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ORDINARY DIFFERENTIAL EQUATIONS

On Asymptotic Behavior of Solutions of Linear Inhomogeneous Stochastic Differential Equations with Correlated Inputs

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Abstract— We analyze the behavior of solutions of linear stochastic differential equations (SDEs) with time-varying coefficients. The underlying SDEs contain correlated additive and multiplicative disturbances as well as external input in the form of stochastic process. We obtain functions serving as upper bounds on solutions in the mean-square and almost sure sense as time increases. The results are used to study the subdiffusion modeling problem in which the velocity process is determined by the solution of a linear stochastic differential equation.

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INTRODUCTION

Let us analyze the behavior of solutions of linear inhomogeneous stochastic differential equations (SDEs) with correlated noises in the underlying dynamics as the time parameter tends to infinity.

Suppose that, on a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$, we are given a scalar process X_t, $t \geq 0$, that is a solution of a linear SDE of the form

$$dX_t = a_t X_t\,dt + f_t\,dt + G_t\,dW_t + \sigma_t X_t\,dw_t \tag{1}$$

with the nonrandom initial condition $X_0 = x$; let a_t, G_t, and σ_t be piecewise continuous deterministic functions of time; let f_t, $t \geq 0$, be a measurable $\mathcal{F}_t$-adapted stochastic process with the property $E\int_0^t f_s^2\,ds < \infty$, $t \geq 0$; let W_t and w_t, $t \geq 0$, be correlated one-dimensional $\mathcal{F}_t$-adapted Wiener processes; i.e., $dW_t\,dw_t = \rho\,dt$, where ρ is a constant such that $-1 \leq \rho \leq 1$. In this case, by a solution of Eq. (1) we mean an $\mathcal{F}_t$-adapted stochastic process X_t, $t \geq 0$, that almost surely (a.s.) has continuous sample paths such that for any $t \geq 0$ with probability 1 one has the relations

$$\int\limits_0^t \left(|a_s X_s| + |f_s| + \sigma_s^2 X_s^2\right) ds < \infty$$

and

$$X_t = x + \int\limits_0^t a_s X_s\,ds + \int\limits_0^t f_s\,ds + \int\limits_0^t G_s\,dW_s + \int\limits_0^t \sigma_s X_s\,dw_s$$

(see [1, Definition 6.15, p. 48]); here $|\cdot|$ stands for the absolute value. Obviously, here we are talking about a so-called strong solution of the SDE, whose existence and uniqueness in the pathwise sense are discussed below.

Equation (1) is an equation with time-varying coefficients satisfying the assumptions stated below. Here we note that the situations of both unboundedness and singularity of parameters are allowed as $t \to \infty$. Equations of this type are widely used for modeling in various fields of applications (see [2–4] and the references in [5]). In particular, X_t may specify a velocity process, and models of anomalous diffusion then arise on its basis (see [4, 6, 7]) for special cases of Eq. (1); this will also be discussed within this article. In addition, it is important to note that (1) can be obtained from a number of nonlinear equations by a change of variable, say, in the field of population dynamics [8] or economic models [9]. The assumption about the correlation between the Wiener processes that define additive and multiplicative perturbations is also motivated by taking into account various factors that affect the evolution of real processes (in engineering, see [10], and in economics [11]). External inputs f_t of random nature can also play a significant role in describing the dynamics, for example, in stochastic control theory (see [12, Sec. 1.2, p. 8]), in particular, for mean field systems [12, Sec. 3.6, p. 106] as well as in particular models (see [3] and [13]). The main assumptions about the coefficients of Eq. (1) are stated below.

Assumption A. There exists a monotone deterministic function $\delta_t > 0$, $t \geq 0$, such that

$$\int\limits_0^t \delta_v\, dv \to \infty, \quad t \to \infty, \quad \limsup_{t\to\infty} \left\{(G_t^2 + \sigma_t^2)/\delta_t\right\} < \infty, \tag{2}$$

and moreover, the function $\bar{\Phi}(t,s) = \exp\left(\int_s^t a_v\, dv\right)$ satisfies the inequalities

$$\kappa_2 \exp\left(-2\bar{\kappa}\int\limits_s^t \delta_v\, dv\right) \leq \bar{\Phi}^2(t,s) \leq \kappa_1 \exp\left(-2\int\limits_s^t \delta_v\, dv\right), \quad s \leq t, \tag{3}$$

$$\bar{\Phi}^2(t,s)\exp\left(\int\limits_s^t \sigma_v^2\, dv\right) \leq \kappa_3 \exp\left(-2\kappa\int\limits_s^t \delta_v\, dv\right), \quad s \leq t, \tag{4}$$

with some constants $\kappa_i > 0$ $(i = 1,2,3)$, $\bar{\kappa}$, and κ such that $\bar{\kappa} \geq 1$ and $0 < \kappa \leq 1$.

Condition (2) and the right inequality in (3) imply the asymptotic convergence to zero of the solution of the deterministic version of Eq. (1) $(G_t = f_t = \sigma_t \equiv 0)$ at the rate δ_t. If $f_t \equiv 0$, then, in the stochastic case, the presence of conditions (4) and (2) ensures the boundedness of EX_t^2, $t \geq 0$. We also note that the left inequality in (3) is similar to the Lyapunov estimate (see [14, p. 132]).

