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ORDINARY DIFFERENTIAL EQUATIONS

On the Generalization of Logarithmic Upper Function for Solution of a Linear Stochastic Differential Equation with a Nonexponentially Stable Matrix

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Abstract—The problem of finding the upper function for the squared norm of the solution of a linear stochastic differential equation with a nonexponentially stable matrix is solved. A novel characteristic of a nonconstant stability rate of the matrix is introduced. The determined upper function generalizes the previously known logarithmic estimate and is expressed in closed form in terms of the rate of matrix stability. Examples of determining the upper function for different stability rates are provided.

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1. THE NOTION OF UPPER FUNCTION AND PROBLEM FORMULATION

Suppose that on a complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$, an n-dimensional random process X_t, $t \geq 0$, described by the linear stochastic differential equation

$$dX_t = A_t X_t\, dt + G_t\, dw_t, \qquad X_0 = x, \tag{1}$$

is defined, where the initial state x is nonrandom; w_t, $t \geq 0$ is a d-dimensional standard Wiener process; A_t, G_t, $t \geq 0$, are such matrices of appropriate dimensions that there exists a solution of Eq. (1). It is also assumed that

$$\int_0^\infty \|G_t\|^2\, dt > 0$$

($\|\cdot\|$ is the matrix Euclidean norm).

In the present paper, we consider a situation where matrix A_t possesses a type of stability that is more general than exponential stability. This type of stability is characterized by a rate $\delta_t > 0$ that is rigorously formulated as follows.

Definition 1. A matrix A_t is called stable with a rate $\delta_t > 0$ (or δ_t-stable) if the following conditions are satisfied:

(i) $\limsup_{t\to\infty}(\|A_t\|/\delta_t) < \infty$;

(ii) there exists a constant $\varkappa > 0$ such that

$$\|\Phi(t,s)\| \leq \varkappa \exp\left\{-\int_s^t \delta_v\, dv\right\}, \qquad s \leq t,$$

where $\Phi(t,s)$ is the Cauchy matrix of a deterministic linear differential equation with the coefficient matrix A_t, i.e., a solution of the problem

$$\frac{\partial\Phi(t,s)}{\partial t}=A_t\Phi(t,s),\qquad \frac{\partial\Phi(t,s)}{\partial s}=-\Phi(t,s)A_s,\qquad \Phi(s,s)=\Phi(t,t)=I$$

(hereinafter I is the identity matrix);

(iii) $\int_0^t \delta_s\,ds\to\infty$ as $t\to\infty$.

Let us comment the requirements introduced in Definition 1. Condition (i) means that the stability rate of a given matrix A_t cannot be improved, as follows from the Lyapunov inequality (see, for example, [1, p. 108])

$$\|\Phi(t,s)\|\ge\hat{\varkappa}\exp\biggl\{-\int\limits_s^t\|A_v\|\,dv\biggr\},\qquad s\le t,$$

where $\hat{\varkappa}>0$ is a constant.

Inequality (ii), together with relation (iii), implies the asymptotic stability of the solution of a deterministic linear differential equation [2, p. 241]. Note that exponential stability corresponds to the case of $\delta_t\equiv\varkappa_1$ ($\varkappa_1>0$ is some constant). If $\delta_t\to0$ as $t\to\infty$, then we have weak subexponential stability, while if $\delta_t\to\infty$ as $t\to\infty$ then the type of stability strengthens to the so-called superexponential (see the work [3], devoted to the analysis of nonlinear differential equations, for relevant terminology).

