{bmatrix} 1 & 0 & 0 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}. \qquad (2)$$

We now describe the structure of hierarchical network systems studied ahead. The lower level of the hierarchy is a *linear macro-vertex*, which is a specific subdigraph whose $n \geq 1$ nodes are identified with indexed dynamical agents while the top level is a Hamiltonian cycle on[2] $m \geq 1$ macro-vertices.

Definition 3: A *linear macro-vertex* $\mathcal{G}_n^i = (\mathcal{V}^i, \mathcal{E}^i)$ of a digraph $\mathcal{G}_N = (\mathcal{V}, \mathcal{E})$ is a subdigraph of $\mathcal{G}_N$ with $\mathcal{V}^i = \{v_1^i, \ldots, v_n^i\}$ $(n \geq 1)$ obtained from the directed path $v_n^i \to v_{n-1}^i \to \cdots \to v_1^i$ (*main direction*; no arcs when $n = 1$) by adding the reverse path $v_1^i \to v_2^i \to \cdots \to v_n^i$ from which any subset of arcs is dropped.

The following definition introduces a topology consisting of m identical macro-vertices on disjoint subsets of nodes along with a top-level Hamiltonian cycle that forms a Hamiltonian cycle on the whole set of $N = mn$ nodes together with the main direction paths traversing the macro-vertices. We will associate the term *ring digraph* with such a topology.

Definition 4: A *ring digraph* denoted by $\mathcal{G}_{m,n} = (\mathcal{V}, \mathcal{E})$ is a digraph such that $\mathcal{V} = \bigcup_{i=1}^m \mathcal{V}^i$, $\mathcal{V}^i = \{n(i-1)+1, \ldots, ni\}$, $\mathcal{E} = (\bigcup_{i=1}^m \mathcal{E}^i) \cup \{e_1, \ldots, e_m\}$,$(\mathcal{V}^i, \mathcal{E}^i) = \mathcal{G}_n^i$ are identical linear macro-vertices on $n \geq 1$ nodes, and the arcs $e_i = (ni+1, ni)$ $(i \in \{1, \ldots, m-1\})$ and $e_m = (1, nm)$ link the first node of each macro-vertex with the nth node of the previous one (which is the same macro-vertex when $m = 1$).

It can be observed that each macro-vertex of a ring digraph is its *induced*[3] subdigraph whenever $m > 1$ while for $m = 1$, it drops the arc $(1, n)$. The arcs $e_1, \ldots, e_m$ form a Hamiltonian cycle on m macro-vertices.

An example of a ring digraph with $n = 2$ and $m = 4$ is presented in Fig. 2. It is constructed from the Hamiltonian cycle shown in Fig. 1(a) and the macro-vertex (it is the complete digraph on two nodes) shown in Fig. 1(b), where a pair of opposite arcs is represented by a line segment without arrows.

Remark 1: A ring digraph can be considered as a Hamiltonian cycle $\{(1, N), (N, N-1), \ldots, (2, 1)\}$ supplemented by the path $\{(1, 2), (2, 3), \ldots, (N-1, N)\}$ in which ν $(0 \leq \nu \leq N-1)$ arcs are dropped in a regular fashion. In a sense, ring digraphs fill the gap between the Hamiltonian cycle and the bidirectional ring. Obviously, every ring digraph contains a spanning converging tree. It should be noted that this condition is necessary and sufficient for attaining asymptotic consensus in the system consisting of first-order agents. In Section IV, we consider a more general setting with high-order agent models and derive a consensus condition that does not depend on the number of nodes in the network.

We now introduce cooperating agents and then formulate the problem. The agents are assumed to have identical high-order (double integrator or higher) SISO linear models. Let $x_i \in \mathbb{R}$ represent the position of agent i, $i \in \{1, \ldots, N\}$. Therefore, the consensus-seeking communication over the network $\mathcal{G}_{m,n}$ can be described as

$$\mathbf{a}(s)x_i = u_i \qquad (3)$$

$$u_i = \mathbf{b}(s)\left(\sum_{k \in \mathcal{N}_i} a_{ik}(x_k - x_i)\right), \quad i \in \{1, \ldots, N\} \qquad (4)$$

where a_{ik} are the elements of the adjacency matrix $\mathcal{A}_N$ and $\mathcal{N}_i$ is the set of neighbors of node i, i.e., the set of nodes k such that $a_{ik} \neq 0$. Here $s := d/dt$ denotes the differentiation operator, the scalar polynomials

$$\mathbf{a}(s) = s^d + \mathbf{a}_{d-1}s^{d-1} + \ldots + \mathbf{a}_1 s + \mathbf{a}_0$$

$$\mathbf{b}(s) = \mathbf{b}_q s^q + \mathbf{b}_{q-1}s^{q-1} + \ldots + \mathbf{b}_1 s + \mathbf{b}_0$$

determine agent's dynamics and communications, and u_i is the control signal. For convenience, we assume $d > q$.

Let us introduce the vector $\xi_i = [x_i, \dot{x}_i, \ldots, x_i^{(d-1)}]^\top$ and transform (3), (4) into the state-space form

$$\dot{\xi}_i = A\xi_i + Bu_i \qquad (5)$$

$$u_i = K\sum_{k \in \mathcal{N}_i} a_{ik}(\xi_k - \xi_i), \quad i \in \{1, \ldots, N\} \qquad (6)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\mathbf{a}_0 & -\mathbf{a}_1 & -\mathbf{a}_2 & \cdots & -\mathbf{a}_{d-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$K = \begin{bmatrix} \mathbf{b}_0 & \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_q & 0 & \ldots & 0 \end{bmatrix}.$$

The entire closed-loop dynamics can thus be written as

$$\dot{\xi} = (I_N \otimes A - \mathcal{L}_N \otimes BK)\xi \qquad (7)$$

[2]The shortest Hamiltonian cycle consists of one node (in our construction, it is a macro-vertex) and one directed loop.

[3]An induced subdigraph of a digraph is a subdigraph whose arc set consists of all of the arcs of the digraph that have both endpoints in the node set of the subdigraph.

where $\xi = [\xi_1^\top, \xi_2^\top, \dots, \xi_N^\top]^\top$ and $\otimes$ is the Kronecker product.

In Section IV, we will obtain a consensus criterion for ring-shaped networks of agents (3), (4).

Let us formulate a definition of consensus for the systems under study.

Definition 5: We say that the network system (5) with feedback control (6) *reaches consensus* if

$$\lim_{t\to\infty} \|\xi_i(t) - \xi_k(t)\| = 0 \quad \forall i, k \in \{1, \dots, N\} \tag{8}$$

for any initial condition $\xi(0) = [\xi_1^\top(0), \dots, \xi_N^\top(0)]^\top$.

