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A new model of opinion dynamics for social actors with multiple interdependent attitudes and prejudices

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Abstract— **Unlike many complex networks studied in the literature, social networks rarely exhibit regular cooperative behavior such as synchronization (referred usually as *consensus* or *agreement* of the opinions). This requires a development of mathematical models that capture the complex behavior of real social groups, where opinions and the actions related to them form clusters of different size, and yet are sufficiently simple to be examined. One such model, proposed in [1], deals with scalar opinions and extends the idea in [2] of iterative pooling in a way to take into account the actors' prejudices, caused by some exogenous factors and leading to disagreement in the final opinions. In this paper, we offer two extensions, where opinions are multidimensional, representing the agents' attitudes on several topics, and those topic-specific attitudes are interrelated. We examine convergence of the proposed model and find explicitly the steady opinions of the agents. Although our model assumes synchronous communication among the agents, we show that the same final opinion may be achieved "on average" via asynchronous randomized gossip-based protocol.**

I. INTRODUCTION

Real-world social networks are captivating classes of complex multi-agent systems that are attracting more and more attention from the research community. Unlike many natural and man-made complex networks with cooperative behaviors motivated by the attainment of *consensus* between the nodes, opinions of social actors often do not reach any agreement but rather form highly irregular factions (clusters) of different sizes. A challenging problem is to develop a model of opinion dynamics, which admits mathematically rigorous analysis and yet is sufficiently instructive to capture the main properties of real social networks.

The backbone of many mathematical models, explaining the clustering of continuous opinions, is the idea of *homophily* or *biased assimilation* [3]: a social actor readily accepts opinions of like-minded individuals, examining the deviant opinions with discretion. This principle is prominently manifested by various *bounded confidence* models, where the agents completely ignore the opinions outside their confidence intervals [4]–[7]. Demonstrating opinion polarization or clustering, the models from [3]–[7] are however quite complicated from the mathematical point of view and their nonlinear dynamics are far from being fully investigated. Another possible explanation for opinion disagreement is presence of *antagonism* or *negative ties* among the agents [8]. A simple yet instructive dynamics of this type, leading to opinion polarization, was addressed in [9]–[12]. It should be noticed, however, that no experimental evidence securing the postulate of ubiquitous negative interpersonal influences (referred to as *boomerang effects*) seems to be available.[1]

It is known that even a network with positive and linear couplings may exhibit persistent disagreement and clustering, if its nodes are heterogeneous, e.g. some agents are "informed" (have some external input) [14]. One of the first models of opinion dynamics, taking into account such a heterogeneity, was suggested by N.E. Friedkin and E.C. Johnsen [1], [15], henceforth referred to as the Friedkin-Johnsen (FJ) model. The FJ model promotes and extends the idea of DeGroot's iterative pooling [2], where each agent updates its opinion, based not only on its own and neighbors' opinions, but in general also on its initial opinion, or *prejudice*. In other words, some (possibly, all) of the agents are *stubborn* in the sense that they never forget their prejudices, and factor their initial opinions into every iteration of opinion. This can be otherwise treated as constant influence of exogenous conditions under which those prejudices were formed [1], [15]. In recent papers [16], [17] a condition for stability of the FJ model was obtained, which requires any agent to be influenced by at least one stubborn one. Furthermore, although the original FJ model is based on synchronous communication, in [16], [17] its "lazy" version was proposed that is based on asynchronous gossip influence and provides the same steady opinion *on average*, no matter if one considers the probabilistic average (that is, the expectation) or time-average (the solution Cesàro mean). It should be noticed that both the "simultaneous" deterministic FJ model and its randomized gossip modification are closely related to the PageRank computation algorithms [17]–[21].

The models of opinion dynamics from [1], [16], [17] deal with scalar opinions. However, during social interactions each actor usually changes its attitudes to several topics, which makes it natural to consider *vector-valued* opinions

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[1]Since the first definition of boomerang effects [13], the empirical literature has concentrated on the special conditions under which these effects might arise; there is no assertion in this literature that such odd effects, sometimes observed in $n = 2$ dyad systems, are non-ignorable components of $n > 2$ interpersonal influence systems.