Obviously, one important question in studying Eq. (1) is the examination of its solutions as the time parameter increases; in particular, the possibility of X_t tending to zero is of interest. In this paper, such an analysis is carried out by deriving upper bounds in some probabilistic sense as functions of the parameters in (1). More precisely, the problem is to find nonnegative functions h_t and $\bar{h}_t$, $t \geq 0$, such that, with probability 1,

$$\limsup_{t\to\infty} \frac{EX_t^2}{\bar{h}_t} < \infty \quad \text{and} \quad \limsup_{t\to\infty} \frac{X_t^2}{h_t} < \infty.$$

If the form of h_t and $\bar{h}_t$ is known, then one can find conditions on the coefficients under which $EX_t^2 \to 0$, i.e., we have a mean square convergence, or $X_t^2 \to 0$ with probability 1, which means that the process tends to zero a.s. as $t \to \infty$. Previously, the problem of finding the functions h_t and $\bar{h}_t$ was studied for special cases of Eq. (1). For example, the paper [15] considered time-invariant a_t and σ_t, the coefficient $G_t \equiv 0$, and a nonnegative deterministic function f_t with a growth constraint. In [16], for $a_t \equiv a$, $\sigma_t \equiv \sigma$, $W_t = w_t$, and nonrandom f_t, a condition for the a.s. convergence $X_t \to 0$ as $t \to \infty$ was found. Under the superexponential decay of the coefficient G_t, $f_t \equiv 0$, and $W_t = w_t$, superexponentially decaying functions h_t and $\bar{h}_t$ were obtained in [5]. The cases when only one type of disturbances was included and no external inputs in (1), i.e., the cases in which $f_t \equiv 0$ and either G_t or σ_t is identically zero, were studied in [17] for additive noise and in [18, Sec. 4.2, p. 117] for multiplicative noise.

The remaining part of the paper is organized as follows. Section 1 presents the main results on the form of the functions h_t and $\bar{h}_t$ under Assumption A as well as a number of auxiliary results and examples. Section 2 deals with modeling subdiffusion processes, which are one class of anomalous diffusion, with the use of Eq. (1).

1. MAIN RESULTS

First, we present a number of auxiliary results.

Lemma 1. *Equation* (1) *has a unique strong solution with*

$$X_t = \Phi(t,0)x + \Phi(t,0)\int_0^t \Phi^{-1}(s,0)f_s\,ds + \Phi(t,0)\int_0^t \Phi^{-1}(s,0)G_s\,dW_s - \rho\Phi(t,0)\int_0^t \Phi^{-1}(s,0)\sigma_s G_s\,ds, \tag{5}$$

where

$$\Phi(t,s) = \exp\left(\int_s^t a_v\,dv - \frac{1}{2}\int_s^t \sigma_v^2\,dv + \int_s^t \sigma_v\,dw_v\right) \tag{6}$$

and

$$\Phi^{-1}(s,0) = \exp\left(-\int_0^s a_v\,dv + \frac{1}{2}\int_0^s \sigma_v^2\,dv - \int_0^s \sigma_v\,dw_v\right).$$

Proof. Under the assumptions made, the existence of a unique strong solution of a linear SDE of the form (1) follows from well-known results (see, e.g., [19, Lemma 7.1]), while the Lebesgue integrals in formula (5) and the Itô stochastic integrals are well defined by virtue of the conditions imposed on the coefficients of Eq. (1), in particular, under the condition $E\int_0^t f_s^2\,ds < \infty$, $t \ge 0$. In addition (see [20]), almost all trajectories X_t are continuous functions of time, and $EX_t^2 < \infty$, $t \ge 0$. Further, we find the stochastic differential for the function (5). Here the formula $d(\xi_t\eta_t) = \eta_t\,d\xi_t + \xi_t\,d\eta_t + d\xi_t\,d\eta_t$ is applied (see [21, p. 168]), and the relations

$$d\Phi(t,0) = a_t\Phi(t,0)\,dt + \sigma_t\Phi(t,0)\,dw_t,$$
$$d\Phi(t,0)\,dW_t = \rho\sigma_t\Phi(t,0)\,dt$$

are taken into account. Thus, we can see that

$$dX_t = a_tX_t\,dt + f_t\,dt + G_t\,dW_t + \sigma_tX_t\,dw_t + \rho\sigma_tG_t\,dt - \rho\sigma_tG_t\,dt$$

for $X_0 = x$, which is the same as (1). The proof of the lemma is complete.

Lemma 2. *Let Assumption* A *be true. Then one has the estimate*

$$\limsup_{t\to\infty}\{EX_t^2/\bar h_t\} < \infty, \tag{7}$$

where the function $\bar h_t$ *is given in the form*

$$\bar h_t = \exp\left\{-2\kappa(1-\lambda)\int_0^t \delta_v\,dv\right\}x^2 + \int_0^t \exp\left\{-2\kappa(1-\lambda)\int_s^t \delta_v\,dv\right\}\left(G_s^2 + \frac{Ef_s^2}{\delta_s}\right)ds \tag{8}$$

for each constant λ, $0<\lambda<1$, *with the constant* κ, $0<\kappa\le 1$, *being taken from condition* (4).