In the simplest case of $\mathbf{a}(s) = s$ and $\mathbf{b}(s) = 1$, we face the classical first-order consensus model; e.g., the cyclic pursuit if $a_{ik} = 1$ for $k = i - 1 \pmod N$ and $a_{ik} = 0$ otherwise. The corresponding Laplacian matrix $\mathcal{L}_N$ is given by (2), and its characteristic polynomial $\Delta(\lambda)$ has the form

$$\Delta(\lambda) = (\lambda - 1)^N - 1.$$

The roots of $\Delta(\lambda)$ can be found using Lemma 1, which follows from De Moivre's Theorem.

Lemma 1: The roots of the cyclotomic equation

$$\sigma^N - 1 = 0 \tag{9}$$

are

$$\sigma_k = \mathrm{e}^{j\frac{2\pi k}{N}}, k \in \{0, \dots, N-1\} \tag{10}$$

and the roots of

$$\sigma^N + 1 = 0 \tag{11}$$

are

$$\sigma_k = \mathrm{e}^{j\frac{2\pi k + \pi}{N}}, k \in \{0, \dots, N-1\}. \tag{12}$$

The roots in both sets are uniformly distributed on the unit circle centered at $(0, j0)$ in the complex plane $\mathbb{C}$.

Therefore, the spectra of the Laplacian matrices (2) with all $N \in \mathbb{N}$ are jointly dense on the unit circle centered at $(1, j0)$.

The equation of the corresponding unit circle in $\mathbb{R}^2$ is

$$(x-1)^2 + y^2 - 1 = 0. \tag{13}$$

This circle is a basic example of a curve that contains the Laplacian spectrum of a ring digraph; it entirely lies in $\mathbb{C}^+ \cup \{0\}$. The spectrum of any such a digraph contains 0 with multiplicity 1, which guarantees consensus in the first-order cyclic pursuit process according to the well-known consensus criterion.

Remark 2: The dynamic system (3), (4) can be considered from different points of view: Its coordinates can have different physical meanings, and the signal u_i can contain both the plant dynamics and elements of a local or/and a distributed controller. In addition, the right-hand side can also contain other external signals and disturbances that do not affect the form of the state matrix of the closed-loop system (7). A particular example of such a system is a leaderless vehicle platoon moving on a ring, e.g., see [27], [28], [29], and [30]. In such problems, two types of stability are studied: The classical stability of a closed-loop system and string stability associated with the amplification of a disturbance propagating through the system [35], [36]. With an increase in the number of vehicles N in the platoon, the system

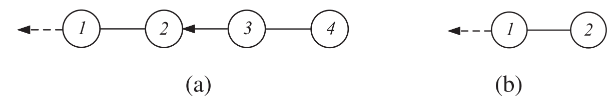


Fig. 3. Macro-vertex (a) on four nodes can be obtained by connecting two macro-vertices of type (b) by a directed arc. (a) Macro-vertex on 4 nodes. (b) Macro-vertex on 2 nodes.

may exhibit *eventual instability* [35]. Therefore, the problem of stabilization regardless of the number N is important.

The article aims at the following:

1) localizing the Laplacian spectra of the ring digraphs defined above;
2) obtaining a necessary and sufficient consensus condition applicable to any number of agents in the network.

III. Laplacian Spectra of Ring Digraphs

In this section, we propose a method for the exact localization of Laplacian spectra for ring digraphs. It turns out that these spectra always lie on algebraic curves whose expressions can be found in a closed form. Thus, equations of these curves are among the main results of the work. First, we classify ring digraphs and discuss their properties. After that we

1) derive a general form of the characteristic polynomial of the corresponding Laplacian matrices;
2) present a way to obtain the equations of algebraic curves that contain the roots of the characteristic polynomial regardless of the number of nodes in $\mathcal{G}_{m,n}$.

A. Simple and Complex Rings

Let us find out how the set of ring digraphs is organized. Clearly, different macro-vertices can give rise to isomorphic ring digraphs. For instance, consider the two macro-vertices depicted in Figs. 3(a) and (b), where each macro-vertex has an unattached dotted arc of a Hamiltonian cycle connecting macro-vertices within a ring digraph. Obviously, two macro-vertices of type (a) form the same digraph (shown in Fig. 2) as four macro-vertices of type (b).

By construction, ring digraphs are scalable, i.e., they can be "inflated" by cloning macro-vertices. To distinguish the types of such digraphs and characterize their simplest components, we introduce the following definition.

Definition 6: A ring digraph will be called a *complex ring* if it can be represented as a Hamiltonian cycle on two or more macro-vertices. If this is not the case, we call it a *simple ring*. A complex ring $\mathcal{G}_{m,n}$ is said to be a *round replication* of a simple ring $\mathcal{G}_{1,n}$ if the representations of $\mathcal{G}_{m,n}$ and $\mathcal{G}_{1,n}$ involve identical macro-vertices.

While examples of simple and complex rings are shown in Fig. 4, the theorem ahead recursively counts the number of nonisomorphic simple rings with a given number of nodes.

Theorem 1: The number $Y(N)$ of nonisomorphic simple rings on N nodes satisfies the relationship

$$Y(N) = \frac{2^N - \sum_{n \in D(N)} nY(n)}{N} \tag{14}$$

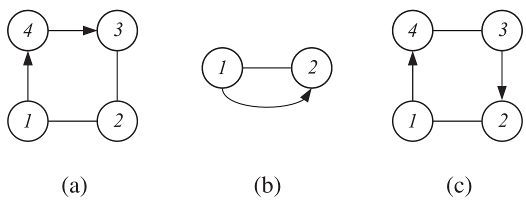


Fig. 4. (a) and (b) Two simple rings and (c) a complex ring constructed as the round replication of the simple ring (b). (a) Simple ring on 4 nodes. (b) Simple ring on 2 nodes. (c) Complex ring on 4 nodes.

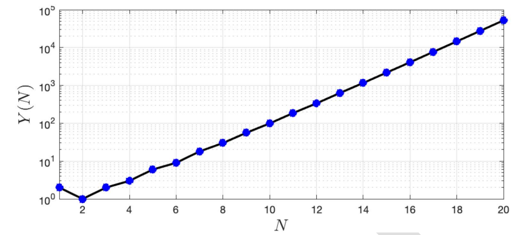


Fig. 5. Quantity $Y(N)$ as function of the number of nodes.

where $D(N)$ is the set of all divisors of N excluding N and $Y(1)$ is set to be 2.