[22], which may be e.g. subjective distributions of outcomes in some random experiment [2], [23]. In this paper we suggest a multidimensional extension of the FJ model, where each opinion is constituted by an agent's attitudes or beliefs on several interdependent issues. This multidimensional extension cannot be obtained by mechanical replication of the scalar FJ model on each issue, nevertheless, as we show, the condition for its convergence remains the same as in the scalar case. We also develop a randomized asynchronous protocol, which provides convergence to the same steady opinion as the original deterministic dynamics on average.

II. Preliminaries and Notation

Henceforth we denote matrices with capital letters $A = (a_{ij})$, using lower case letters for their scalar entries and for vectors. Given a square matrix $A = (a_{ij})_{i,j=1}^n$, let $\operatorname{diag} A = \operatorname{diag}(a_{11}, a_{22}, \dots, a_{nn})$ stand for its main diagonal and $\rho(A)$ be its spectral radius. The matrix is *Schur stable* if $\rho(A) < 1$. The matrix A is *row-stochastic* if $a_{ij} \geq 0$ and $\sum_{j=1}^n a_{ij} = 1\,\forall i$. Given matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, the matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & & \ddots & \vdots \\ a_{m1}B & a_{n2}B & \dots & a_{mn}B \end{pmatrix} \in \mathbb{R}^{mp\times nq}.$$

is called their *Kronecker product* [24]. A (directed) *graph* is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ stands for the finite set of *nodes* or *vertices* and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of *arcs* or *edges*. We say the node i is connected to the node j (written $i \mapsto j$) if $(i, j) \in \mathcal{E}$; a sequence $i = i_0 \mapsto i_1 \mapsto \dots \mapsto i_r = i'$ is called a *walk* from i to i'. The graph is *strongly connected* if a walk between any two distinct nodes exists.

III. The Friedkin-Johnsen model and its stability

The FJ model [1] deals with a community of n social actors (agents), whose interpersonal influences are determined by a row-stochastic matrix $W = (w_{ij}) \in \mathbb{R}^{n\times n}$. We associate this matrix to a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the set of nodes $\mathcal{V} = \{1, 2, \dots, n\}$ is in one-to-one correspondence with the agents and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ consists of all such pairs (i, j) that $w_{ij} > 0$. The diagonal entry w_{ii} is considered as a measure of *stubborness* or *closure* of the ith agent to interpersonal influence. If $w_{ii} = 1$ then $w_{ij} = 0\,\forall j \neq i$ which means that the agent is maximally stubborn and completely ignores opinions of its neighbors. We call such agents *totally stubborn* in the sense that they keep their opinions unchanged: $x_i(k) = u_i$. Conversely, if $w_{ii} = 0$, then the agent is completely open to interpersonal influence, attaches no weight to its own opinions, and fully relies on others' opinions. In the case where $w_{ii} \in (0; 1]$ we call the agent *stubborn* in the sense that the prejudice u_i is factored into any iteration of its opinion. The entries $\lambda_{ii} = 1 - w_{ii}$ of the diagonal matrix[2] $\Lambda = I - \operatorname{diag} W$ may be treated as agents' *susceptibilities* to neighbors' opinions.

[2]Here we follow the notation from [16].

Introducing the vector of scalar opinions on kth stage $x(k) = (x_1(k), \dots, x_n(k))^\top$, the FJ opinion dynamics is

$$x(k+1) = \Lambda W x(k) + (I - \Lambda)u, \quad u := x(0). \qquad (1)$$

The linear system (1) is stable if and only if ΛW is Schur stable, in which case the opinion vector converges to [3]

$$x' := \lim_{k\to\infty} x(k) = (I - \Lambda W)^{-1}(I - \Lambda)u. \qquad (2)$$

The steady opinions x'_j basically *disagree*, e.g. due to presence of several totally stubborn agents. Moreover, unlike the DeGroot dynamics [2], consensus is usually not reached even for irreducible and aperiodic matrix W. The stability of the FJ model (1) may be reformulated in graph-theoretic terms.

Assumption 1: Any node of the graph $\mathcal{G}$ is connected by a walk[4] to at least one node r with $w_{rr} > 0$. In other words, any agent is influenced by at least one stubborn agent.

Assumption 1 holds e.g. if the graph is strongly connected.