Proof. For the process X_t, $t \ge 0$, we have the representation

$$X_t = \bar\Phi(t,0)x + \int_0^t \bar\Phi(t,s)f_s\,ds + \int_0^t \bar\Phi(t,s)G_s\,dW_s + \int_0^t \bar\Phi(t,s)\sigma_sX_s\,dw_s, \tag{9}$$

where $\bar\Phi(t,s)=\exp\left(\int_s^t a_v\,dv\right)$ and the corresponding integrals in (9) are well defined by virtue of the fact that $E\int_0^t f_s^2\,ds<\infty$ and $EX_t^2<\infty$, $t\ge 0$. After squaring equality (9), finding the expectation of both sides according to the rules of stochastic calculus, in particular, taking into account the formula

$$E\left(\int\limits_0^t \bar\Phi(t,s)G_s\,dW_s\int\limits_0^t \bar\Phi(t,s)\sigma_sX_s\,dw_s\right)=\int\limits_0^t \bar\Phi^2(t,s)\rho\sigma_sG_sEX_s\,ds,$$

we obtain the relation

$$EX_t^2=\bar\Phi^2(t,0)x^2+E\left(\int\limits_0^t \bar\Phi(t,s)f_s\,ds\right)^2+\int\limits_0^t \bar\Phi^2(t,s)G_s^2\,ds+\int\limits_0^t \bar\Phi^2(t,s)\sigma_s^2EX_s^2\,ds+I_0(t),$$

where

$$I_0(t)=2\bar\Phi(t,0)xE\left(\int\limits_0^t \bar\Phi(t,s)f_s\,ds\right)+2E\left(\int\limits_0^t \bar\Phi(t,s)f_s\,ds\int\limits_0^t \bar\Phi(t,s)G_s\,dW_s\right)$$
$$+2\int\limits_0^t \bar\Phi^2(t,s)\rho\sigma_sG_sEX_s\,ds+2E\left(\int\limits_0^t \bar\Phi(t,s)f_s\,ds\int\limits_0^t \bar\Phi(t,s)\sigma_sX_s\,dw_s\right).$$

Further, applying the elementary inequality $AB\le A^2/c+cB^2$, $A,B,c>0$, to the terms in the formula for $I_0(t)$ and the Cauchy–Schwarz inequality to the expression $\left(\int_0^t \bar\Phi(t,s)f_s\,ds\right)^2$, after collecting like terms, we have the estimate

$$EX_t^2\le c_x\bar\Phi^2(t,0)x^2+c_f\int\limits_0^t \delta_s\bar\Phi^{2\tilde\varepsilon}(t,s)\,ds\int\limits_0^t \bar\Phi^{2-2\tilde\varepsilon}(t,s)\delta_s^{-1}Ef_s^2\,ds$$
$$+c_G\int\limits_0^t \bar\Phi^2(t,s)G_s^2\,ds+c_\sigma\int\limits_0^t \bar\Phi^2(t,s)\sigma_s^2EX_s^2\,ds,$$

where $c_x=1+\varepsilon_{xf}^{-1}$, $c_f=1+\varepsilon_{xf}+\varepsilon_{fG}^{-1}+\varepsilon_{f\sigma}^{-1}$ $c_G=1+\varepsilon_{fG}+\varepsilon_{f\sigma}^{-1}$, and $c_\sigma=1+\varepsilon_{f\sigma}+\varepsilon_{G\sigma}$ with some arbitrarily small positive constants ε_{xf}, ε_{fG}, $\varepsilon_{f\sigma}$, $\varepsilon_{G\sigma}$, and $0<\tilde\varepsilon<1$. Note that $I_0(t)\equiv 0$ for $f_t\equiv 0$ and $\rho=0$, and then we can set $c_\sigma=1$. We introduce the variable $y_t=\bar\Phi^2(0,t)EX_t^2$ and pass to the inequality

$$y_t\le c_xx^2+c_fl_t^{(1)}+c_Gl_t^{(2)}+\int\limits_0^t c_\sigma\sigma_s^2y_s\,ds,$$

where

$$l_t^{(1)}=\int\limits_0^t \delta_s\bar\Phi^{2\tilde\varepsilon}(0,s)\,ds\int\limits_0^t \bar\Phi^{2-2\tilde\varepsilon}(0,s)\delta_s^{-1}Ef_s^2\,ds,$$
$$l_t^{(2)}=\int\limits_0^t \bar\Phi^2(0,s)G_s^2\,ds;$$

based on this, using the Gronwall–Bellman inequality in integral form (see [22, Lemma 2.7, p. 42]), we obtain the estimate

$$y_t\le c_x\exp\left(\int\limits_0^t c_\sigma\sigma_v^2\,dv\right)x^2+c_f\int\limits_0^t \exp\left(c_\sigma\int\limits_s^t \sigma_v^2\,dv\right)dl_s^{(1)}+c_G\int\limits_0^t \exp\left(\int\limits_s^t c_\sigma\sigma_v^2\,dv\right)dl_s^{(2)}.$$