Proof: First, to simplify the proof, we redefine ring digraph on $N = 1$ node (cyclic pursuit of a single agent makes no sense, so this redefinition does not affect the application) as a multidigraph that has either 1 or 2 directed loops. Then, $Y(1) = 2$, as stated in Theorem 1. Next, for any $N > 1$, let us supplement the set of ring digraphs on N nodes with all digraphs of the same form that additionally have arc $(N, 1)$, where $N = mn$ (this arc is absent in ring digraphs by definition). The supplemented set of ring digraphs will be called the set of *necklace digraphs*.

Any necklace digraph on the node set $\mathcal{V} = \{1, \dots, N\}$ can be identified with a vector $(a_1, \dots, a_N)$, where $a_i = 2$ if and only if there are two opposite arcs between nodes i and $i+1 \pmod N$ and $a_i = 1$ otherwise. A necklace digraph is *periodic* if its vector representation is periodic in the sense that $(a_1, \dots, a_N) = (a_1, \dots, a_n, a_1, \dots, a_n, \dots, a_1, \dots, a_n)$ with $n < N$ being the minimum length of a subvector whose replication gives the whole vector.

Denote by $\widetilde{Y}(N)$ the number of nonisomorphic nonperiodic necklace digraphs on N nodes. Obviously, there is a bijection between such digraphs and distinct cycles of minimal period N (in the case of two contractivity factors) enumerated[4] in [40, Sec. 4.8, Lemma 1]. Consequently, $\widetilde{Y}(N) = (2^N - \sum_{n \in D(N)} n\widetilde{Y}(n))/N$. Finally, we prove that $Y(N) = \widetilde{Y}(N)$ for all $N \in \mathbb{N}$. We have $Y(1) = \widetilde{Y}(1)$ by redefinition. For $N > 1$, consider any nonperiodic necklace digraph. Its vector representation contains at least one $a_i = 1$. Therefore, it can be transformed into the representation of a simple ring by a number of cyclic shifts transferring $a_i = 1$ to the position a_N corresponding to the pair of nodes $(N, 1)$. This defines a one-to-one correspondence between the equivalence classes of isomorphic nonperiodic necklace digraphs and the classes of isomorphic simple rings (all on N nodes). Hence, the number of the latter classes is given by (14). ■

Corollary 1: 1. If N is prime, then $Y(N) = (2^N - 2)/N$. 2. If $N = 2^p$, $p \in \mathbb{N}$, then $Y(N) = (2^N - 2^{N/2})/N$.

Proof: The first statement is a direct consequence of Theorem 1. To prove the second one by induction, first observe that in the base case, $p = 1$, it follows from the first part. Assume that it is true for all $N = 2^k$, $k < p$ and prove it for $N = 2^p$. In this case, $D(N) = \{1, 2, \dots, N/2\}$. By Theorem 1 and the induction hypothesis, it holds that $Y(N) = (2^N - 2^1 - (2^2 - 2^1) - \dots - (2^{N/2} - 2^{N/4}))/N = (2^N - 2^{N/2})/N$, as desired.

Some values of the function $Y(N)$ (modified for $N = 1$) are given in Table I. Fig. 5 illustrates its growth graphically using base-10 logarithmic scale on the vertical axis. ■

Remark 3: In the proof of Theorem 1, we reduced the enumeration of nonisomorphic simple rings on N nodes to that of distinct cycles of minimal period N. Essentially, the same numerical sequence appeared as a solution to a number of other equivalent enumeration problems including those of dimensions of the homogeneous parts of the free Lie algebras, irreducible polynomials of degree N over the field GF(2), binary Lyndon words of length N, etc. (see sequences A001037 and A059966 in [41]).

It is worth mentioning that expression (14) has significant consequences regarding the divisibility of numbers. Say, part 1 of Corollary 1 implies a special case of Fermat's little theorem ($a^p \equiv a \pmod p$, where p is prime) for $a = 2$ while extending (14) to multigraphs gives a proof of this theorem in its general form.

B. Laplacian Spectra and Algebraic Curves

We now consider complex rings with $N > 3$ nodes and characterize the locus of the corresponding Laplacian spectra.

Theorem 2: For any simple ring $\mathcal{G}_{1,n}$ on n nodes, the Laplacian eigenvalues of all complex rings $\mathcal{G}_{m,n}$ obtained by m-fold round replication of $\mathcal{G}_{1,n}$ belong to a bounded algebraic curve of order $2n$ in $\mathbb{C}^+ \cup \{0\}$.

Proof: In accordance with [42, Th. 4], the Laplacian characteristic polynomial of $\mathcal{G}_{m,n}$ has the form

$$\Delta(\lambda) = (P_n(\lambda))^m - (-1)^N \tag{15}$$

where $P_n(\lambda) = \prod_{k=1}^{K} Z_{i_k}$ is an nth-order polynomial and $i_1, \dots, i_K$ are the path lengths in the decomposition of the cycle $\{(1, n), (n, n-1), \dots, (2, 1)\}$ into the paths linking the consecutive nodes of indegree 1 in $\mathcal{G}_{1,n}$. The polynomials Z_i are the modified Chebyshev polynomials of the second kind

$$Z_n(\lambda) := (\lambda - 2)Z_{n-1}(\lambda) - Z_{n-2}(\lambda)$$

where $Z_0(\lambda) \equiv 1$ and $Z_1(\lambda) \equiv \lambda - 1$.

By Lemma 1, the roots $\alpha_k + j\beta_k$, $k \in \{0, \dots, m-1\}$ of $\sigma^m - (-1)^N = 0$ are roots of unity (the roots of $\sigma^m = -1$ are also roots of $\sigma^{2m} = 1$) lying on the unit circle in $\mathbb{C}$. Therefore,

[4] Problem 3.5 "How many different necklaces of length m can be made from beads of q given colors?" appeared earlier in [38], although without the desired formula; see also [39].