The following result, extending Proposition 1 in [16], gives necessary and sufficient condition for the FJ model stability.

Theorem 1: If W is row-stochastic and $\Lambda = I - \operatorname{diag} W$, then $\rho(\Lambda W) < 1$ if and only if Assumption 1 holds.

Proof: The sufficiency part was proved in [16]. To prove necessity, suppose on the contrary that a node i exists such that $w_{rr} = 0$ whenever $r = i$ or r is reachable from i by a walk. Denoting the set of such r with R, one easily notices that $w_{jk} = 0$ whenever $j \in R$ and $k \notin R$ (indeed, otherwise an arc (j, k) would exist and hence k would belong to R). Furthermore, as $w_{jj} = 0\,\forall j$, one has $\lambda_{jj} = 1$. Defining a vector $\xi \in \mathbb{R}^n$ as $\xi_j = 1$ if $j \in R$ and $\xi_j = 0$ otherwise, one has $\Lambda W \xi = \xi$ and thus arrives at the contradiction. ■

Remark 1: Convergence of general dynamics (1), where Λ may be an arbitrary diagonal matrix with $0 \leq \lambda_{ii} \leq 1$, is addressed in the companion paper [25], where a general condition for the Schur stability of a matrix ΛW is offered.

IV. A multidimensional extension

In this section, we propose an extension of the FJ model, dealing with vector opinions $x_1(k), \dots, x_n(k) \in \mathbb{R}^m$. The elements of each vector $x_i(k) = (x_i^1(k), \dots, x_i^m(k))$ stand for the *attitudes* of the ith agent to m different topics, which we call *issues*. In the simplest situation where agents communicate on m completely unrelated topics, it is natural to assume that the particular issues $x_1^j(k), x_2^j(k), \dots, x_n^j(k)$ satisfy the FJ model (1) for any $j = 1, \dots, m$, that is

$$x_i(k+1) = \lambda_{ii} \sum_{j=1}^n w_{ij} x_j(k) + (1 - \lambda_{ii})u_i, \; u_i := x_i(0). \qquad (3)$$

However, if these topics are related to each other, one may expect dependencies between corresponding issues. Consider, for instance, a group of people discussing two topics,

[3]The convergence to the equilibrium (2) may also take place if the system (1) is neutrally stable, i.e. $\rho(\Lambda W) = 1$, being however non-robust to numerical errors. Necessary and sufficient conditions for (2) are beyond the scope of this paper and will be addressed in its extended version [25].

[4]By definition, if $w_{ii} > 0$ then $\mathcal{G}$ has a self-loop (i, i), so that a walk from i to itself exists; that is, any stubborn agent is influenced by itself.

namely, fish in general and salmon. Salmon is nested in fish. If someone dislikes fish, then he/she will dislike salmon. If the influence process changes individuals' attitudes toward fish, say promoting fish as a healthy part of a diet, then the door is opened for influences on salmon as a part of this diet. If, on the other hand, the influence process changes individuals' attitudes against fish, say warning that fish are now contaminated by toxic chemicals, then the door is closed for influences on salmon as part of this diet.

In order to take the dependencies between different issues into account, we modify dynamics (3) as follows

$$x_i(k+1) = \lambda_{ii} \sum_{j=1}^{n} w_{ij} y_j(k) + (1-\lambda_{ii}) u_i,\ y_j(k) := C x_j(k), \tag{4}$$

and $u_i = x_i(0)$ is a prejudice of the ith agent. Here C is a row-stochastic matrix of *multi-issues dependence structure* (hereinafter called the MiDS matrix) and we will refer to $y_j(k)$ as the *impact* of the jth opinion on the kth stage. For $C = I_n$ the model (4) coincides with (3), and the impact is just an opinion vector. In general, its components are "mixed" issues, i.e. convex combinations (weighted sums) of attitudes of the jth agent on several topics.