Returning to the original variables, we can write

$$EX_t^2 \le c_x \tilde{\Phi}^2(t,0) x^2 \exp\left(\int_0^t \varepsilon_0 \sigma_v^2\, dv\right) + c_G \int_0^t \tilde{\Phi}^2(t,s) \exp\left(\int_s^t \varepsilon_0 \sigma_v^2\, dv\right) G_s^2\, ds + I_1(t) + I_2(t),$$

where $\varepsilon_0 > 0$ is an arbitrarily small number,

$$\tilde{\Phi}^2(t,s) = \bar{\Phi}^2(t,s) \exp\left(\int_s^t \sigma_v^2\, dv\right),$$

$$I_1(t) = c_f \int_0^t \tilde{\Phi}^2(t,s) \exp\left(\int_s^t \varepsilon_0 \sigma_v^2\, dv\right) \bar{\Phi}^{2\tilde{\varepsilon}}(s,0) \delta_s^{-1} E f_s^2\, ds \int_0^s \delta_\tau \bar{\Phi}^{2\tilde{\varepsilon}}(0,\tau)\, d\tau\, ds,$$

$$I_2(t) = c_f \int_0^t \tilde{\Phi}^2(t,s) \exp\left(\int_s^t \varepsilon_0 \sigma_v^2\, dv\right) \delta_s \bar{\Phi}^{2-2\tilde{\varepsilon}}(s,0) \int_0^s \bar{\Phi}^{2-2\tilde{\varepsilon}}(0,\tau) \delta_\tau^{-1} E f_\tau^2\, d\tau\, ds.$$

Note that the condition $\limsup\limits_{t\to\infty}\{\sigma_t^2/\delta_t\} < \infty$ implies an inequality of the form

$$\exp\left(\int_s^t \varepsilon_0 \sigma_v^2\, dv\right) \le \hat{\kappa} \exp\left(\int_s^t \tilde{\varepsilon}_0 \delta_v^2\, dv\right),$$

which holds for an arbitrarily small number $\tilde{\varepsilon}_0 > 0$ and some constant $\hat{\kappa} > 0$. Then, considering properties (3) and (4) in Assumption A, the terms $I_1(t)$ and $I_2(t)$ satisfy the inequalities

$$I_1(t) \le \tilde{c}_f \int_0^t \exp\left\{-2(1-\lambda)\kappa \int_s^t \delta_v\, dv\right\} \int_0^s \delta_\tau \exp\left\{-2\tilde{\varepsilon} \int_\tau^s \delta_v\, dv\right\} d\tau\, \delta_s^{-1} E f_s^2\, ds,$$

$$I_2(t) \le \tilde{c}_f \int_0^t \exp\left\{-2(1-\lambda)\kappa \int_s^t \delta_v\, dv\right\} \delta_s \int_0^s \exp\left\{-(2-2\tilde{\varepsilon}) \int_\tau^s \delta_v\, dv\right\} \delta_\tau^{-1} E f_\tau^2\, d\tau\, ds,$$

where $0 < \lambda < 1$, with some constant $\tilde{c}_f > 0$. Further, choosing the value of $\tilde{\varepsilon}$ so that $2 - 2\tilde{\varepsilon} > 2(1-\lambda)\kappa$, after integration by parts we can pass to the estimate

$$I_1(t) + I_2(t) \le \hat{c}_f \int_0^t \exp\left\{-2(1-\lambda)\kappa \int_s^t \delta_v\, dv\right\} \delta_s^{-1} E f_s^2\, ds$$

with some constant $\hat{c}_f > 0$. Combining the above results, we finally obtain

$$EX_t^2 \le \hat{c}_x \exp\left\{-2(1-\lambda)\kappa \int_0^t \delta_v\, dv\right\} x^2 + \hat{c}_G \int_0^t \exp\left\{-2(1-\lambda)\kappa \int_s^t \delta_v\, dv\right\} G_s^2\, ds$$
$$+ \hat{c}_f \int_0^t \exp\left\{-2(1-\lambda)\kappa \int_s^t \delta_v\, dv\right\} \delta_s^{-1} E f_s^2\, ds,$$

where $\hat{c}_x$ and $\hat{c}_G$ are some positive constants; the latter implies that the estimate (2) holds true. The proof of the lemma is complete.

The appearance of the constant λ, which reduces the rate of the exponential function in (8), is due to the presence of a nonzero correlation $\rho \neq 0$ between the Wiener processes defining the additive and multiplicative disturbances in (1) as well as the contribution of the external inputs f_t in the dynamics (1). It is easily seen (cf. the proof of Lemma 2) that if $\rho = 0$ and $f_t \equiv 0$, then the estimate (2) remains valid for $\lambda = 0$.

As was mentioned earlier, in the pathwise analysis of the behavior of X_t, $t \to \infty$, it becomes a challenging task to obtain conditions on the coefficients that guarantee the convergence of X_t to zero with probability 1. The result in Lemma 2 can also be used for problems of this kind. Note that the following representation holds with probability 1 for a process X_t, $t \geq 0$, that is a solution of Eq. (1) with probability 1:

$$X_t = \exp\left(\int_0^t a_v\,dv\right)x + \int_0^t \exp\left(\int_s^t a_v\,dv\right)G_s\,dW_s + \int_0^t \exp\left(\int_s^t a_v\,dv\right)f_s\,ds + \int_0^t \exp\left(\int_s^t a_v\,dv\right)\sigma_s X_s\,dw_s;$$

here the Lebesgue integrals and the stochastic Itô integrals are well defined by virtue of the conditions imposed on the coefficients of Eq. (1) and the fact that $EX_t^2 < \infty$, $t \geq 0$ (see the proof of Lemma 1). Then, defining

$$Y_t^{(0)} = \int_0^t \exp\left(\int_s^t a_v\,dv\right)\sigma_s X_s\,dw_s,$$

$$X_t^{(0)} = \int_0^t \exp\left(\int_s^t a_v\,dv\right)G_s\,dW_s,$$

$$\xi_t = \int_0^t \exp\left(\int_s^t a_v\,dv\right)f_s\,ds,$$

we arrive at the representation

$$X_t = Y_t^{(0)} + X_t^{(0)} + \xi_t + \exp\left(\int_0^t a_v\,dv\right)x, \tag{10}$$