TABLE I
FIRST VALUES OF THE FUNCTION $Y(N)$, THE NUMBER OF NONISOMORPHIC RING DIGRAPHS ON N NODES

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$Y(N)$	2	1	2	3	6	9	18	30	56	99	186	335	630	1161	2182	4080	7710	14532	27594	52377

by (15), the zeros of $\Delta(\lambda)$ satisfy

$$P_n(\lambda) = \alpha_k + j\beta_k, \quad k \in \{0, \ldots, m-1\} \tag{16}$$

where

$$\alpha_k^2 + \beta_k^2 = 1. \tag{17}$$

Varying m we obtain a countable set of roots of unity, which is everywhere dense on the unit circle. This means that for any $u, v \in \mathbb{R}$ such that $u^2 + v^2 = 1$, there exist sequences $u_i \to u$ and $v_i \to v$ such that $u_i + jv_i$ are roots of unity ($i \in \mathbb{N}$). Based on this we apply [43, Th. 11.1] on the continuous dependence of the roots of a polynomial with leading coefficient 1 on its other coefficients (cf. [44], [45]). Due to this theorem, if λ_k, $k \in \{0, \ldots, n-1\}$, are the roots of equation $P_n(\lambda) = u + jv$, then the roots $\lambda_{k,i}$ of equations $P_n(\lambda) = u_i + jv_i$ ($k \in \{0, \ldots, n-1\}, i \in \mathbb{N}$) can be numbered in such a way that $\lambda_{k,i} \to \lambda_k$, $k \in \{0, \ldots, n-1\}$. This justifies the following method for determining the curve (in the implicit form $f(x, y) = 0$) on which the Laplacian eigenvalues of complex rings $\mathcal{G}_{m,n}$ are everywhere dense. Setting $\lambda = x + jy$ for (16) and substituting $\mathrm{Re}[P_n(x + jy)] = \alpha_k$ and $\mathrm{Im}[P_n(x + jy)] = \beta_k$ into (17) yields an equation of order $2n$, which determines the desired algebraic curve of order $2n$ in the form $f(x, y) = 0$. Indeed, this curve contains the roots of (16) for all $\alpha_k + j\beta_k$ that belong to the unit circle. According to the above continuity theorem, any neighborhood of each such a root contains infinitely many roots of (16) in which $\alpha_k + j\beta_k$ are roots of unity. The latter roots lie on the same curve and are the Laplacian eigenvalues of ring digraphs $\mathcal{G}_{m,n}$. By the properties of the Laplacian spectra of digraphs, they lie in $\mathbb{C}^+ \cup \{0\}$. Substituting $\lambda = |\lambda|(\cos\varphi + j\sin\varphi)$ into $P_n(\lambda) = \lambda^n + \sum_{k=0}^{n-1} p_k \lambda^k$ for $\lambda \neq 0$ we have $|P_n(\lambda)| = |\lambda|^n |1 + \sum_{k=0}^{n-1} p_k \lambda^{-n+k}(\cos k\varphi + j\sin k\varphi)|$. Therefore, it is easy to specify $h > 0$ such that $|\lambda| > h$ implies $|P_n(\lambda)| > 1$. Consequently, λ with $|\lambda| > h$ cannot satisfy (16) and thus the Laplacian spectra locus of ring digraphs $\mathcal{G}_{m,n}$ is bounded.

Let us emphasize that an unbounded "inflation" of a ring digraph $\mathcal{G}_{m,n}$ by increasing m leaves the Laplacian eigenvalues on the same algebraic curve and only increases their density on it. ■

Corollary 2: For a fixed $n \in \mathbb{N}$, the number of distinct algebraic curves of order $2n$ containing the Laplacian spectra of ring digraphs obtained by round replication of simple rings on n nodes does not exceed the number of nonisomorphic simple rings on n nodes determined by Theorem 1.

C. Quartic and Sextic Curves

In this section, we consider several special cases that allow relatively simple closed-form expressions of the corresponding algebraic curves mentioned in Theorem 2.

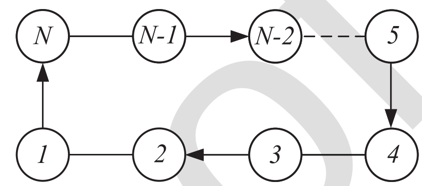


Fig. 6. Round replication of the simple ring shown in Fig. 4(b).

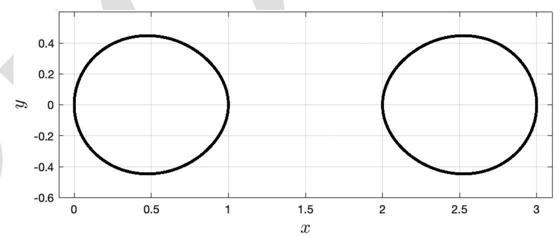


Fig. 7. Cassini ovals.

The case $n = 2$: We first consider a complex ring with the following structure: It has $N = 2m$ nodes, $m \geq 2$, and contains a Hamiltonian cycle supplemented by the inverse cycle, where every other arc is dropped (see Fig. 6). This digraph is a round replication of the simple ring depicted in Fig. 4(b); the ring digraph in Fig. 2 belongs to this class with $m = 4$.

The Laplacian matrix of this digraph has the form

$$\mathcal{L}_N = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{18}$$

and by (15), its characteristic polynomial is $(Z_2)^{\frac{N}{2}} - 1 = (\lambda^2 - 3\lambda + 1)^m - 1$. Its roots satisfy

$$\lambda^2 - 3\lambda + 1 - \alpha_k - j\beta_k = 0, \quad k \in \{0, \ldots, m-1\}.$$

From $(x + jy)^2 - 3(x + jy) + 1 - \alpha_k - j\beta_k = 0$, it follows $\alpha_k = (x - 1.5)^2 - y^2 - 1.25$ and $\beta_k = 2xy - 3y$. Substituting the last expressions into (17) gives the equation of the curve.

In this case, the eigenvalues of the Laplacian matrix (18) lie on the quartic Cassini curve (Cassini ovals) defined by

$$[(\tilde{x} - \sqrt{5})^2 + \tilde{y}^2][(\tilde{x} + \sqrt{5})^2 + \tilde{y}^2] = 2^4 \tag{19}$$

where $\tilde{x} = 2(x - 3/2)$ and $\tilde{y} = 2y$, see [46] for the details. This curve is shown in Fig. 7.

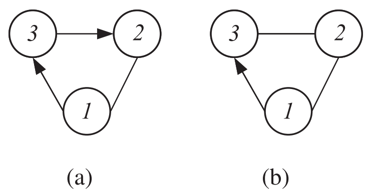


Fig. 8. Two simple rings on $n = 3$ nodes. (a) Simple ring #1. (b) Simple ring #2.

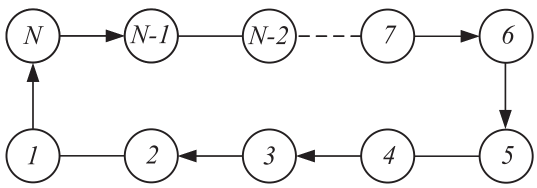


Fig. 9. Ring digraph obtained by round replication of the simple ring in Fig. 8(a).