To clarify the roles of the MiDS matrix and impacts, consider for the moment a network with star-shape topology where all the agents follow one totally stubborn leader, i.e. there exists $j \in \{1, 2, \dots, n\}$ such that $w_{ij} = 1\, \forall i$ and hence $x_i(k+1) = y_j(k) = C u_j$. The opinion changes in this system are movements of the opinions of the followers toward the initial opinions of the leader, and these movements are strictly based on the direct influences of the leader. The entries of the MiDS matrix govern the relative contributions of each of the leader's opinions on multiple issues to the formation of followers' opinions on each issue. In general, since $y_i^p(k+1) = \sum_{q=1}^{m} c_{pq} x_i^q(k)$, the weight c_{pq} measures the effect of the qth issue of the opinion to the pth issue of impact. In our example, c_{pq} is a contribution of the qth issue of the leader's opinion to the pth issue of the follower's one.

Introducing stack vectors of opinions $x(k) = (x_1(k)^\top, \dots, x_n(k)^\top)^\top$ and prejudices $u = (u_1^\top, \dots, u_n^\top)^\top$, the dynamics (4) may be rewritten as

$$x(k+1) = (\Lambda W) \otimes C\, x(k) + (I_n - \Lambda) \otimes I_m\, u. \tag{5}$$

Two natural questions, addressed below, are concerned with the stability of model (5) and identification of the MiDS matrix C, given information on W and opinions. Measurement models for W are discussed in [1], [15], [26].

A. Convergence and the steady opinions

Stability of the system (5) reduces to the question when the matrix $A = \Lambda W \otimes C$ is Schur stable, i.e. $\rho(A) < 1$. To answer it, we recall that the eigenvalues of A are products $\lambda_i \mu_j$, where $\lambda_1, \dots, \lambda_n$ are eigenvalues of ΛW and $\mu_1, \dots, \mu_m$ are those of C [24, Theorem 13.12]; therefore, $\rho(A) = \rho(\Lambda W)\rho(C)$. This yields the following.

Theorem 2: The system (5), where $\Lambda = I - \operatorname{diag} W$ and C is row-stochastic, is stable if and only if Assumption 1 holds. If this holds, then for any prejudice $u = x(0)$ a limit exists

$$x'_C := \lim_{k\to\infty} x(k) = (I_{nm} - (\Lambda W) \otimes C)^{-1} ((I_n - \Lambda) \otimes I_m) u. \tag{6}$$

Proof: Since the matrix C is row-stochastic, one has $\rho(C) = 1$ and hence $\rho(\Lambda W \otimes C) = \rho(\Lambda W)$. The stability criterion now follows from Theorem 1. The formula for the limit opinion is immediate from (5). ■

Theorem 2 shows that introducing the interdependencies among the issues does not change the stability condition, provided that the MiDS matrix is row-stochastic. Moreover, examining the proof one may note that the stability in fact does not require C being stochastic and may take place even for some *Schur unstable* matrices C, provided that $\rho(C) < \frac{1}{\rho(\Lambda W)}$. However, an important property of the model with row-stochastic MiDS matrix, we are confined to, is the solution boundedness independently of the system stability: for any $i = 1, \dots, n,\ j = 1, \dots, m$ one has $\underline{M} \le x_i^j(k) \le \overline{M}$, where $\underline{M} = \min_{i,j} x_i^j(0)$ and $\overline{M} = \max_{i,j} x_i^j(0)$.

B. Design of the MiDS matrix

A key problem, related to the feasibility of research on MiDS matrices, is whether they may be estimated based on measures of agents' opinions and their influence network. Suppose that we know the matrix of social influences W and hence the matrix of susceptibilities $\Lambda = I - \operatorname{diag} W$, depending on the agents and the network topology. The question is how to find the MiDS matrix C (assuming that it exists). A typical experiment [1], during which the agents communicate on one issue, starting at known initial opinions, may be elaborated to include several issues. Let $\hat{x}'$ be an estimated final opinion vector. A natural idea is to find C (being row-stochastic) in a way to minimize the distance (in some norm) between x'_C, given by (6), and $\hat{x}'$: $\|\hat{x}' - x'_C\| \to \min$. This problem is, however, not easy to solve since x'_C is non-convex in C. To avoid non-convex optimization, we modify the problem. Let $\varepsilon = [I_{mn} - \Lambda W \otimes C]\hat{x}' - [(I_n - \Lambda) \otimes I_m]u$. It may be noticed that if $\hat{x}' = x'_C$, then $\varepsilon = 0$, so the idea is to minimize the norm of ε subject to all row-stochastic C, arriving thus at a convex optimization problem as follows:

$$\|\varepsilon\| \to \min \tag{7}$$

$$\varepsilon = [I_{mn} - \Lambda W \otimes C]\hat{x}' - [(I_n - \Lambda) \otimes I_m]u \tag{8}$$

$$\sum_{j=1}^{m} c_{ij} = 1 \quad \forall i, \quad c_{ij} \ge 0 \quad \forall i, j. \tag{9}$$

It should be noticed that even minimum (7) equals to zero, the system of linear equations (8),(9) (where C is unknown) is overdetermined unless $n \le m - 1$, having in total $mn + m = (n+1)m$ equations for m^2 unknowns.