where the processes $Y_t^{(0)}$, $X_t^{(0)}$, and ξ_t, $t \geq 0$, admit the stochastic differentials

$$dY_t^{(0)} = a_t Y_t^{(0)}\,dt + \sigma_t X_t\,dw_t, \tag{11}$$

$$dX_t^{(0)} = a_t X_t^{(0)}\,dt + G_t\,dW_t, \tag{12}$$

$$d\xi_t = a_t \xi_t\,dt + f_t\,dt \tag{13}$$

under the zero initial conditions $Y_0^{(0)} = X_0^{(0)} = \xi_0 = 0$. The next statement gives a condition that allows passing from the analysis of the convergence of the process X_t as $t \to \infty$ to the examination of the behavior of the processes $X_t^{(0)}$ and ξ_t, whose underlying dynamics contains no multiplicative disturbances.

Lemma 3. *Let Assumption* A *be true, and let* $\limsup_{t\to\infty}\{|a_t|/\delta_t\} < \infty$. *If* $\int_0^\infty \sigma_t^2 \bar{h}_t\,dt < \infty$, *where the function* $\bar{h}_t$ *is defined in* (8), *then in* (10) *the process* $Y_t^{(0)} \to 0$ *a.s. as* $t \to \infty$. *Further,*

$X_t \to 0$, $t \to \infty$, *with probability* 1 *if* $X_t^{(0)} \to 0$ *and* $\xi_t \to 0$ *a.s. as* $t \to \infty$. *In this case, the processes* $Y_0^{(0)}$, $X_0^{(0)}$, *and* ξ_t, $t \ge 0$, *are given by formulas* (11)–(13).

Proof. The integral representation corresponding to Eq. (11) has the form $Y_t^{(0)} = I_t + M_t$, where $I_t = \int_0^t a_s Y_s^{(0)}\,ds$ and $M_t = \int_0^t \sigma_s X_s\,dw_s$. One has $M_t \to M_\infty$ as $t \to \infty$ a.s. and in mean square; here M_∞ is a random variable, because the condition $\limsup_{t\to\infty}\{EM_t^2\} < \infty$ implied by (2), the estimates $EM_t^2 = \int_0^t \sigma_s^2 EX_s^2\,ds \le c\int_0^t \sigma_t^2 \bar{h}_t$, and the assumptions in the statement of the lemma being proved hold true. (Here and in the following, $c > 0$ is some constant whose specific value is of no importance.) For the term I_t, it is known that $I_t \to I_\infty$, $t \to \infty$, in mean square and a.s. (I_∞ is an random variable) if $EI_\infty^2 < \infty$ (see [23, p. 97 of the Russian translation]), where $EI_t^2 = \int_0^t\int_0^t a_s a_\tau E(Y_s^{(0)}Y_\tau^{(0)})\,ds\,d\tau$. The covariance $E(Y_s^{(0)}Y_\tau^{(0)})$ is found as

$$E(Y_s^{(0)}Y_\tau^{(0)}) = \exp\left(\int_\tau^s a_v\,dv\right)\left[E(Y_\tau^{(0)})^2\right], \quad \tau \le s,$$

$$E(Y_s^{(0)}Y_\tau^{(0)}) = \exp\left(\int_s^\tau a_v\,dv\right)\left[E(Y_s^{(0)})^2\right], \quad \tau > s.$$

From the equation

$$dE[Y_t^{(0)}]^2 = 2a_t E[Y_t^{(0)}]^2\,dt + \sigma_t^2 EX_t^2\,dt$$

with the initial condition $E[Y_0^{(0)}]^2 = 0$, we determine the function

$$E[Y_t^{(0)}]^2 = \int_0^t \exp\left(\int_s^t a_v\,dv\right)\sigma_s^2 EX_s^2\,ds.$$

Considering the form of $E[Y_t^{(0)}]^2$, the assumption $\limsup_{t\to\infty}\{|a_t|/\delta_t\} < \infty$, and inequalities (3), when estimating the above expression for EI_t^2, we conclude that

$$EI_t^2 \le 2c\int_0^t \delta_t \int_0^s \delta_s \exp\left(-\int_s^t \delta_v\,dv\right)E[Y_s^{(0)}]^2\,ds\,dt \le 2c\int_0^t \delta_s E[Y_s^{(0)}]^2\,ds \le 2\tilde{c}\int_0^t \sigma_s^2 EX_s^2\,ds.$$

(Here and in the following, $\tilde{c} > 0$ is some constant.) Thus, $EI_\infty^2 < \infty$ and $I_t \to I_\infty$, $t \to \infty$, a.s. and in mean square. Consequently, $Y_t^{(0)} \to Y_\infty^{(0)}$, $t \to \infty$, where $Y_\infty^{(0)}$ is some random variable, with probability 1. Then the condition $\int_0^\infty \sigma_s^2 EX_s^2\,ds < \infty$ implies $E[Y_t^{(0)}]^2 \to 0$, $t \to \infty$; i.e., $Y_t^{(0)} \to 0$ in mean square. It is known from [22, p. 49] that then the limit random variables coincide a.s.; i.e., $Y_\infty^{(0)} = 0$. Accordingly, it follows from the representation (10) that for the property $X_t \to 0$ to hold a.s., it suffices to require that $X_t^{(0)} \to 0$ and $\xi_t \to 0$ with probability 1 as $t \to \infty$. The proof of the lemma is complete.