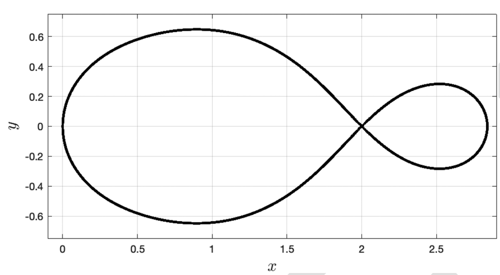


Fig. 10. Sextic curve defined by (21).

The case $n = 3$. Observe that there are exactly two nonisomorphic simple rings on $n = 3$ nodes; these are depicted in Fig. 8.

Consider two complex rings on $N = 3m$ nodes ($m > 1$) constructed by round replication of these simple rings. The one obtained from simple ring #1 is shown in Fig. 9.

Its Laplacian matrix has the form

$$\mathcal{L}_N = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{20}$$

and by (15), its characteristic polynomial is $(Z_1Z_2)^m - (-1)^N$.

According to Theorem 2, the eigenvalues of matrix (20) lie on a sextic curve. Its equation is

$$\left(\tilde{x}^2 + \tilde{y}^2\right)^3 + (4 + 4\tilde{x})\left(\tilde{x}^2 + \tilde{y}^2\right)^2 - 2\tilde{x}^3 - 4\tilde{x}^2 + 6\tilde{x}\tilde{y}^2 + 4\tilde{y}^2 = 0 \tag{21}$$

where $\tilde{x} = x - 2$ and $\tilde{y} = y$. This curve is depicted in Fig. 10.

The complex ring constructed by round replication of simple ring #2 [see Fig. 8(b)] is shown in Fig. 11.

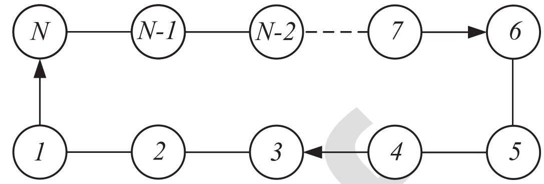


Fig. 11. Ring digraph obtained by round replication of the simple ring in Fig. 8(b).

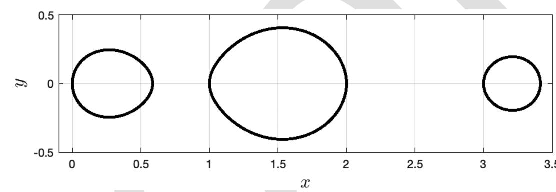


Fig. 12. Sextic curve defined by (23).

Its Laplacian matrix is of the form

$$\mathcal{L}_N = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{22}$$

and by (15), its characteristic polynomial is $(Z_3)^m - (-1)^N$.

By Theorem 2, the eigenvalues of matrix (22) lie on a sextic curve; it is defined by equation

$$\left(\tilde{x}^2 + \tilde{y}^2\right)^3 + 2\tilde{x}\left(\tilde{x}^2 + \tilde{y}^2\right)^2 - 3\tilde{x}^4 - 6\tilde{x}^3 + 2\tilde{x}^2\tilde{y}^2 + 2\tilde{x}^2 + 2\tilde{x}\tilde{y}^2 + 4\tilde{x} + 5\tilde{y}^4 + 6\tilde{y}^2 = 0 \tag{23}$$

where $\tilde{x} = x - 2$ and $\tilde{y} = y$. This curve is depicted in Fig. 12.

Graphs with a more complex structure based on simple rings on $4, 5, \ldots,$ nodes can be obtained in the same way along with the corresponding expressions for higher-order curves that contain the spectrum loci.

In Section III-D, we present a result involving a weighted necklace digraph. Such a structure generalizes the topology of cyclic pursuit in a different way: There are no macro-vertices, but the arcs of one of the directions are weighted and have the same weight.

Due to the presence of this variable weight, the corresponding Laplacian spectra belong to a certain drop-shaped *region* rather than lie on an algebraic curve.

D. Weighted Ring

Consider a *weighted necklace* digraph on N nodes consisting of a Hamiltonian cycle and the inverse one.

Assume that all arcs of one of the cycles have the same weight a, and the arcs in the opposite direction have weight b. Without loss of generality, we can restrict ourselves to the case where one weight is unity and the other one is $c \in [0,\ 1]$.

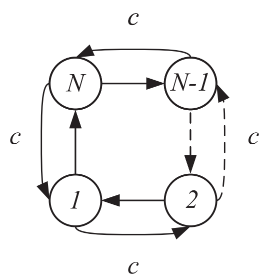


Fig. 13. Two-cycle weighted digraph.

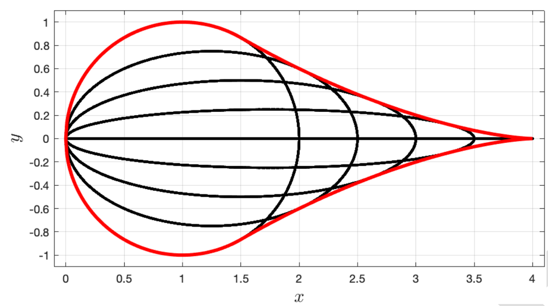


Fig. 14. Sequence of five ellipses that contain the spectrum loci of the Laplacian matrices (24) as c increases from 0 to 1, including a unit circle ($c = 0$) and a segment ($c = 1$); the boundary $f_{1,2}(x)$ of a drop-shaped region, which is the union of all the ellipses (see Theorem 3), is shown in red.

A digraph of this type is shown in Fig. 13.

Its Laplacian matrix has the form

$$\mathcal{L}_N = \begin{bmatrix} 1{+}c & -c & 0 & 0 & \cdots & 0 & -1 \\ -1 & 1{+}c & -c & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1{+}c & -c & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & 1{+}c & -c & 0 \\ 0 & \cdots & 0 & 0 & -1 & 1{+}c & -c \\ -c & \cdots & 0 & 0 & 0 & -1 & 1{+}c \end{bmatrix}. \quad (24)$$

Lemma 2: For any weight $c \in [0,\, 1]$ and any $N \in \mathbb{N}$, the eigenvalues of matrix (24) lie on the ellipse

$$\frac{(x-(1+c))^2}{(1+c)^2} + \frac{y^2}{(1-c)^2} = 1. \quad (25)$$

Proof: Obviously, $\mathcal{L}_N = (1+c)I_N - \mathcal{P}_N - c\mathcal{P}_N^{N-1}$, where $\mathcal{P}_N$ is the counterclockwise principal circulant permutation matrix (1). Therefore, the eigenvalues of the Laplacian matrix are $\lambda_k = (1+c) - \mathrm{e}^{j\frac{2\pi k}{N}} - c\,\mathrm{e}^{j\frac{2\pi(N-1)k}{N}}$, $k \in \{1, \dots, N\}$. Rewriting this expression in a trigonometric form leads to the parametric equation of the ellipse (25) in $\mathbb{R}^2$.