Note that Eq. (1) belongs to the class of equations that define an Ornstein–Uhlenbeck process with time-varying parameters. Studying the asymptotic behavior of solutions of Eq. (1) is motivated by the wide usage of these processes in various applications. In this case, of significance are the nonexponential type of stability of matrix A_t and time-dependent elements of the diffusion matrix G_t (in particular, when modeling anomalous diffusions [4, 5] as well as in reliability theory [6], climatology [7], finance [8, 9], and others). For instance, when investigating the dynamics of material deformation, Liu et al. [10] considered a subexponentially stable matrix $A_t\sim-1/(t+a)$ and a diffusion coefficient $\|G_t\|^2\sim k/(t+a)$ ($a,k>0$ are constants). Hereinafter, the following is assumed to hold true with regard to the coefficients in Eq. (1).

Assumption $\mathcal{AG}$. A matrix A_t is stable with a rate δ_t, and the condition

$$\limsup_{t\to\infty}(\|G_t\|^2/\delta_t)<\infty$$

is fulfilled for the diffusion matrix G_t.

In this work, the well-known approach that consists in constructing upper functions for random processes is used when analyzing the asymptotic behavior of solutions of linear stochastic differential equations. Let us provide the corresponding

Definition 2. A deterministic function $h_t>0$ is said to be the upper function of a scalar process Y_t, $t\ge0$ if with the probability of 1, for a certain constant $\overline{c}>0$ the following relation is true:

$$\limsup_{t\to\infty}\frac{Y_t}{h_t}<\overline{c}<\infty. \tag{2}$$

The upper function makes it possible to estimate the order of variation of a random process over time. Indeed, relation (2) means that there exist a constant $c_0>0$ and almost surely (a.s.) a finite time moment t_0 such that with the probability of 1, the inequality $Y_t\le c_0h_t$ is fulfilled for any $t>t_0$. For example, it follows from the law of the iterated logarithm for a Wiener process (see [11, p. 91]) that the function $h_t=\sqrt{t\ln\ln t}$ (with $Y_t=\|w_t\|$). Having the form of function h_t known, one can define a sufficient normalization $g_t>0$ that, when applied, leads to the inequality

$\limsup_{t\to\infty}(Y_t/g_t) \le 0$ a.s., $t \to \infty$. It is clear that in this case, g_t is any positive function for which we have the relation $\limsup_{t\to\infty}(h_t/g_t) = 0$. If $h_t \to 0$ then $\limsup_{t\to\infty} Y_t \le 0$ with probability 1.

The goal of the present paper is to establish the upper function for process $Y_t = \|X_t\|^2$. It is known that $h_t = \ln t$ in the case of an exponentially stable matrix A_t and a bounded diffusion matrix G_t (see [12]); under the same assumptions, the form of the upper function was refined in [13] with dependence on G_t taken into account. In that case, it was additionally required that the upper function h_t be nondecreasing, in view of it being used in the problem of a stochastic linear regulator over an infinite time interval.

In this work, the upper function of a more general logarithmic form is obtained, with information on the stability rate and the diffusion matrix elements taken into account. Along with estimating the quadratic forms of Ornstein–Uhlenbeck type processes, which have financial [14], biological [15], and physical [16] applications, upper function is used when studying integral quadratic functionals. To be precise, it makes it possible to estimate the deviation of the functionals from their mean values. Results derived in this area refer to the case of an exponentially stable matrix A_t and concern assertions about sufficient normalizations with applications to control theory [17, 18]. In Section 2, the main result on the form of the upper function is provided. In Section 3, the established result is discussed and examples of determining the upper function for different stability rates are given.

2. MAIN RESULT

Equation (1) has a solution

$$X_t = \Phi(t,s)x + \int_0^t \Phi(t,s)G_s\,dw_s,$$

where $\Phi(t,s)$ is the Cauchy matrix of a deterministic linear differential equation with the coefficient matrix A_t. It is known that the main characteristics of process X_t can be expressed in terms of matrix $\Phi(t,s)$. For example (see [19, p. 100]), for the mean value we have relation $EX_t = \Phi(t,0)x$, while matrix $C_t = E(X_tX_t')$ has the form

$$C_t = \Phi(t,0)xx'\Phi'(t,0) + \int_0^t \Phi(t,s)G_sG_s'\Phi'(t,s)\,ds \tag{3}$$