It may be noticed that for the case of Euclidean norm $\|\cdot\| = \|\cdot\|_2$ the optimization problem (7)-(9) is a convex quadratic programming, whereas for l^∞- and l^1-norms it is reducible to linear programming. The only feature hindering the use of standard solvers is a non-standard form of the equality constraint (8), employing unknown matrix C and the Kronecker product operation, whereas standard QP and LP problems deal with constraints $A\xi = b$, where A is a matrix, b is a known vector and ξ is a column vector of unknowns. To rewrite constraints in this standard form, one may use the following technical lemma.

Given a matrix M, its *vectorization* $\operatorname{vec} M$ is a column vector obtained by stacking the columns of M on top of one another [24], e.g. $\operatorname{vec}\left(\begin{smallmatrix}1 & 0\\ 2 & 1\end{smallmatrix}\right) = [1, 2, 0, 1]^\top$.

Lemma 1: [24] For any three matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$ such that the product $\mathcal{ABC}$ is defined, one has

$$\operatorname{vec} \mathcal{ABC} = (\mathcal{C}^\top \otimes \mathcal{A}) \operatorname{vec} \mathcal{B}. \tag{10}$$

In particular, for $\mathcal{A} \in \mathbb{R}^{m\times l}$ and $\mathcal{B} \in \mathbb{R}^{l\times n}$ one obtains

$$\operatorname{vec} \mathcal{AB} = (I_n \otimes \mathcal{A}) \operatorname{vec} \mathcal{B} = (\mathcal{B}^\top \otimes I_m) \operatorname{vec} \mathcal{A}. \tag{11}$$

Let $\hat{x}'_i$ be the estimated final opinion of the ith agent and the matrix $\hat{X} = [\hat{x}'_1, \dots, \hat{x}'_n]$ have these vectors as columns, so that $\hat{x}' = \operatorname{vec} X$. Applying (11) for $\mathcal{A} = C$ and $\mathcal{B} = \hat{X}$ entails that $[I_n \otimes C]\hat{x}' = [\hat{X}^\top \otimes I_m] \operatorname{vec} C$, thus $[\Lambda W \otimes C]\hat{x}' = [\Lambda W \otimes I_m][I_n \otimes C]\hat{x}' = [\Lambda W \hat{X}^\top \otimes I_m] \operatorname{vec} C$. Introducing a vector $c = \operatorname{vec} C$, eq. (8) can be rewritten [25] in the following vector form

$$\varepsilon + [\Lambda W \hat{X}^\top \otimes I_m]c = \hat{x}' - [(I_n - \Lambda) \otimes I_m]u, \tag{12}$$

where the vector in the right-hand side is known and $\Lambda W \hat{X}^\top \otimes I_m$ is a known matrix.

V. A RANDOMIZED GOSSIP-BASED MODEL

A restriction of the model (5), inherited from the original Friedkin-Johnsen model, is *synchronous* communication. On each step the actors simultaneously communicate to all of their neighbors that is improbable in a large-scale social network. A more realistic is *gossip*-based communication, assuming that only one pair of agents interact during each step. A randomized version of the Friedkin-Johnsen model, based on the idea of gossiping, was proposed in [16], [17].

