Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 55-13 (2022) 246–251

Laplacian Spectra of Two-Layer Hierarchical Cyclic Pursuit Schemes

Sergei Parsegov [*] **Pavel Shcherbakov** [*,**] **Pavel Chebotarev** [**] **Victoria Erofeeva** [***] **Alexander Rogozin** [**]

[*] *Institute of Control Sciences of Russian Academy of Sciences, Moscow, Russia (e-mail: s.e.parsegov@gmail.com, cavour118@mail.ru)*
[**] *Moscow Institute of Physics and Technology (e-mail: cavour118@mail.ru, pavel4e@gmail.com, aleksandr.rogozin@phystech.edu)*
[***] *Skolkovo Institute of Science and Technology, Moscow, Russia (e-mail: v.erofeeva@skoltech.ru)*

Abstract: Cyclic pursuit is one of the oldest multi-agent strategies with many interesting features. The vast majority of the papers dedicated to this strategy cover various extensions related to the models of interacting agents, delays, uncertainties, asynchronous communication, etc. A certain line of research studies hierarchical topologies that extend the conventional single-layer scheme. Our paper contributes to this line. Motivated by the fact that such structures are scalable, we study the spectral properties of their Laplacian matrices. First, we consider a two-layer cyclic pursuit strategy and analyze its Laplacian spectrum as the number of agents tends to infinity. Next, we propose a more sparse two-layer topology, study its spectrum, and describe the curves that contain a limit location of the eigenvalues of the corresponding Laplacian matrix.

Keywords: multi-agent systems, hierarchy, cyclic pursuit, Laplacian matrix, spectrum locus

1. INTRODUCTION

Distributed control and learning over networks has become an essential research topic due to increasing number of applications related to transportation systems (Hu et al. (2020)), cooperative control of unmanned vehicles (Ren and Beard (2008)), distributed sensor networks (Iyengar and Brooks (2016)), and others. The distributed algorithms rely on local information exchange and a cooperative consensus-based protocol driving all nodes to a common global solution (Olfati-Saber et al. (2007)). In the literature, the researches distinguish at least two types of strategies, namely, leader-following and leaderless. The scope of this paper is limited to leaderless consensus, where no leader and no external control inputs are presented in the system.

In the area of leaderless problems, cyclic pursuit is a specific and interesting multi-agent strategy with rich history (see Darboux (1878); Bruckstein et al. (1991); Klamkin and Newman (1971); Behroozi and Gagnon (1979); Segall and Bruckstein (2020) and references therein) and a variety of generalizations. The most common formulation of cyclic pursuit is as follows: Each node (agent) pursues its nearest neighbor, and all the nodes communicate in the same direction thus forming a Hamiltonian cycle. Obviously, such a digraph contains a spanning converging tree and its Laplacian matrix has a single zero eigenvalue. This guarantees that the system achieves consensus.

[⋆] The research of P. Shcherbakov and A. Rogozin in Section 4 was supported by the Russian Science Foundation (project No. 21-71-30005).

The topic has been explored by researchers in different ways related to agent models, coupling between them, and time-varying topology to name a few. Despite great scalability of single-layer cyclic pursuit, the increasing number of nodes strongly affects the rate of convergence to a common point slowing down the underlying control algorithms incorporating such a scheme. This drawback gave rise to research aimed at convergence rate improvements. Recent advances have been obtained for hierarchical cyclic pursuit schemes, where agents are divided into groups, subgroups of a group, and so on. The basic model presented in Smith et al. (2005) has become a cornerstone in the analysis of hierarchical schemes and it has been later extended in various directions. In particular, the authors of Iqbal et al. (2018) weakened the assumption requiring the system matrix to be block circulant by using Cartesian product-based hierarchical scheme and showed the same convergence rate as in the classic one. Tsubakino et al. (2013) focused on the design of control laws for systems possessing hierarchical structures. In a series of works, e.g. Tsubakino and Hara (2012), the researchers generalized the hierarchical cyclic pursuit scheme and focused on the intergroup connection reducing information exchange among the groups. Iqbal et al. (2017) considered hierarchical block circulant strategy to solve the rendezvous problem and analyzed its convergence properties. Some papers, see Mukherjee and Ghose (2016), explored a multi-layer hierarchical structure generalizing the works featuring single- or two-layer schemes.