Referring to the assertion in Lemma 3, note that (12) and (13) are linear equations, and therefore, when studying the asymptotic behavior of their solutions, one can use known results. The paper [17] gives the form of a deterministic function $h_t^{(0)}$ such that $\limsup_{t\to\infty}\{X_t^{(0)}/h_t^{(0)}\} < \infty$, and obviously, the convergence $h_t^{(0)} \to 0$ also implies the convergence $X_t^{(0)} \to 0$ a.s. as $t \to \infty$. In addition, if $h_t^{(f)}$ is a nonrandom function such that $\limsup_{t\to\infty}\{|f_t|/h_t^{(f)}\} < \infty$ with probability 1, then, to verify the convergence $\xi_t \to 0$ a.s. as $t \to \infty$, one turns to the study of the function $d_t = \int_0^t \exp\left(-\int_s^t \delta_v\,dv\right)h_s^{(f)}\,ds$ as $t \to \infty$.

Remark 1. Assumption A and the estimate (2) imply that the condition in Lemma 3 is satisfied under the inequalities $\int_0^\infty \sigma_t^2\,dt < \infty$ and $\int_0^\infty (Ef_t^2/\delta_t^2)\,dt < \infty$. For these inequalities to hold, it suffices that the conditions $\int_0^\infty G_t^2\,dt < \infty$ and $\int_0^\infty (Ef_t^2/\delta_t)\,dt < \infty$ be satisfied. Moreover, the condition $\int_0^\infty G_t^2\,dt < \infty$ implies the convergence $X_t^{(0)} \to 0$ a.s. as $t \to \infty$. If it is additionally known that $\int_0^\infty (Ef_t^2/\delta_t)\,dt < \infty$ or $\int_0^\infty \sqrt{Ef_t^2}\,dt < \infty$, then $\xi_t \to 0$ a.s., and hence also $X_t \to 0$, $t \to \infty$, with probability 1.

The conditions obtained in Lemma 3 permit one to find out whether the asymptotic pathwise convergence of the process X_t to zero takes place, but this does not answer the question about the rate of such convergence. The corresponding statement is one of the main results of this paper and is formulated below.

Let $\varepsilon > 0$ be a real number. Set

$$\mathcal{N}_t(\varepsilon) = \int_0^t \exp\left\{2\bar\kappa \int_0^s \delta_v\,dv + (1+\varepsilon)\int_0^s \sigma_v^2\,dv\right\} G_s^2\,ds, \tag{14}$$

where $\bar\kappa \geq 1$ is the constant in condition (3).

We also define an $\alpha > 0$ as follows:

$$\alpha = \frac{2}{1 + (1-\kappa/\bar\kappa)^{-1}} + \beta, \tag{15}$$

where $\beta > 0$ is an arbitrarily small number and the constants $\bar\kappa$ and κ are taken from (3) and (4) in Assumption A.

Theorem 1. *Let Assumption* A *be true. Then* $\limsup_{t\to\infty}\{X_t^2/h_t\} < \infty$ *a.s. for the function*

$$h_t = h_t^{(0)} + \int_0^t \exp\left\{-2(1-\lambda_2)\int_s^t \delta_v\,dv\right\}\left(G_s^2 + \frac{f_s^2}{\delta_s}\right)ds\left(\int_0^t \sigma_v^2\,dv\right)^{\alpha}, \tag{16}$$

where the function $h_t^{(0)}$ *is defined by the formula*

$$h_t^{(0)} = \exp\left\{-2\int_0^t \delta_v\,dv - 2(1-\lambda_0)\int_0^t \sigma_v^2\,dv\right\}x^2 + \int_0^t \exp\left\{-2(1-\lambda_1)\int_0^t \delta_v\,dv\right\}G_s^2\,ds\left(\int_0^t \sigma_v^2\,dv\right)^{\alpha}\ln\left(\int_0^t \delta_v\,dv\right)$$

if the condition $\mathcal{N}_t(\varepsilon) \to \infty$, $t \to \infty$, *is satisfied for each* $\varepsilon > 0$ *and by the formula*

$$h_t^{(0)} = \exp\left\{-2\int_0^t \delta_v\,dv - 2(1-\lambda_0)\int_0^t \sigma_v^2\,dv\right\}(1+x^2)$$

if $\mathcal{N}_\infty(\varepsilon) < \infty$ *for some* $\varepsilon > 0$.

Here the function $\mathcal{N}_t(\varepsilon)$ *is defined in* (14), *and the* λ_i *are arbitrary constants*, $0 < \lambda_i < 1$, $i = 0, 1, 2$. *The number* α *is defined in* (15) *if* $\limsup_{t\to\infty}\{\sigma_t^2/\delta_t\} > 0$, *and* $\alpha > 0$ *is an arbitrarily small number if* $\limsup_{t\to\infty}\{\sigma_t^2/\delta_t\} = 0$. *Moreover, one can set* $\lambda_1 = 0$ *if* $\int_0^\infty \sigma_t^2\,dt < \infty$.