The idea of the model from [16], [17] is as follows. Each actor starts with some initial opinion $u_i = x_i(0)$. On each step an arc is randomly sampled with the uniform distribution from the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, corresponding to the matrix of social influences W. If this arc is (i, j), then the ith agent meets the jth one and updates its opinion in accordance with

$$x_i(k+1) = h_i\left((1-\gamma_{ij})x_i(k) + \gamma_{ij}x_j(k)\right) + (1-h_i)u_i. \tag{13}$$

Hence, the new opinion of the agent is a weighted average of its previous opinion, the prejudice and the neighbor's previous opinion. The opinions of other agents remain unchanged

$$x_l(k+1) = x_l(k) \quad \forall l \neq i. \tag{14}$$

It was shown in [16], [17] that under proper choice of the coefficients h_i and γ_{ij}, the expectation $\mathbb{E}x(k)$ converges to the same steady value x' as the Friedkin-Johnsen model and, moreover, the process is *ergodic* in both mean-square [16] and almost sure [17] sense. In other words, both probabilistic averages (expectations) and time averages (referred to as the *Cesàro* or *Polyak* averages) of the random opinions converge to the final opinion in the FJ model. It should be noticed that opinions themselves are *not convergent* (see numerical simulations below) but oscillate around their expected values. In this section we discuss the extensions of these scheme to the case of multidimensional opinions.

We consider a modification of the aforementioned algorithm (13),(14) as follows. An arc $e \in \mathcal{E}$ is uniformly randomly distributed; if an arc $e = (i, j)$ is sampled, the ith agent updates its opinion in accordance with

$$x_i(k+1) = (1-\gamma^1_{ij}-\gamma^2_{ij})x_i(k) + \gamma^1_{ij}Cx_j(k) + \gamma^2_{ij}u_i. \tag{15}$$

Here $\gamma^1_{ij}, \gamma^2_{ij} \geq 0$ and $\gamma^1_{ij} + \gamma^2_{ij} \leq 1$, hence during each interaction the agent's opinion is averaged with its own *prejudice* and the neighbor's *impact* (see Section IV). The other agents do not change their opinions, i.e. (14) holds.

The following theorem shows that under Assumption 1, which guarantees stability of the deterministic multidimensional model (6), and proper choice of Γ^1, Γ^2 the model (15),(14) mimics the limit behavior of the deterministic model (5) in the aforementioned sense.

Theorem 3: Let Assumption 1 hold, C be row-stochastic, $\Gamma^1 = \Lambda W$ and $\Gamma^2 = (I - \Lambda)W$ with $\Lambda = I - \operatorname{diag} W$. Then the limit $x_* = \lim_{k\to\infty} \mathbb{E}x(k)$ exists and equals to the steady-state opinion (6) of the FJ model (1): $x_* = x'_C$. Moreover, the random process $x(k)$ is *almost sure ergodic* and *mean-square ergodic*: $\bar{x}(k) \to x_*$ with probability 1 and $\mathbb{E}\|\bar{x}(k) - x_*\|_2^2 \xrightarrow[k\to\infty]{} 0$, where

$$\bar{x}(k) := \frac{1}{k+1}\sum_{l=0}^{k} x(l). \tag{16}$$

Proof: The proof is just outlined here due to space limitation, see [25] for the missing details and calculations. As for the scalar opinion case in [16], system (15),(14) shapes into

$$x(k+1) = A(k)x(k) + B(k)u, \tag{17}$$

where $A(k)$, $B(k)$ are independent identically distributed (i.i.d.) random matrices. If arc (i, j) is sampled, then $A(k) = A^{(i,j)}$ and $B(k) = B^{(i,j)}$, where by definition

$$A^{(i,j)} = \left(I_{mn} - (\gamma^1_{ij} + \gamma^2_{ij})e_ie_i^\top \otimes I_m + \gamma^1_{ij}e_ie_j^\top \otimes C\right),$$
$$B^{(i,j)} = \gamma^2_{ij}e_ie_i^\top \otimes I_m.$$

Taking into account that $\mathbb{E}A(k) = |\mathcal{E}|^{-1}\sum_{(i,j)\in\mathcal{E}} A^{(i,j)}$ and $\mathbb{E}B(k) = |\mathcal{E}|^{-1}\sum_{(i,j)\in\mathcal{E}} B^{(i,j)}$, one finally arrives at

$$\begin{aligned}\mathbb{E}A(k) &= I_{mn} - \frac{1}{|\mathcal{E}|}\left[I_{nm} - \Lambda W \otimes C\right],\\ \mathbb{E}B(k) &= \frac{1}{|\mathcal{E}|}(I - \Lambda) \otimes I_m.\end{aligned} \tag{18}$$