Some other works analyze the spectral properties of Laplacian matrices to study their influence on the convergence

10.1016/j.ifacol.2022.07.267

rate of the corresponding consensus-based algorithms and system stability (see, e.g., Iqbal et al. (2018); Sharma et al. (2013)). It was found that hierarchical structures have specific Laplacian spectra. As an example, in Parsegov and Chebotarev (2018), the authors consider the cyclic pursuit of macro-vertices and study the Laplacian spectra in the case where each macro-vertex is represented by an undirected connected graph defined on two nodes. In the classical coupling case, when the number of agents goes to infinity, the eigenvalues densely fill a unit circle in the right half-plane. As shown by Parsegov and Chebotarev (2018), as the number of macro-vertices tends to infinity, the eigenvalues fill the Cassini ovals. In this paper, we analyze the relationship between system scalability and the Laplacian spectra of the hierarchical cyclic pursuit scheme presented in Smith et al. (2005) as well as its modification. More precisely, we study the asymptotic behavior of the Laplacian eigenvalues of two hierarchical schemes. First, for a two-layer hierarchical scheme, we extend the results of Smith et al. (2005) by determining the region holding the Laplacian spectrum for any number of agents. Second, we propose a more sparse representation of the two-layer hierarchical scheme and carry out the similar analysis showing the limit location of the corresponding eigenvalues. The obtained results can be helpful for stability (or consensusability) analysis of the systems under study.

The remainder of the paper is organized as follows. Section 2 introduces the classical cyclic pursuit and some properties of circulant matrices. Section 3 defines a two-layer hierarchical scheme and shows how the spectrum locus of the corresponding Laplacian matrix evolves as the scheme parameters tend to infinity. Section 4 proposes a more sparse hierarchical scheme and provides an analysis of its Laplacian spectrum including its asymptotics. Section 5 concludes the paper.

We use the following notation throughout the paper: $(\cdot)^\top$ is the matrix or vector transpose; $j := \sqrt{-1}$ denotes the imaginary unit, whereas the letters i and k are reserved for indices. The symbol $\otimes$ denotes the Kronecker product of two matrices, and $(\cdot)^*$ stands for the complex conjugate. The identity matrix is denoted by $I_n \in \mathbb{R}^{n\times n}$.

2. PROBLEM STATEMENT

2.1 Single-Layer Cyclic Pursuit

To begin with, we introduce the classic cyclic pursuit scheme using a first-order differential equation model, as in Marshall et al. (2004). Consider a communication network of n ordered agents. Let $s_i(t)$ be the state of the i-th agent that pursues the agent $i-1$ modulo n. The latter means that we refer to the index $(i-1) \bmod n$ instead of $i-1$ whenever it appears in the text. Suppose the agents start at arbitrary initial conditions and their dynamics is described by

$$\dot{s}_i = u_i$$

with control inputs

$$u_i = s_{i-1} - s_i.$$

Thus, the n-agent system of these equations can be arranged into the following linear form:

$$\dot{\mathbf{s}} = -\mathcal{L}_n \mathbf{s},$$

where $\mathbf{s} = [s_1, \dots, s_n]^\top$, and $\mathcal{L}_n$ is a special *circulant* matrix. Necessary definitions along with the spectral properties of this matrix are presented in the following subsection.

2.2 Circulant Matrices

As a first step towards discussing hierarchical cyclic schemes, we provide mathematical representation of general cyclic schemes through circulant matrices. The section includes a brief introduction needed for the subsequent analysis; a more detailed information on circulant matrices can be found in Davis (2013).

A circulant matrix $\mathcal{C}_n$ is a Toeplitz matrix having the form

$$\mathcal{C}_n = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_2 & c_3 & \cdots & c_1 \end{bmatrix} \qquad (1)$$

with the rows formed by the vector $\mathbf{c} = [c_1, c_2, \dots, c_n]$ and its $n-1$ circular permutations. In the sequel, we define such matrices as the operator $\mathrm{circ}(\cdot)$; i.e., $\mathcal{C}_n := \mathrm{circ}(\mathbf{c})$.

A useful property of circulant matrices is that they are easy to diagonalize using the Fourier matrix $\mathcal{F}_n \in \mathbb{C}^{n\times n}$ given by

$$\mathcal{F}_n = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}, \qquad (2)$$

where $\omega = \mathrm{e}^{j\frac{2\pi}{n}}$. Note that ω depends on the matrix dimension, which will be clear from the context. This property will be used within the analysis of "sparse" two-layer hierarchical cyclic pursuit proposed in Section 4.