Proof. Consider the representation (5) and write the estimate

$$X_t^2 \leq 2(I_t + J_t^{(1)} + J_t^{(2)}) + 2\Phi^2(t,0)x^2, \tag{17}$$

where

$$I_t = \Phi^2(t,0)M_t^2 \quad \text{for} \quad M_t = \int_0^t \Phi^{-1}(s,0)G_s\,dW_s, \tag{18}$$

$$J_t^{(1)} = \Phi^2(t,0)\left(\int_0^t \Phi^{-1}(s,0)f_s\,ds\right)^2,$$

$$J_t^{(2)} = \rho^2\Phi^2(t,0)\left(\int_0^t \Phi^{-1}(s,0)\sigma_s G_s\,ds\right)^2.$$

By the Cauchy–Schwarz inequality and the condition $\limsup_{t\to\infty}\{\sigma_t^2/\delta_t\} < \infty$, the terms $J_t^{(1)}$ and $J_t^{(2)}$ are estimated as follows:

$$J_t^{(1)} + J_t^{(2)} \le c\Phi^2(t,0)\big(\bar\Phi(t,0)\big)^{-2\varepsilon}\int_0^t \Phi^{-2}(s,0)\big(\bar\Phi^{-1}(s,0)\big)^{2\varepsilon}\left(\frac{f_s^2}{\delta_s} + \rho^2 G_s^2\right)ds; \tag{19}$$

here $\bar\Phi(t,s) = \exp\big(\int_0^t a_v\,dv\big)$ and $\varepsilon > 0$ is an arbitrarily small number. Next, two cases must be analyzed depending on the behavior of $\int_0^t \sigma_v^2\,dv$ as $t \to \infty$.

I. Consider the term $I_t = \Phi^2(t,0)M_t^2$ in (17). Since M_t is a martingale (see (18)), for the function $\mathcal{N}_t(\varepsilon)$ defined in (14) the fact that the condition $\mathcal{N}_\infty(\varepsilon) < \infty$ is satisfied for some $\varepsilon > 0$ will suffice for the existence of M_∞ by virtue of the relation $\limsup_{t\to\infty}\{\langle M_t\rangle/\mathcal{N}_t(\varepsilon)\} < \infty$; here $\langle M_t\rangle = \int_0^t \Phi^{-2}(s,0)G_s^2\,ds$ is the quadratic variation of M_t. Moreover, it follows from the law of the iterated logarithm and formula (6) that the following estimate holds:

$$\Phi^2(t,0) \le c\exp\left\{-2\int_0^t \delta_v\,dv - (1-\varepsilon)\int_0^t \sigma_v^2\,dv\right\}, \quad t > t_0(\omega), \quad \omega \in \Omega,$$

where $\varepsilon > 0$ is an arbitrarily small number and $t_0(\omega)$ is a.s. a finite moment.

Consequently, $\limsup_{t\to\infty}\{|I_t|/h_t^{(0)}\} < \infty$ a.s. for the function

$$h_t^{(0)} = \exp\left\{-2\int_0^t \delta_v\,dv - (1-\varepsilon)\times\int_0^t \sigma_v^2\,dv\right\}.$$

Now assume that for each $\varepsilon > 0$ we have $\mathcal{N}_t(\varepsilon) \to \infty$ as $t \to \infty$. Let us apply the law of the iterated logarithm for martingales (see [24]) to the process M_t in (18) to obtain the estimate $M_t^2 \le c\langle M_t\rangle \ln\ln\langle M_t\rangle$ a.s. for $t > t_0(\omega)$. In view of Assumption A and formula (6), we conclude that $\limsup_{t\to\infty}\{\ln\ln\langle M_t\rangle/\ln(\int_0^t \delta_v\,dv)\} < \infty$. Then with probability 1 we have the estimate

$$I_t^2 \le c\tilde J_t \ln\left(\int_0^t \delta_v\,dv\right), \quad t > t_0(\omega), \tag{20}$$

where

$$\tilde J_t = \Phi^2(t,0)\int_0^t \Phi^{-2}(s,0)G_s^2\,ds. \tag{21}$$

Further, let $\int_0^t \sigma_v^2\,dv \to \infty$ as $t \to \infty$, and assume that for a fixed $\omega \in \Omega$ the laws of iterated logarithm are used for the increments of the Wiener process when estimating $\Phi^2(t,s)$ [25, 26]. Let us make the change of time variable $\tilde t = \int_0^t \sigma_v^2\,dv$ to obtain the process $\hat w_{\tilde t} = \int_0^t \sigma_v\,dw_v$, where $\hat w$ is some Wiener process. Then for small $\varepsilon > 0$ the inequalities

$$|\hat w_{\tilde t} - \hat w_{\tilde s}| \le \sqrt{2(1+\varepsilon)}\sqrt{\tilde t - \tilde s}\sqrt{\ln \frac{\tilde t}{\tilde t - \tilde s} + \ln\ln\left\{(\tilde t - \tilde s) \vee e\right\}} \tag{22}$$

hold with probability 1 for $\tilde t > \tilde t_0(\omega)$ and $0 \le \tilde s \le \tilde t - (1/e - \varepsilon)$,

$$|\hat w_{\tilde t} - \hat w_{\tilde s}| \le \sqrt{2}N(\omega)\sqrt{\tilde t - \tilde s}\sqrt{\ln\ln \frac{1}{(\tilde t - \tilde s)}}, \quad 0 \le \tilde t - \tilde s \le 1/e - \varepsilon, \tag{23}$$

where $\tilde t_0(\omega)$ is a.s. a finite time moment, $N(\omega)$ is a.s. a finite random variable, and the notation $A \vee B$ means $\max\{A, B\}$. As a result, for the function (21) we have the estimate

$$\tilde J_t \le \tilde J_t^{(1)} + \tilde J_t^{(2)} + \tilde J_t^{(3)}, \quad t > \hat t_0 = \max(t_0, t_1), \quad t_1 = \inf\left\{ s : \int_0^s \sigma_v^2\,dv > \tilde t_0 \right\}.$$