Remark 4: The limit cases of (25) are the unit circle centered at $(1, 0)$ (for $c = 0$) and the segment $[0, 4]$ of the real axis (for $c = 1$). These two limit shapes are shown in Fig. 14 along with the three ellipses of the form (25).

Theorem 3: Every eigenvalue of matrix (24) for any $c \in [0,\, 1]$ and $N \in \mathbb{N}$ lies in the drop-shaped region bounded by the functions

$$f_{1,2}(x) = \begin{cases} \pm\sqrt{1-(x-1)^2} & \text{if } x \in [0, 1.5] \\ \pm\frac{1}{\sqrt{2}}(3-\sqrt{1+2x})\sqrt{\sqrt{1+2x}-x+1} & \text{if } x \in (1.5, 4]. \end{cases} \quad (26)$$

Proof: For ellipses (25), we have $x \in [0, 2(1+c)]$ and $y \in [-(1-c), (1-c)]$, with the maximum and minimum at $x = 1+c$ (cf. Fig. 14). Thus, for any two different ellipses of this family, each one extends beyond the other. Let us fix $c \in (0, 1)$. Suppose that $(x_{cz}, \pm y_{cz})$ with $x_{cz} \neq 0$ are the intersection points of the two ellipses corresponding to arc weights c and $z \neq c$. Then, x_{cz} increases in z. Let $f_z(x)$ be the function representing the upper (nonnegative) part of the ellipse corresponding to $z \in (0, 1)$. We have

$$f_z(x) > f_c(x) \text{ whenever}$$
$$((z < c)\,\&\,(0 < x < x_{cz})) \text{ or } ((c < z)\,\&\,(x_{cz} < x \le 2+2c)). \quad (27)$$

Let

$$x_c = \lim_{z' \to c-0,\, z'' \to c+0} x_{z'z''} = \lim_{z' \to c-0} x_{z'c} = \lim_{z'' \to c+0} x_{cz''}.$$

It follows from (27) that the only x for which $f_c(x) = \max_z f_z(x)$ is x_c.

Let us find x_c as a function of c. To this end, we first find x_{cz} as a function of c and z. Using (25), it is straightforward to verify that

$$x_{cz} = 2\frac{\frac{(1-z)^2}{1+z} - \frac{(1-c)^2}{1+c}}{\left(\frac{1-z}{1+z}\right)^2 - \left(\frac{1-c}{1+c}\right)^2}. \quad (28)$$

Now it can be shown that

$$x_c = \lim_{z \to c} x_{cz} = \frac{(1+c)(3+c)}{2} \quad (29)$$

and by (25) it holds that

$$f_c(x_c) = \frac{1}{2}(1-c)\sqrt{(1-c)(3+c)}\,. \quad (30)$$

Substitution of the expression for c from (29) into (30) yields the form of $f_{1,2}(x)$ given in Theorem 3.

In the following section, we show how the localization of the Laplacian spectra helps to analyze the stability of networks of high-order agents.

IV. Consensus Criterion

A. Consensus Region

A system composed of agents (3) controlled by distributed protocol (4) can be equivalently represented as

$$\mathbf{a}(s)x = \mathbf{b}(s)\,(-\mathcal{L}_N x) \quad (31)$$

where $s := d/dt$, $x = [x_1,\, x_2, \dots, x_N]^\top$, and $\mathcal{L}_N$ is the Laplacian matrix of the dependence digraph $\mathcal{G}_N$ containing a spanning converging tree.

The following condition simplifies the analysis of reaching consensus in system (31) by dividing the problem into two subproblems.

Definition 7 ([32], [33], [34]): The consensus region (or Ω-region) of the function $\phi(s) = \mathbf{a}(s)/\mathbf{b}(s)$ in the Laplace variable s is the set of points λ in $\mathbb{C}$ for which the function $\phi(s) - \lambda$ has no zeros in the closed right half-plane

$$\Omega = \{\lambda \in \mathbb{C} : \phi(s) - \lambda \neq 0 \text{ whenever } \mathrm{Re}(s) \geq 0\}.$$

The function $\phi(s)$ is sometimes referred to as the *generalized frequency variable* [34], [47].

Such a set can be found using the general D-decomposition method.[5] In accordance with [33], to do this, we construct a curve $z = \phi(j\omega)$ on the complex plane $\mathbb{C}$. We say that this curve encircles l times the point λ (the number l may not necessarily be integer) if the increment of the argument of the function $\phi(j\omega)$ is $2\pi l$ as ω changes from $-\infty$ to $+\infty$. Typically, for a fixed domain Λ_i, the number of encirclements does not depend on the choice of $\lambda \in \Lambda_i$. Therefore, we can talk about the encirclements about a domain. Thus, the following result on the consensus (stability) region Ω of a hierarchical system consisting of subsystems with identical transfer functions $\phi(s)$ is valid.

Lemma 3 ([33]): Let $\phi(s)$ have the form $\phi(s) = \mathbf{a}(s)/\mathbf{b}(s)$ (the degrees of the polynomials $\mathbf{a}(s)$ and $\mathbf{b}(s)$ are equal to d and q, respectively), $\mathbf{b}(j\omega) \neq 0$, $\omega \in \mathbb{R}$, and let $\mathbf{b}(s)$ have l right zeros. Then, the Ω-region is the domain Λ_i encircled exactly N times by the curve $z = \phi(j\omega)$. Here, the following statements hold.

1) $N = l$ if $\phi(s)$ is a proper function ($d \leq q$).
2) $N = (d - q)/2 + l$ if $\phi(s)$ is not proper ($d > q$).

Thus, we can formulate the following necessary and sufficient consensus condition.

Lemma 4 ([32], [33], [34]): The network system with agents described by (3) reaches consensus under protocol (4) if and only if

$$\lambda_i \in \Omega, \quad i \in \{2, \ldots, N\}$$

where $\lambda_i, i \in \{2, \ldots, N\}$, are the nonzero eigenvalues of $-\mathcal{L}_N$.

The details of determining the consensus region may be found in [33]. In the case of $\phi(s) = s^2 + \gamma s$, $\gamma > 0$, this region has the form of the interior of a parabola in the complex plane: $\phi(j\omega) = -\omega^2 + j\gamma\omega, -\infty < \omega < \infty$, and if $\phi(s) = s$, then the Ω-region is the open left half-plane of the complex plane.