Let $\mathbf{e}_i \in \mathbb{R}^n$ be a canonical basis column-vector that has a unit entry at position i and zeros elsewhere. Then, the counter-clockwise principal circulant permutation matrix $\mathcal{P}_n \in \mathbb{R}^{n\times n}$ is defined as follows, see Johnsen (1973):

$$\mathcal{P}_n = \mathrm{circ}(\mathbf{e}_n^\top) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & & 0 & 1 & 0 \end{bmatrix}.$$

Obviously, the characteristic polynomial of $\mathcal{P}_n$ is $p(\lambda) = \lambda^n - 1$. Its n roots are the roots of unity:

$$\lambda_k = \mathrm{e}^{j\frac{2\pi k}{n}}, \quad k \in \{0, \dots, n-1\}.$$

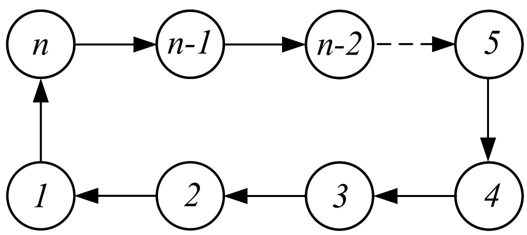
This matrix is directly related to the Laplacian matrix of the cyclic pursuit scheme. For the general case of n agents, this matrix can be defined through the matrix $\mathcal{P}_n$ as follows:

$$\mathcal{L}_n = I_n - \mathcal{P}_n.$$

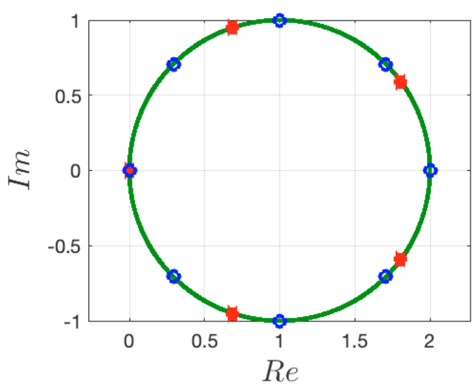
Its explicit form is given by

$$\mathcal{L}_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & & 0 & -1 & 1 \end{bmatrix}.$$

The following property of this matrix is crucial: All its eigenvalues lie on the unit circle on the complex plane centered at $(1,\ j0)$ regardless of the number of nodes constituting the graph. Figure 1a illustrates a single-layer cyclic pursuit scheme, whereas Fig. 1b shows the spectra of the corresponding matrix $\mathcal{L}_n$ for two different values of n.



(a) Single-layer cyclic pursuit scheme



(b) The eigenvalues of $\mathcal{L}_n$, $n = 5$ (red) and $n = 8$ (blue), and the unit circle that contains them

Fig. 1. Cyclic pursuit: The scheme and the Laplacian spectra

Parsegov and Chebotarev (2018) proposed and studied a strategy with special macro-vertices in cyclic pursuit. In particular, it was shown that the fourth-order algebraic curve known as Cassini ovals contains the spectrum of the corresponding Laplacian matrix regardless of the number of macro-vertices in the graph. In both cases (the unit circle and Cassini ovals), as the number of agents tends to infinity, this only increases the density of eigenvalues on the curves, but the curves remain the same.

Motivated by this fact, our goal is to

- analyze two strategies that extend the conventional cyclic pursuit scheme. Both strategies use the concept of hierarchy in the sense that they include two communication layers based on the cyclic topology;
- investigate the asymptotic behavior of the strategies as the number of agents tends to infinity. Such an analysis will make it possible to get closer to the problem of localizing the spectrum of the Laplacian matrices of these hierarchical systems, regardless of their dimension. The localization of the spectrum of such matrices is important for analyzing consensusability of groups of high-order agents; e.g., see Polyak and Tsypkin (1996); Hara et al. (2013); Li and Duan (2017); Parsegov and Chebotarev (2018).

3. TWO-LAYER HIERARCHICAL SCHEME

We start our analysis with a two-layer hierarchical scheme that generalizes the classic cyclic pursuit strategy presented and studied in Smith et al. (2005). Such an update of the conventional cyclic pursuit scheme was performed to obtain a higher rate of convergence. The scheme is represented by the graph depicted in Fig. 2. We can construct the graph in two steps. Suppose we have m groups with n nodes in each group. First, assume that the structure of each group is initially a Hamiltonian cycle (the ith node "pursues" its neighbor $i-1$). Next, we add an extra arc to each node i, $i \in \{1, \dots, n\}$, of the kth group linking it to the ith node of group $k-1$, $k \in \{1, \dots, m\}$.