Denoting $\alpha := |\mathcal{E}|^{-1} \in (0; 1]$, one has

$$\mathbb{E}A(k) = (1-\alpha)I + \alpha\Lambda W \otimes C, \mathbb{E}B(k)u = \alpha(I-\Lambda)\otimes I_m u.$$

In view of this and and Schur stability of $\Lambda W \otimes C$, Theorem 1 in [17] implies now the convergence $\mathbb{E}x(k) \to x_*$ and almost sure ergodicity. The mean-square ergodicity follows from the dominant convergence theorem as $x(k)$ are bounded. ■

VI. Numerical Simulations

In this section, we give a few numerical tests which confirm the convergence of the "synchronous" multidimensional FJ model and its "lazy" gossip version. We consider a social network of $n = 4$ actors, addressed in [1] and describe by the matrix of interpersonal influences

$$W = \begin{bmatrix} 0.220 & 0.120 & 0.360 & 0.300 \\ 0.147 & 0.215 & 0.344 & 0.294 \\ 0 & 0 & 1 & 0 \\ 0.090 & 0.178 & 0.446 & 0.286 \end{bmatrix}.$$

One may easily notice that the third agent is totally stubborn, whereas the remaining ones are "partially" stubborn ($w_{ii} > 0$), satisfying thus Assumption 1.

We assume that the agents discuss two topics, say the fish (as a part of diet) in general and salmon, and hence their opinions are two-dimensional $x_i(k) = (x_i^1(k),\ x_i^2(k))^\top \in \mathbb{R}^2$. We choose the following initial conditions

$$u = x(0) = (25, 25, 25, 15, 75, -50, 85, 5)^\top \in \mathcal{D}. \qquad (19)$$

In other words, agents $1, 2$ have modest positive liking for fish and salmon; the third (totally stubborn) agent has a strong liking for fish, but dislikes salmon; the agent 4 has a strong liking for fish and a weak positive liking for salmon.

In our simulations we compared the opinion dynamics (5) in the case of independent issues (Fig. 1) with more realistic situation (Fig. 2) where issues are interdependent and

$$C = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}. \qquad (20)$$

In all the figures, solid and dashed lines of the same color correspond to the same agent. One can see that introducing the MiDS matrix C from (20), with its dominant main diagonal, imposes a substantial drag in opinions of the "open-minded" agents 1,2 and 4. Their attitudes toward fish become more positive and those toward salmon become less positive, compared to the initial values. However, in the case of dependent issues their attitudes toward salmon do not become negative as they did in the case of independence.

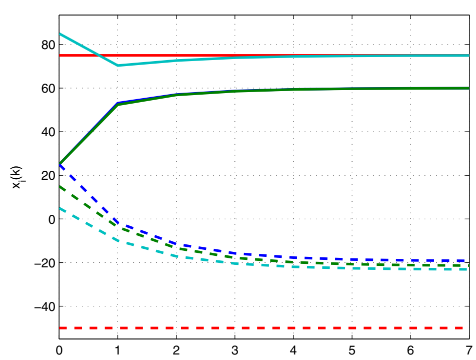

Fig. 1. Opinion dynamics (5) with independent issues

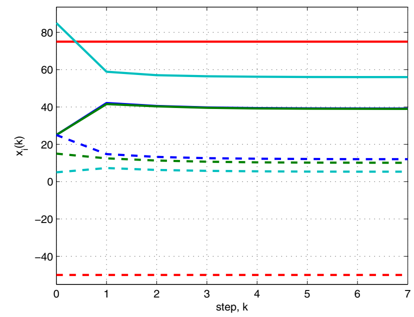

Fig. 2. Opinion dynamics (5) with interrelated issues

In Figs. 3 and 4 we simulated the dynamics of the Cesàro (Polyak) averages of the opinions under the gossip-based protocol Theorem 3. One can see that these averages converge to the same limits as in the model (5). This is *not the case* for opinions themselves which oscillate (Fig. 5).