(prime denotes transposition). Relation (3) and Assumption $\mathcal{AG}$ imply the boundedness of the second moments of the components of vector X_t, $t \ge 0$, i.e.,

$$E\|X_t\|^2 = \mathrm{tr}\,(C_t) \le \varkappa_0\biggl(\int_0^t \exp\biggl\{-2\int_s^t \delta_v\,dv\biggr\}\|G_s\|^2\,ds + \exp\biggl\{-2\int_0^t \delta_v\,dv\biggr\}\|x\|^2\biggr) \tag{4}$$

for a certain constant $\varkappa_0 > 0$; this leads to the boundedness of matrix C_t, too ($\mathrm{tr}\,(\cdot)$ is the matrix trace).

In what follows, we will need an expression similar to the right-hand side of inequality (4) at $x = 0$ to describe the upper function of process $Y_t = \|X_t\|^2$. Under Assumption $\mathcal{AG}$, let us define a bounded function d_t for some positive constant $\gamma < 1/2$ with the relation

$$d_t = \int_0^t \exp\biggl\{-2\gamma\int_s^t \delta_v\,dv\biggr\}\|G_s\|^2\,ds. \tag{5}$$

The following is the main result of the present paper.

Theorem 1. *Let Assumption $\mathcal{AG}$ hold true. Then the upper function h_t for process $Y_t = \|X_t\|^2$ has the form*

$$h_t = d_t \ln\Bigl(\int_0^t \delta_v \, dv\Bigr)$$

if function $d_t \exp\{2\gamma \int_0^t \delta_v \, dv\} \to \infty$, $t \to \infty$, and the form

$$h_t = \exp\Bigl\{-2\alpha\gamma \int_0^t \delta_v \, dv\Bigr\}, \tag{6}$$

if function $d_t \exp\{2\gamma \int_0^t \delta_v \, dv\}$ is bounded. The constants $0 < \alpha < 1$, $0 < \gamma < 1/2$, with d_t being defined by relation Eq. (5).

Proof. The theorem is proved in two steps. First, we consider a process $\hat{X}_t$, $t \geq 0$, with a dynamics equation of the form (1) for $A_t = -\hat{\delta}_t I$ and zero initial state $x = 0$. An upper function $\hat{h}_t$ is determined for process $\|\hat{X}_t\|^2$ ($\hat{\delta}_t$ is a certain stability rate, $\limsup\limits_{t\to\infty}(\hat{\delta}_t/\delta_t) < \infty$). Then it is shown that the difference process $X_t - \hat{X}_t$ can also be estimated using function $\hat{h}_t$ with an appropriate choice of $\hat{\delta}_t$.

Let us write a solution $\hat{X}_t$ of the equation

$$d\hat{X}_t = -\hat{\delta}_t \hat{X}_t \, dt + G_t \, dw_t, \qquad \hat{X}_0 = 0, \tag{7}$$

component by component; for the component $\hat{X}_t^{(i)}$ $(i = 1, \ldots, n)$ we obtain

$$\hat{X}_t^{(i)} = \exp\Bigl\{-\int_0^t \hat{\delta}_v \, dv\Bigr\} \sum_{j=1}^d \int_0^t \exp\Bigl\{\int_0^s \hat{\delta}_v \, dv\Bigr\} G_t^{(ij)} \, dw_t^{(j)} = \exp\Bigl\{-\int_0^t \hat{\delta}_v \, dv\Bigr\} \mathcal{M}_t^{(i)}, \tag{8}$$

where $w_t^{(j)}$ are the components of the Wiener process w_t in Eq. (7); $G_t^{(ij)}$ are the elements of the diffusion matrix G_t; the martingale $\mathcal{M}_t^{(i)}$ is defined by the relation