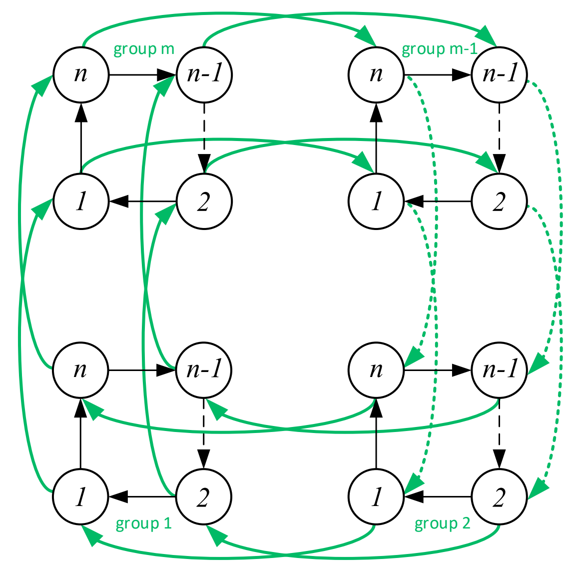


Fig. 2. Two-layer hierarchical scheme. Links within groups are depicted in black and links between groups in green

The corresponding Laplacian matrix has the following form:

$$\mathcal{L}_{nm} = I_m \otimes \mathcal{L}_n + \mathcal{L}_m \otimes I_n, \tag{3}$$

where $\mathcal{L}_n = I_n - \mathcal{P}_n$ and $\mathcal{L}_m = I_m - \mathcal{P}_m$ are circulant matrices defining interactions within each group and between groups, respectively.

Let us study what happens to the eigenvalues of $\mathcal{L}_{nm}$ as $m, n \to \infty$ and describe a domain that contains the spectrum. In the following lemma we find a more visual representation of the spectrum of $\mathcal{L}_{mn}$ as compared to the one presented in Smith et al. (2005) and study the asymptotics of the resulting expression as the number of groups and the number of agents in each group tend to infinity.

Lemma 1. Suppose that the graph has a two-layer structure shown in Fig. 2 with the corresponding Laplacian matrix (3). Then, for any number n of agents within groups and any number m of groups, the eigenvalues of $\mathcal{L}_{nm}$ lie within the disk of radius $\rho = 2$ centered at $(2,\ j0)$ in the complex plane and fill the disk densely as $m, n \to \infty$.

Proof. First, the matrix $\mathcal{L}_{nm}$ can be represented as the Kronecker sum of the matrices $\mathcal{L}_n$ and $\mathcal{L}_m$:

$$\mathcal{L}_n \oplus \mathcal{L}_m := I_m \otimes \mathcal{L}_n + \mathcal{L}_m \otimes I_n.$$

According to the properties of the Kronecker sum, any eigenvalue λ_i of $\mathcal{L}_{nm}$, $i \in \{1, \ldots, nm\}$, is the sum of two eigenvalues of $\mathcal{L}_n$ and $\mathcal{L}_m$. Using the properties of circulant matrices, the eigenvalues of the two latter matrices are given by

$$\lambda_k = 1 - \mathrm{e}^{j\frac{2\pi k}{n}},\ k \in \{0, \ldots, n-1\},$$
$$\lambda_l = 1 - \mathrm{e}^{j\frac{2\pi l}{m}},\ l \in \{0, \ldots, m-1\},$$

respectively. Hence, we have

$$\lambda_i = \lambda_k + \lambda_l = 2 - \mathrm{e}^{j\frac{2\pi k}{n}} - \mathrm{e}^{j\frac{2\pi l}{m}}. \tag{4}$$

We now analyze the locus of λ_i. Note that the locus of λ_k lies on the unit circle centered at $(1, j0)$. Then, the first two terms appearing in (4) produce the numbers lying on the unit circle centered at $(2, j0)$. Now, we define $z_k = 2 - \mathrm{e}^{j\frac{2\pi k}{n}}$ and substitute it into (4). The parameter z_k becomes a shifting one. Then, the eigenvalues λ_i belong to the union of k unit circles centered at $(2 - \cos\frac{2\pi k}{n}, \sin\frac{2\pi k}{n})$, see Fig. 3.

Next, fix the parameter m and let the other parameter n tend to infinity (no matter which parameter we fix, as they are interchangeable in (4)). Obviously, the density of the points located on the circles centered at $(2 - \cos\frac{2\pi k}{n}, \sin\frac{2\pi k}{n})$ increases, whereas the circles themselves stay the same. Now, with the growth of the other parameter m, the number of circles increases to densely fill the disk of radius $\rho = 2$ centered at $(2,\ j0)$, see Fig. 4.