B. Consensus in Systems on Ring Digraphs

In this section, we formulate and prove a consensus criterion for systems (31).

Theorem 4: System (31), where $\mathcal{L}_N$ is the Laplacian matrix of a ring dependence digraph, reaches consensus in the sense of (8) for all numbers of agents if and only if the locus of the spectrum of $-\mathcal{L}_N$ lies entirely in the open consensus region Ω defined by $\phi(s)$ and shares only the point $(0, j0)$ with its boundary.

Proof: By Theorem 2, the Laplacian spectra of ring digraphs $\mathcal{G}_{m,n}$ obtained by round m-fold replication from a given simple ring $\mathcal{G}_{1,n}$ lie on a certain algebraic curve of order $2n$, irrespective of m. Taking this fact into account, it suffices to apply Lemma 4 to prove Theorem 4.

Remark 5: As mentioned above, Theorem 4 applies to systems whose ring topology always contains a spanning converging tree, which guarantees consensus in the case of first-order agents. Thus, this theorem gives additional conditions that ensure consensus at a higher order of agents.

C. Consensus in Networks of Second-Order Agents

Consensus problems in networks of second-order agents have been widely studied; e.g., see [1], [51], [52], and [53]. Here, we consider the cases with absolute and relative velocity gain from the point of view of the consensus criterion of Theorem 4. Thus, the consensus conditions derived for the examples ahead are based on finding the intersection of the consensus region and the curve that contains the spectrum of system matrix $-\mathcal{L}_N$. In some cases, we will use Vieta's theorem.

Example 1: Consider the following system of N interconnected second-order agents with absolute velocity gain $\gamma > 0$ (see [46] for the details)

$$\ddot{x} + \gamma\dot{x} = -r\mathcal{L}_N x \tag{32}$$

where $r > 0$ is a scaling factor. This factor is introduced for the sake of generality and can be considered either as part of agent's dynamics or as a parameter of the communication Laplacian matrix. In any case, the matrix $-r\mathcal{L}_N$ now plays the role of $-\mathcal{L}_N$ in Theorem 4.

The consensus region of system (32) is bounded by the curve $\phi(j\omega) = -\omega^2 + j\gamma\omega$, and the corresponding curve in $\mathbb{R}^2$ has the form $y^2 = -\gamma^2 x$. By Theorem 4, the system reaches consensus if and only if the spectrum of $-r\mathcal{L}_N$ belongs to the interior of the parabola $y^2 = -\gamma^2 x$ (except for the intersection at the origin) for all N.

Consider the communication topology represented by a Hamiltonian cycle [the classical cyclic pursuit illustrated by Fig. 1(a)] as the dependence digraph. The corresponding Laplacian matrix is given by (2); therefore, the eigenvalues of $-r\mathcal{L}_N$ are located on the circle of radius r centered at $(-r, j0)$. It is straightforward to check that this circle has no intersection with the above parabola except for the origin point whenever $r/\gamma^2 \leq 1/2$. Note that this result for the "predecessor–follower" topology corresponds to the condition of asymptotic stability of the platoon solution in [27, Th. 2], as N tends to infinity.

If the dependence digraph has the form shown in Fig. 6, then the system reaches consensus in the sense of (8) if and only if the Cassini ovals (19) (see Fig. 7) reflected about the vertical axis and r-scaled, belong to the consensus region. This is satisfied whenever $r/\gamma^2 \leq 7/6$. In terms of the vehicular platoon control problem, this result means that the system becomes eventually unstable when the inequality above does not hold.

[5] The D-decomposition method proposed by Neimark [48], [49] allows one to construct a stability region in the parameter space of a linear system that depends on the parameters. The history of the method and an overview of the results on its generalization can be found in [50].

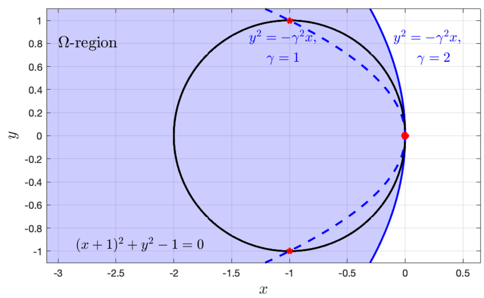


Fig. 15. Ω-region bounded by $y^2 = -\gamma^2 x$ and the unit circle ($r = 1$), where $\gamma \in \{1,\ 2\}$.

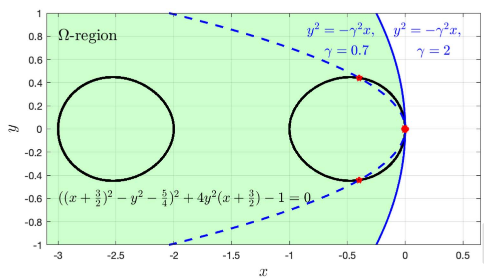


Fig. 16. Ω-region bounded by $y^2 = -\gamma^2 x$ and the reflected Cassini ovals (19) ($r = 1$), where $\gamma \in \{0.7,\ 2\}$.

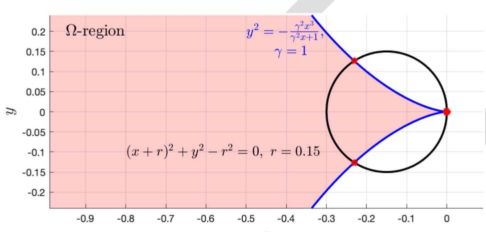


Fig. 17. Ω-region bounded by $y^2 = -\frac{\gamma^2 x^3}{\gamma^2 x+1}$ and the circle containing the spectrum of $-r\mathcal{L}_N$, where $\gamma = 1$ and $r = 0.15$.

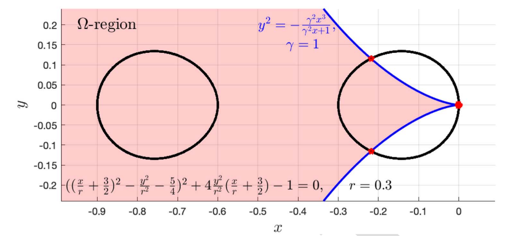


Fig. 18. Ω-region bounded by $y^2 = -\frac{\gamma^2 x^3}{\gamma^2 x+1}$ and the Cassini ovals containing the spectrum of $-r\mathcal{L}_N$, where $\gamma = 1$ and $r = 0.3$.