$$\mathcal{M}_t^{(i)} = \sum_{j=1}^d \int_0^t \exp\Bigl\{\int_0^s \hat{\delta}_v \, dv\Bigr\} G_t^{(ij)} \, dw_t^{(j)}$$

and has the quadratic characteristic

$$\langle \mathcal{M}_t^{(i)} \rangle = \int_0^t \exp\Bigl\{\int_0^s 2\hat{\delta}_v \, dv\Bigr\} \Bigl(\sum_{j=1}^d (G_s^{(ij)})^2\Bigr) \, ds,$$

that allows one to derive a representation for the components of Eq. (8) with change of time as

$$\hat{X}_t^{(i)} = \exp\Bigl\{-\int_0^t \hat{\delta}_v \, dv\Bigr\} \hat{w}^{(i)}_{\langle \mathcal{M}_t^{(i)} \rangle}, \tag{9}$$

where $\hat{w}_t^{(i)}$, $t \geq 0$, is a new Wiener process.

Thus, if $\langle \mathcal{M}^{(i)}_\infty \rangle < \infty$ then, by virtue of condition (iii) in Definition 1, the relationship $(\hat{X}^{(i)}_t)^2 \to 0$ holds true a.s. as $t \to \infty$. To be precise (see relation (9)), we have the asymptotic estimate

$$(\hat{X}^{(i)}_t)^2 \le c \exp\Big\{ -\int_0^t 2\alpha \hat{\delta}_v \, dv \Big\} \tag{10}$$

for a certain constant $0 < \alpha < 1$. Hereinafter, c stands for a positive constant, with its particular value being of no significance and varying from formula to formula. If the quadratic characteristic increases without limit, i.e., $\langle \mathcal{M}^{(i)}_t \rangle \to \infty$, then for the Wiener process $\hat{w}^{(i)}$ with the changed time, the law of the iterated logarithm is valid, i.e., the upper function $\hat{h}^{(i)}_t$ of process $(\hat{w}^{(i)}_{\langle \mathcal{M}^{(i)}_t \rangle})^2$ can then be defined in the form

$$\hat{h}^{(i)}_t = \langle \mathcal{M}^{(i)}_t \rangle \ln \ln \langle \mathcal{M}^{(i)}_t \rangle.$$

Let $\sum_{i=1}^n \langle \mathcal{M}^{(i)}_t \rangle \to \infty$, $t \to \infty$. As the norm $\|G_t\|^2$ satisfies Assumption $\mathcal{AG}$, we have $\ln \langle \mathcal{M}^{(i)}_t \rangle \le c \int_0^t \hat{\delta}_v \, dv$ and function $\hat{h}_t = \hat{d}_t \ln(\int_0^t \delta_v \, dv)$ is the upper function of process $\|\hat{X}_t\|^2$, with the bounded function $\hat{d}_t$ defined by relation (5) with $\gamma\delta_t$ replaced by $\hat{\delta}_t$. If $\sum_{i=1}^n \langle \mathcal{M}^{(i)}_\infty \rangle < \infty$, then, according to inequality (10) we define function $\hat{h}_t$ by the relation $\hat{h}_t = \exp\{-\int_0^t 2\alpha \hat{\delta}_v \, dv\}$.

Further, let us consider the difference process $Z_t = X_t - \hat{X}_t$; the equation for its dynamics does not contain explicitly any perturbations

$$dZ_t = A_t Z_t dt + (A_t + \hat{\delta}_t I)\hat{X}_t dt, \qquad Z_0 = x.$$

Let us write the solution of this linear inhomogeneous differential equation

$$Z_t = \Phi(t,0)x + \int_0^t \Phi(t,s)(A_s + \hat{\delta}_s I)\hat{X}_s \, ds.$$