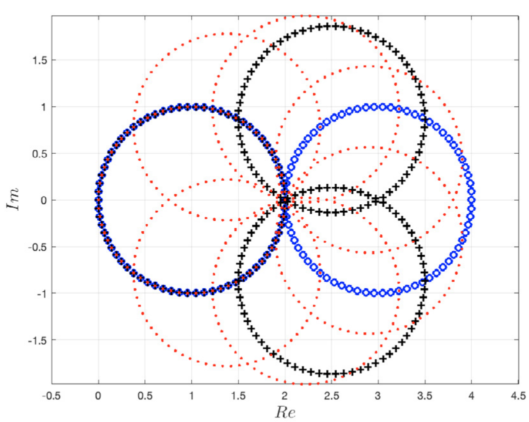


Fig. 3. The eigenvalues of $\mathcal{L}_{nm}$ for $n = 70$ and $m = 2$ (blue), $m = 3$ (black), and $m = 7$ (red)

Remark 2. The asymptotics of the spectrum of matrix (3) can be studied using the basic result obtained in Smith et al. (2005). We illustrate the fact that, regardless of the dimension of the hierarchical multi-agent system, the spectrum of its Laplacian matrix remains in the described disk-shaped region. This can be used for stability (or consensusability) analysis of linear dynamical systems with generalized frequency variable; e.g., see Polyak and Tsypkin (1996); Hara et al. (2013); Li and Duan (2017). Using the obtained localization of the spectrum, stability (or consensus) conditions can be verified or derived irrespective of the number of agents forming such a hierarchical system.

4. A MORE SPARSE TWO-LAYER HIERARCHY

An essential feature of the two-layer hierarchical scheme described in the previous section is a necessity for *each*

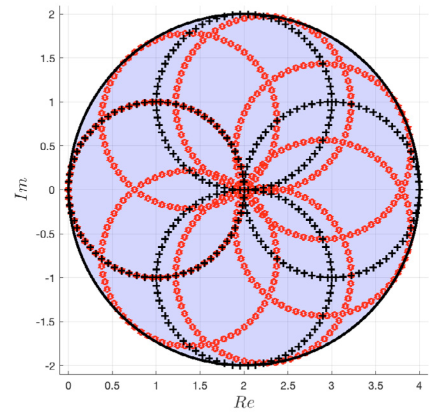


Fig. 4. The disk region that contains the spectrum loci of the Laplacian matrix (3) and the eigenvalues of $\mathcal{L}_{nm}$ for $m = 4$ (black), $m = 7$ (red), and $n = 70$

agent in the system to pursue its neighbors in both layers of hierarchy. In such a structure, it is easy to observe the redundancy of links between agents for reaching consensus. Even a single link between the groups is enough to reach consensus, though at the cost of slower convergence. In what follows, we propose a new two-layer hierarchical topology, which is more sparse.

As in the previous section, we consider m groups of n single-layered agents. Without loss of generality, we assume that the first agent in each group is a "negotiator" with the first agent in the previous one. The communication topology is presented in Fig. 5, and the corresponding Laplacian matrix is easily shown to have the following form:

$$\mathcal{L}_{nm} = I_m \otimes \mathcal{L}_n + \mathcal{L}_m \otimes \mathcal{B}, \tag{5}$$

where $\mathcal{L}_n$ and $\mathcal{L}_m$ are the matrices of cyclic pursuit introduced above, and $\mathcal{B}$ is the rank-one matrix $\mathcal{B} = \mathbf{e}_1\mathbf{e}_1^\top$.

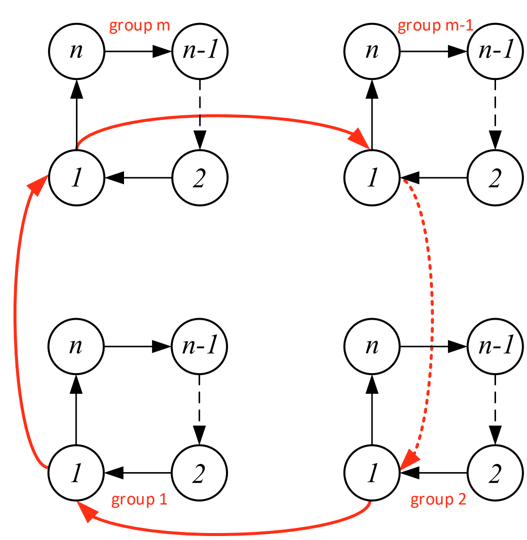


Fig. 5. A more sparse two-layer hierarchical scheme. The links within groups are depicted in black and those between groups are plotted in red