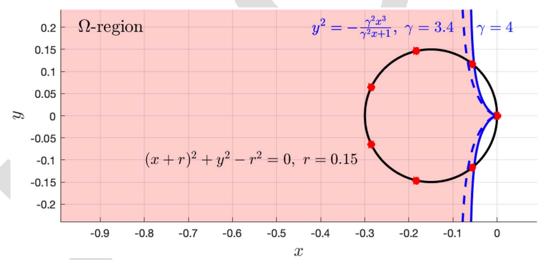


Fig. 19. Ω-region bounded by $y^2 = -\frac{\gamma^2 x^3}{\gamma^2 x+1}$, the circle that contains the spectra locus of $-r\mathcal{L}_N$ ($r = 0.15$, $\gamma \in \{3.4,\ 4\}$), and the eigenvalues of the matrix for $N = 7$.

Figs. 15 and 16 illustrate the cases where the condition of Theorem 4 is satisfied or violated.

Example 2: Now consider the system

$$\ddot{x} = -r\mathcal{L}_N x - \gamma r\mathcal{L}_N \dot{x}, \quad r > 0 \tag{33}$$

with relative velocity gain $\gamma > 0$ and $r > 0$.

Here, the generalized frequency variable is $\phi(s) = s^2/(1 + \gamma s)$. Since $\phi(j\omega) = -\omega^2/(1+\gamma^2\omega^2) + j\gamma\omega^3/(1+\gamma^2\omega^2)$, the boundary of the consensus region of system (33) on $\mathbb{R}^2$ has algebraic expression $y^2 = -\gamma^2 x^3/(\gamma^2 x + 1)$.

Similarly to the previous example, consider two communication topologies and the two corresponding curves containing the spectrum of $-r\mathcal{L}_N$: 1) the circle of radius r centered at $(-r,\ j0)$ and 2) the Cassini ovals (19) reflected about the vertical axis and r-scaled. In the first case, there always exists an intersection at $x = -2r/(1 + 2r\gamma^2)$. In the second case, the corresponding cubic equation always has one negative real root x_0 regardless of the values of r and γ, as illustrated in Figs. 17 and 18.

Corollary 3: For system (33) with predefined relative velocity gain γ, no cyclic topology whose Laplacian spectrum belongs to the curve (13), (19), (21), (23), or (26) guarantees consensus for all $N \in \mathbb{N}$. For vehicle platoons control problems, this means that the system is eventually unstable.

Sketch of the proof: Observe that both the curve $y^2 = -\gamma^2 x^3/(\gamma^2 x + 1)$ bounding the consensus domain of system (33) and the curve containing the Laplacian spectrum of $-r\mathcal{L}_N$ share the origin point $(0,\ j0)$. Near this point, under a negative increment of x, the positive branch of any of the curves under consideration containing the Laplacian spectra of $-r\mathcal{L}_N$ grows faster than that of the curve $y^2 = -\gamma^2 x^3/(\gamma^2 x + 1)$, which can be straightforwardly confirmed by the analysis of derivatives. Therefore, starting from the origin, all the positive branches of the spectra curves lie above the positive branch of the boundary curve. Thus, they do not belong to the Ω-region. Consequently, by Theorem 4, none of the topologies listed in Corollary 3 guarantees consensus for all $N \in \mathbb{N}$.

Remark 6: It follows from the analysis of the spectrum of $-r\mathcal{L}_N$ that system (33) with a certain value of the relative velocity gain γ can reach consensus in the sense of (8), provided that the number of agents N is sufficiently small. For example, for $\gamma = 3.4$, the system with a unidirected topology reaches consensus if and only if $N \leq 6$. With a slightly increased factor $\gamma = 4$, the system always reaches consensus if and only if $N \leq 7$, see Fig. 19 .

Example 3: Let the system have the dynamics

$$\ddot{x} = -\frac{r}{\gamma}\mathcal{L}_N x + \left(r\mathcal{L}_N - \frac{1}{\gamma} I_N\right)\dot{x}, \quad \gamma, r > 0$$

and a more exotic generalized frequency variable $\phi(s) = (s + \gamma s^2)/(1 - \gamma s)$ [34]. For $s = j\omega$, we have

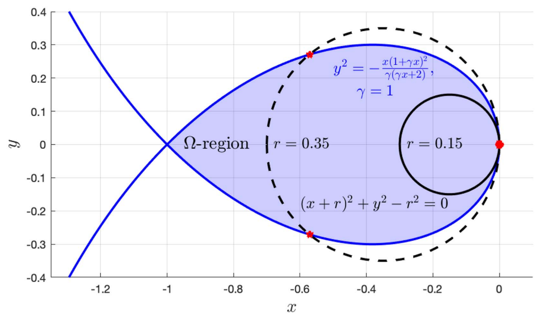


Fig. 20. Ω-region bounded by $y^2 = -x(1+\gamma x)^2/(\gamma(\gamma x+2))$ and the circles that contain the spectra loci of $-r\mathcal{L}_N$, where $r \in \{0.15,\ 0.35\}$ and $\gamma = 1$.

$\phi(j\omega) = -2\gamma^2\omega^2/(1+\gamma^2\omega^2) + j\gamma(\omega - \gamma^2\omega^3)/(1+\gamma^2\omega^2)$, with the boundary of the consensus region Ω in $\mathbb{R}^2$ expressed as $y^2 = -x(1+\gamma x)^2/(\gamma(2+\gamma x))$.

Consider a unidirected topology, whose Laplacian spectrum lies on a circle. It can be shown that the consensus condition of Theorem 4 is satisfied if and only if $r\gamma \leq 0.25$. The consensus region and two versions of the circle that contains the spectra locus of $-r\mathcal{L}_N$ are depicted in Fig. 20.Consensus is reached for $r = 0.15$, but this is not the case with $r = 0.35$.

V. Conclusion

Cyclic pursuit is one of the most attractive and interesting problems of network communication. In this article, its properties are studied using its Laplacian spectrum, which allows for exact localization on the unit circle. In this article, we studied several versions of *hierarchical* cyclic pursuit, where each macro-vertex of the dependence digraph is a sequence of directed and bidirectional arcs.

The contribution of this article is threefold. For the network dynamical systems on ring digraphs, we

1) proved that the corresponding Laplacian spectra lie on certain high-order algebraic curves regardless of the number of macro-vertices in the network;
2) presented an algorithm for obtaining implicit equations of these curves;
3) proposed a consensus condition in the frequency domain applicable to any number of agents in the network.

A characteristic feature of the algebraic curves obtained in this study is that they contain the spectrum loci of specific (Laplacian) matrices associated with network dynamical systems. Some of them, such as the Cassini ovals, have a simple geometric interpretation [54]; some others do not seem to have appeared in handbooks on special functions.

Possible extensions of this work include spectra localization of more general weighted networks that represent hierarchical pursuit. These problems are the subject of continuing research.

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