Using the definition of δ_t-stability, we estimate $\|Z_t\|$ and perform normalization using the previously derived function $\hat{h}_t$ as

$$\frac{\|Z_t\|^2}{\hat{h}_t} \le \frac{c\|x\|^2}{\hat{h}_t} \exp\Big\{ -\int_0^t 2\delta_v \, dv \Big\} + \frac{c}{\hat{h}_t} \int_0^t \delta_s \exp\Big\{ -\int_s^t \delta_v \, dv \Big\} \|\hat{X}_s\|^2 \, ds. \tag{11}$$

The first term in inequality (11) asymptotically tends to zero, while the second term admits the estimate (see the notion of upper function)

$$\frac{1}{\hat{h}_t} \int_0^t \delta_s \exp\Big\{ -\int_s^t \delta_v \, dv \Big\} \|\hat{X}_s\|^2 \, ds \le \frac{1}{\hat{h}_t} \exp\Big\{ -\int_0^t \delta_v \, dv \Big\} \xi_{t_0} + \frac{c}{\hat{h}_t} \int_{t_0}^t \delta_s \hat{h}_s \exp\Big\{ -\int_s^t \delta_v \, dv \Big\} \, ds, \tag{12}$$

where t_0 is a.s. finite time moment, ξ_{t_0} is a random variable that has the form

$$\xi_{t_0} = \int_0^{t_0} \delta_s \exp\Big\{ \int_0^s \delta_v \, dv \Big\} \|\hat{X}_s\|^2 \, ds.$$

Let us set $\hat{\delta}_t = \gamma\delta_t$ with a constant $0 < \gamma < 1/2$. Then, taking the form of $\hat{h}_t$ into account, we obtain $(1/\hat{h}_t) \times \exp\{-\int_0^t \delta_v \, dv\} \to 0$ as $t \to \infty$. The above refinement entails the boundedness of the second term in the right-hand side of inequality (12). Indeed, let $\hat{h}_t = d_t \ln(\int_0^t \delta_v\,dv)$. The logarithmic function being nondecreasing leads to the estimate

$$\frac{1}{\hat{h}_t}\int\limits_{t_0}^{t} \delta_s \hat{h}_s \exp\left\{-\int\limits_{s}^{t}\delta_v\,dv\right\} ds \le \frac{1}{d_t}\int\limits_{t_0}^{t}\delta_s d_s \exp\left\{-\int\limits_{s}^{t}\delta_v\,dv\right\} ds, \tag{13}$$

while calculating the integral in the right-hand side by parts yields the inequality

$$(1-2\gamma)\int\limits_{t_0}^{t}\delta_s d_s \exp\left\{-\int\limits_{s}^{t}\delta_v\,dv\right\} ds \le d_t,$$

which demonstrates the boundedness of the right-hand side in the estimate (13). In case $\hat{h}_t = \exp\{-2\int_0^t \alpha\gamma\delta_v\,dv\}$ $(0<\alpha<1)$, a similar conclusion is reached by straightforward integration of the second term in inequality (12).

Thus, going back to Eq. (11), we have

$$\limsup_{t\to\infty}\frac{\|Z_t\|^2}{\hat{h}_t} < \infty.$$

Hence, the above-defined upper functions $\hat{h}_t$ are upper functions for process $\|X_t\|^2$. This proves the theorem.

3. DISCUSSING THE MAIN RESULT AND EXAMPLES

Note that given information only on the δ_t-stability of matrix A_t and the boundedness of the norm of diffusion matrix G_t by the stability rate δ_t, the upper function can be immediately determined in a logarithmic form as $h_t^{(0)} = \ln(\int_0^t \delta_v\,dv)$. This produces the "roughest" and most rapidly growing estimate, as it has been constructed not taking into account a particular change of norm $\|G_t\|$. In a similar fashion, under these conditions, we can find the exponential lower bound $h_t^{(1)} = \exp\{-\beta\int_0^t \delta_v\,dv\}$ for some constant $0<\beta<1$ (see relation in (6)). Then, for any upper function we have the relation $c_2 h_t^{(1)} \le h_t \le c_1 h_t^{(0)}$ with some positive constants c_1 and c_2.