Obviously, the matrix $\mathcal{L}_{nm}$ defined in (5) is a block circulant one. Therefore, it can be diagonalized using unitary Fourier matrix (2). Block diagonalization of $\mathcal{L}_{nm}$ gives

$$\begin{aligned}&(\mathcal{F}_m \otimes I_n)^*(I_m \otimes \mathcal{L}_n + \mathcal{L}_m \otimes \mathcal{B})(\mathcal{F}_m \otimes I_n) \\ =\,& I_m \otimes \mathcal{L}_n + (\mathcal{F}_m \otimes I_n)^*(\mathcal{L}_m \otimes \mathcal{B})(\mathcal{F}_m \otimes I_n) \\ =\,& I_m \otimes \mathcal{L}_n + \Psi_m \otimes \mathcal{B},\end{aligned} \tag{6}$$

where Ψ_m is the diagonal matrix of the eigenvalues of $\mathcal{L}_m$, namely

$$\Psi_m = \operatorname{diag}(1-\omega^0,\ 1-\omega, \dots, 1-\omega^{m-1}).$$

From now on, we set $\omega = \mathrm{e}^{j\frac{2\pi}{m}}$.

Consequently, for finite n, m, the spectrum of $\mathcal{L}_{nm}$ is the union of the spectra of $\mathcal{L}_n + (1-\omega^k)\mathcal{B}$, $k \in \{0, \dots, m-1\}$:

$$\operatorname{eigs}(\mathcal{L}_{nm}) = \bigcup_{k=1}^{m} \operatorname{eigs}(\mathcal{L}_n + (1-\omega^k)\mathcal{B}). \tag{7}$$

Let us find the characteristic polynomial of the matrix

$$\mathcal{M}(n,k) = \mathcal{L}_n + (1-\omega^k)\mathcal{B} \tag{8}$$

for a fixed k. Apparently, this matrix differs from $\mathcal{L}_n$ only in the first entry. The determinant of $\mathcal{M}(n,k) - \lambda I_n$ can be found by expansion along the first row:

$$\det\begin{bmatrix} 1+(1-\omega^k)-\lambda & 0 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 1-\lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1-\lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 1-\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1-\lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1-\lambda \end{bmatrix} \tag{9}$$
$$\begin{aligned}&= (1-\lambda-(\omega^k-1))(1-\lambda)^{n-1} - 1 \\ &= (1-\lambda)^n - (\omega^k-1)(1-\lambda)^{n-1} - 1.\end{aligned}$$

Then, finding the spectrum of $\mathcal{L}_{nm}$ leads to the analysis of the roots of the family of polynomials $x^n - (\omega^k-1)x^{n-1} - 1$, $k \in \{0, \dots, m-1\}$.

For sufficiently small values of n and m, closed-form expressions for the roots *can* be derived; however, they get overly complicated for higher values, yielding no efficient formulas for the root location of the whole family. Instead, we present the following asymptotic result.

Lemma 3. As $m, n \to \infty$, the limiting location of the roots of the polynomial

$$p(x) = x^n - (\omega^k-1)x^{n-1} - 1, \qquad k \in \{0, \dots, m-1\}, \tag{10}$$

is the union of the unit circle and the arc $\{x \in \mathbb{C} \mid x = \mathrm{e}^{j\varphi} - 1,\ \frac{\pi}{3} < \varphi < \frac{5\pi}{3}\}$ on the complex plane.

Proof. Instead of the multiplier $(\omega^k - 1)$ in (10), consider $(\mathrm{e}^{j\varphi} - 1)$, where φ sweeps the segment $[0, 2\pi]$; this is equivalent to the condition $m \to \infty$.

Now, introduce the new variable $z = \frac{1}{x}$; then $p(x) = 0$ writes as

$$p(z) = z^n + z(\mathrm{e}^{j\varphi} - 1) - 1 = 0. \tag{11}$$

First, consider the case $|z| < 1$; then, as $n \to \infty$, the first term vanishes, and the roots of $p(z)$ are given by $z = (\mathrm{e}^{j\varphi} - 1)^{-1}$. Hence, the (reciprocal) roots of $p(x)$ are equal to $x = \mathrm{e}^{j\varphi} - 1$. From $|z| < 1$ we have $|x| > 1$, so that the condition $|\mathrm{e}^{j\varphi} - 1| > 1$ implies $\frac{\pi}{3} < \varphi < \frac{5\pi}{3}$; i.e., as $m, n \to \infty$, the roots of $p(z)$ fill the arc

$$\left\{x \in \mathbb{C} \mid x = \mathrm{e}^{j\varphi} - 1,\ \frac{\pi}{3} < \varphi < \frac{5\pi}{3}\right\}.$$

In the case of $|z| = 1$, we have $z(\mathrm{e}^{j\varphi} - 1) = 0$, and due to the identity $\mathrm{e}^{j\varphi} = 1$, the zeros of $p(z)$ lie on the unit circle and fill it densely as $m, n \to \infty$.