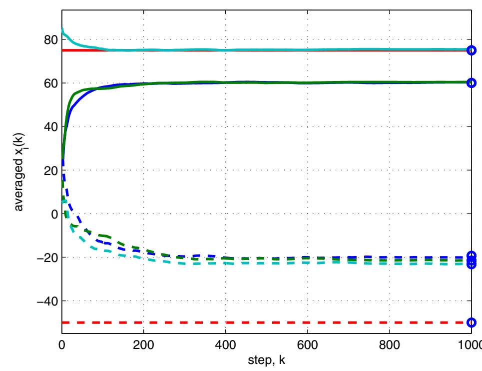

Fig. 3. Gossip-based dynamics with $C = I_2$, Cesàro averages

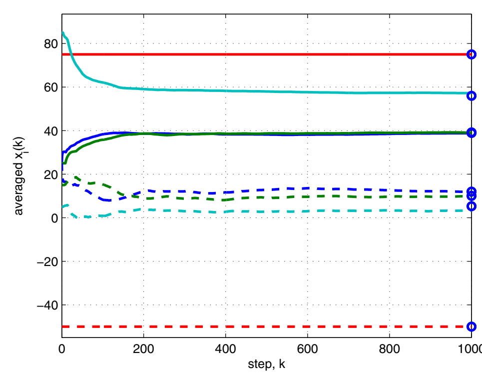

Fig. 4. Gossip-based dynamics with C from (20), Cesàro averages

Finally, we illustrate the use of our identification procedure for the MiDS matrix. Suppose that in the social network just described and starting at the initial opinions (19) one experimentally estimated the vector of steady opinions as

$$\hat{x}' = (35, 11, 35, 10, 75, -50, 53, 5)^\top.$$

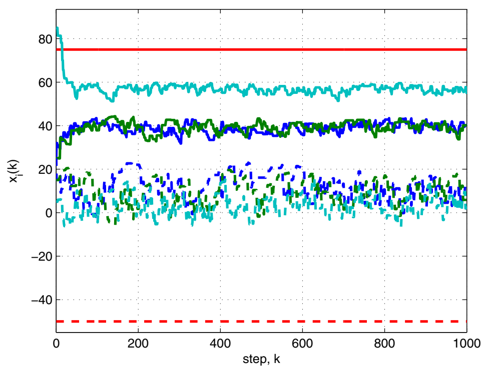

Fig. 5. Gossip-based dynamics with C from (20), opinions

We choose the Euclidean norm of the residual in (7), getting hence a QP problem as follows

$$\|\varepsilon\|_2^2 \to \min \tag{21}$$

$$\text{subject to} \quad (12), \sum_{j=1}^{m} c_{ij} = 1 \quad \forall i, \quad c_{ij} \geq 0 \quad \forall i, j. \tag{22}$$

Solving this problem, one gets the minimal residual $\|\varepsilon\|_2 = 0.9322$, which corresponds to the value of the MiDS matrix

$$C = \begin{bmatrix} 0.7562 & 0.2438 \\ 0.3032 & 0.6968 \end{bmatrix}.$$

Using the formula (6), one can compute the vector of actual steady opinion (under this choice of C)

$$\tilde{x}'_C = (35.316, 11.443, 35.092, 9.483, 75, -50, 52.386, 4.915)^\top.$$

VII. Conclusion

In this paper, we propose a novel model of opinion dynamics in a social network with static topology. Our model is en extension of the Friedkin-Johnsen model [1] to the case where agents' opinions are multidimensional, consisting thus of several attitudes or beliefs, which are referred to as the issues. Furthermore, these issues are interdependent, which is natural if the agent are communicating on several "logically" related topics. In the sociological literature, an interdependent set of attitudes and beliefs on multiple issues is referred to as an ideological or belief system [27]. A specification of the interpersonal influence mechanisms and networks that contribute to the formation of ideological-belief systems has remained an open problem. Our model is just an initial step and this direction. Its extensions, e.g. by considering time- or state-dependent coupling matrix C, and experimental verification using data from laboratory experiments on small groups are subject of ongoing research.

We establish necessary and sufficient conditions for the stability of our model, which guarantee also that opinions converge to finite limit value, depending on the social influences between the agents and their prejudices. We also address the problem of identification of the multi-issue interdependence structure. Although our model requires the agents to communicate synchronously, we show that the same final opinions can be reached by use of the decentralized and asynchronous gossip-based protocol.

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