Based on the established form of the upper function, we can make a number of conclusions about possible conditions on parameters that guarantee with the probability of 1 that the process $\|X_t\|^2$ asymptotically tends to zero as well as on the choice of appropriate normalizations. For example, if $\lim\limits_{t\to\infty}(\|G_t\|^2/\delta_t) = 0$ then in the expression for function h_t, the multiplier $d_t \to 0$ as $t\to\infty$, and, vice versa, if $\liminf\limits_{t\to\infty}(\|G_t\|^2/\delta_t) > 0$ then function d_t is bounded away from zero. The condition

$$\lim_{t\to\infty}\left(\frac{\|G_t\|^2}{\delta_t}\ln\left(\int\limits_0^t \delta_v\,dv\right)\right) = 0$$

is sufficient for $h_t \to 0$ as $t\to\infty$; this can be easily verified using L'Hôpital's rule. Let us formulate the corresponding assertion.

Corollary. *Let the conditions of Theorem* 1 *be satisfied and* $h_t^{(0)} = \ln(\int_0^t \delta_v\,dv)$. *Then*

(a) *if* $(\|G_t\|^2 h_t^{(0)}/\delta_t) \to 0$, *then* $\|X_t\|^2 \to 0$ *a.s. as* $t\to\infty$;

(b) *if* $(\|G_t\|^2/\delta_t) \to 0$, *then* $(\|X_t\|^2/h_t^{(0)}) \to 0$ *a.s. as* $t\to\infty$;

(c) *if* $\liminf\limits_{t\to\infty}(\|G_t\|^2/\delta_t) > 0$, *then* $h_t = c_0 h_t^{(0)}$ ($c_0 > 0$ *is some constant*).

Remark. The relation derived in statement (a) of Corollary can be used to analyze the special case of a constant diffusion matrix $G_t \equiv G$; this case is important in applications (for example, for cognitive models [20] or statistical inference [21]). If the stability rate δ_t does not decrease, then $\limsup_{t\to\infty}(d_t\delta_t) < \infty$ and $h_t^{(2)} = h_t^{(0)}/\delta_t$ is also an upper function of process $\|X_t\|^2$, with $h_t^{(2)} \sim h_t$, where function h_t has been defined in Theorem 1, if $(\dot{\delta}_t/\delta_t) \to 0$ as $t \to \infty$.

Further, we consider some examples of constructing upper functions for different functions δ_t that prescribe stability rate. In all the cases, we will assume that the conditions in Theorem 1 are fulfilled as well as statement (c) of Corollary so as to separate the impact of the stability rate from the dynamics of diffusion matrix when determining the upper function.

Examples:

1. *Power-law* family $\delta_t = a(1+t)^b$, where constants $a > 0$, $b \geq -1$, gives rise to the following stability types: subexponential for $-1 \leq b < 0$; exponential for $b = 0$; superexponential for $b > 0$. As $h_t = c_0 h_t^{(0)} = c_0 \ln(\int_0^t \delta_v\, dv)$, then for $b > -1$ for the upper function we have the equivalence $h_t \sim \ln t$ for large t, i.e., in this case, the type of stability does not affect the growth order of the upper function. For $b = -1$ we have a more slowly increasing function $h_t \sim \ln\ln t$.

2. *Logarithmic* family $\delta_t = a\ln^b(e+t)$, where constants $a > 0$, $b \neq 0$, and e is the exponent, is used to describe subexponential stability for $b < 0$ and superexponential stability for $b > 0$. We find that in all the cases $h_t \sim \ln t$.

3. *Exponential* family $\delta_t = a\exp\{t^b\}$, where constants $a, b > 0$, corresponds to the superexponential type of stability and yields an upper function $h_t \sim t^b$ of a form that depends on the value of b.

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