Finally, with $|z| > 1$ there are clearly no roots of (11) as $n \to \infty$.

An illustration of the result in Lemma 3 is presented in Fig. 6.

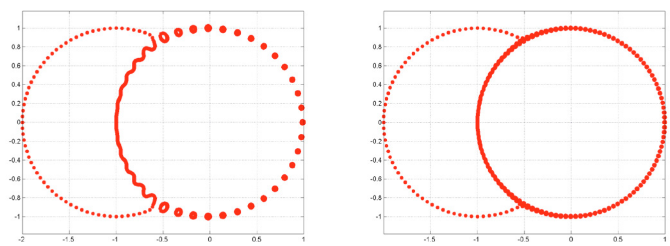


Fig. 6. Root location of polynomials (10) for $m = 100$ and $n = 40$ (left) and $n = 120$ (right)

In particular, it follows from the proof (see the case of $|z| = 1$) that, for any *fixed* n and sufficiently large m, the polynomial family (10) has at least n roots on the unit circle.

Since the eigenvalues λ of $\mathcal{M}(n,k)$ and the zeros x of (10) are related via $\lambda = 1 - x$, as an immediate corollary of Lemma 3, we arrive at

Corollary 4. As $m, n \to \infty$, the limiting location of the eigenvalues of $\mathcal{L}_{nm}$ defined in (5) is the union of the unit circle centered at $(1, j0)$ and the arc $\{x \in \mathbb{C} \mid x = \mathrm{e}^{j\varphi} + 2,\ -\frac{2\pi}{3} < \varphi < \frac{2\pi}{3}\}$ on the complex plane.

Remark 5. Similarly to the hierarchical structure analyzed in the previous section, for n large enough the increase of m leads to a higher density of the points on the corresponding curve, which approaches the limit curve described in Corollary 4 as n grows.

An illustration of Corollary 4 is given in Fig. 7.

Remark 6. Observe the following fact: In contrast to the case of the two-layer hierarchy with a disk region that contains the spectrum of *any* matrix $\mathcal{L}_{nm}$ of the form (3), the curves shown in Fig. 7 represent the *limit location* only. Neither the curves, nor the region they bound contain all possible spectra for all finite m and n. As an example, let us show the eigenvalues of $\mathcal{L}_{nm}$ for a few values of n and $m = 300$ (Fig. 8). It can be seen that the eigenvalues do not belong to the region bounded by the green circles.

5. CONCLUSION

The strategy of cyclic pursuit has been and remains the focus of attention of several scientific communities. An interesting feature of the "classical" multi-agent model of cyclic pursuit is the following property: regardless of

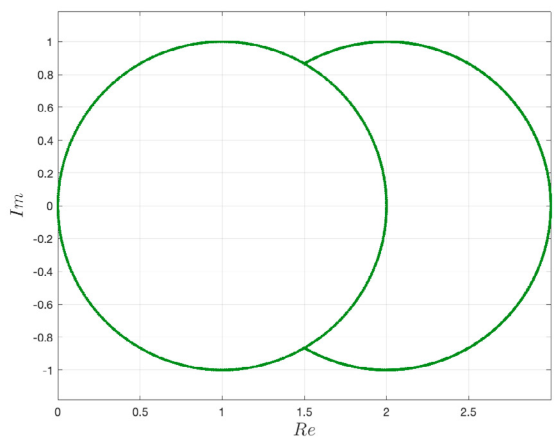


Fig. 7. The limit location of the spectrum of $\mathcal{L}_{nm}$ for $m, n \to \infty$

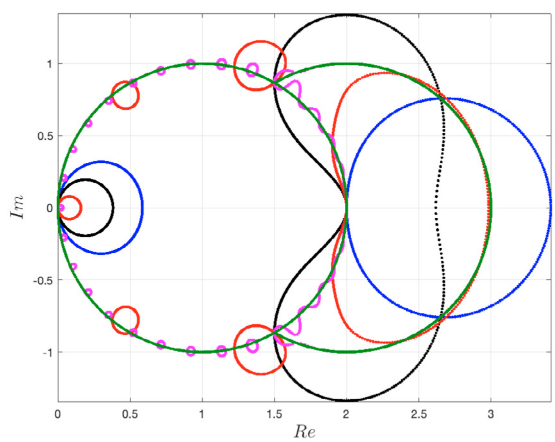


Fig. 8. The eigenvalues of $\mathcal{L}_{nm}$ for $n = 2$ (blue), $n = 3$ (black), $n = 7$ (red), $n = 30$ (magenta), and $m = 300$. The limit location of the eigenvalues is shown in green

the number of agents in the system, the spectrum of the Laplacian matrix lies on a shifted unit circle, which is a second-order curve.

In this paper, we analyzed the behavior of the Laplacian spectrum of two schemes of hierarchical cyclic pursuit and its limiting location.

Future research directions include further studies of more sparse hierarchical schemes. In finite dimensions, we can obtain a commonly known curve, e.g., an ellipse, that covers the eigenvalues for all finite m and n. In addition, we plan to explore potential applications of the developed theoretical results.

REFERENCES

Behroozi, F. and Gagnon, R. (1979). Cyclic pursuit in a plane. *Journal of Mathematical Physics*, 20(11), 2212–2216.

Bruckstein, A.M., Cohen, N., and Efrat, A. (1991). *Ants, crickets and frogs in cyclic pursuit.* Citeseer.

Darboux, G. (1878). Sur un probleme de géométrie élémentaire. *Bulletin des Sciences Mathématiques et Astronomiques*, 2(1), 298–304.

Davis, P.J. (2013). *Circulant matrices.* American Mathematical Soc.

Hara, S., Tanaka, H., and Iwasaki, T. (2013). Stability analysis of systems with generalized frequency variables. *IEEE Transactions on Automatic Control*, 59(2), 313–326.

Hu, J., Bhowmick, P., Arvin, F., Lanzon, A., and Lennox, B. (2020). Cooperative control of heterogeneous connected vehicle platoons: An adaptive leader-following approach. *IEEE Robotics and Automation Letters*, 5(2), 977–984.

Iqbal, M., Leth, J., and Ngo, T.D. (2017). Generalized hierarchical block circulant structure of multi-agent systems. *International Journal of Control, Automation and Systems*, 15(3), 1275–1286.

Iqbal, M., Leth, J., and Ngo, T.D. (2018). Cartesian product-based hierarchical scheme for multi-agent systems. *Automatica*, 88, 70–75.

Iyengar, S.S. and Brooks, R.R. (2016). *Distributed sensor networks: sensor networking and applications (Volume Two).* CRC press.

Johnsen, E.C. (1973). Essentially doubly stochastic matrices. *Linear & Multilinear Algebra*, 1, 33–45.

Klamkin, M. and Newman, D. (1971). Cyclic pursuit or “the three bugs problem”. *The American Mathematical Monthly*, 78(6), 631–639.

Li, Z. and Duan, Z. (2017). *Cooperative control of multi-agent systems: a consensus region approach.* CRC press.

Marshall, J.A., Broucke, M.E., and Francis, B.A. (2004). Formations of vehicles in cyclic pursuit. *IEEE Transactions on automatic control*, 49(11), 1963–1974.

Mukherjee, D. and Ghose, D. (2016). Generalized hierarchical cyclic pursuit. *Automatica*, 71, 318–323.

Olfati-Saber, R., Fax, J.A., and Murray, R.M. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1), 215–233.

Parsegov, S. and Chebotarev, P. (2018). Second-order agents on ring digraphs. In *Proc. 22nd International Conference on System Theory, Control and Computing (ICSTCC)*, 609–614. IEEE.

Polyak, B.T. and Tsypkin, Y.Z. (1996). Stability and robust stability of uniform systems. *Automation and Remote Control*, 57(11), 1606–1617.

Ren, W. and Beard, R.W. (2008). *Distributed consensus in multi-vehicle cooperative control.* Springer.

Segall, I. and Bruckstein, A. (2020). Guidance of agents in cyclic pursuit. *arXiv preprint arXiv:2007.00949.*

Sharma, B.R., Ramakrishnan, S., and Kumar, M. (2013). Cyclic pursuit in a multi-agent robotic system with double-integrator dynamics under linear interactions. *Robotica*, 31(7), 1037–1050.

Smith, S.L., Broucke, M.E., and Francis, B.A. (2005). A hierarchical cyclic pursuit scheme for vehicle networks. *Automatica*, 41(6), 1045–1053.

Tsubakino, D. and Hara, S. (2012). Eigenvector-based intergroup connection of low rank for hierarchical multi-agent dynamical systems. *Systems & Control Letters*, 61(2), 354–361.

Tsubakino, D., Yoshioka, T., and Hara, S. (2013). An algebraic approach to hierarchical lqr synthesis for large-scale dynamical systems. In *Proc. 9th Asian Control Conference*, 1–6. IEEE.