Here $\tilde J_t^{(1)} = c\exp\left(-2\int_0^t \delta_v\,dv - (1-\varepsilon)\int_0^t \sigma_v^2\,dv\right)\chi_{\hat t_0}$, where $\chi_{\hat t_0}$ is a.s. a finite random variable; the remaining integrals are

$$\tilde J_t^{(2)} = \Phi^2(t,0)\int_{\hat t_0}^{t-\hat t_1} \Phi^{-2}(s,0)G_s^2\,ds,$$

$$\tilde J_t^{(3)} = \Phi^2(t,0)\int_{t-\hat t_1}^{t} \Phi^{-2}(s,0)G_s^2\,ds,$$

where $\hat t_1 = t - \inf\{t : \int_0^t \sigma_v^2\,dv > \tilde t - (1/e - \varepsilon)\}$.

When estimating $\tilde J_t^{(2)}$ from above using (22), we notice that the maximum value of the function $g(y) = -b_0 y + \sqrt{y}\sqrt{b - b_1 y + b_1 \ln\ln(y \vee e)}$, where $1/e - \varepsilon \le y \le e^b$, $b \gg b_0, b_1$ ($\gg$ is the "much greater" symbol), does not exceed $b/(4b_0)$. Using (3) and (4) in Assumption A, we write the estimate

$$\exp\left\{2\lambda_1 \int_0^t a_v\,dv\right\} \le \hat\kappa_0 \exp\left\{-\lambda_1(1-\kappa/\bar\kappa)^{-1}\int_0^t \sigma_v^2\,dv\right\}$$

with an arbitrary constant λ_1, $0 < \lambda_1 < 1$, where $\hat\kappa_0 > 0$ is some constant. Setting $b = 8(1+\varepsilon)\ln \tilde t$, $b_1 = 8(1+\varepsilon)$, and $b_0 = 1 + \lambda_1(1-\kappa/\bar\kappa)^{-1}$, we obtain the inequality

$$\Phi^2(t,s) \le c\exp\left\{2(1-\lambda_1)\int_s^t a_v\,dv\right\}\left(\int_0^t \sigma_v^2\,dv\right)^{\alpha}$$

for $\hat t_0 \le s \le t - \hat t_1$, where the degree value $\alpha > 0$ is defined in (15). Note that in the case of $\limsup\limits_{t\to\infty}\{\sigma_t^2/\delta_t\} = 0$, the ratio $\kappa/\bar\kappa$ can be replaced by a number close to one, and then α becomes arbitrarily small.

To determine an asymptotic upper bound for the integral $\tilde J_t^{(3)}$, we take into account (23) and the fact that the function $g_1(y) = -y + \sqrt{2}N(\omega)\sqrt{y\ln\ln(1/y)}$ is a.s. bounded for $0 \le y \le 1/e - \varepsilon$; more precisely, $g_1(y) \le \tilde{c}N(\omega)$. Then $\Phi^2(t,s) \le c\exp\{-2\int_s^t \delta_v\,dv + \tilde{c}N(\omega)\}$, and when integrating, the corresponding expression will be majorized by the upper bound for $\tilde{J}_t^{(2)}$ by virtue of the assumption that $\int_0^t \sigma_v^2\,dv \to \infty$ as $t \to \infty$.

Taking into account the above estimates, we can write the form of the upper function for $\tilde{J}_t$ in (21),

$$\tilde{J}_t \le c\int\limits_0^t \exp\left\{-2(1-\lambda_1)\int\limits_s^t \delta_v\,dv\right\} G_s^2\,ds \left(\int\limits_0^t \sigma_v^2\,dv\right)^{\alpha},$$

where $0 < \alpha < 2$ and α is defined in (15) if $\limsup\limits_{t\to\infty}\{\sigma_t^2/\delta_t\} > 0$; otherwise, α is an arbitrary positive number.

The above argument is also used when constructing an estimate for $J_t^{(1)} + J_t^{(2)}$ in (19) with λ_1 replaced by λ_2. Returning to (17) and (20), we have the inequality

$$X_t^2 \le ch_t^{(0)} + c\int\limits_0^t \exp\left\{-2(1-\lambda_2)\int\limits_s^t \delta_v\,dv\right\}\left(\rho^2 G_s^2 + \frac{f_s^2}{\delta_s}\right) ds \left(\int\limits_0^t \sigma_v^2\,dv\right)^{\alpha}, \qquad (24)$$

where $t > \max\{t_0(\omega), \hat{t}_0(\omega)\}$ and

$$h_t^{(0)} = \int\limits_0^t \exp\left\{-2(1-\lambda_1)\int\limits_s^t \delta_v\,dv\right\} G_s^2\,ds \left(\int\limits_0^t \sigma_v^2\,dv\right)^{\alpha} \ln\left(\int\limits_0^t \delta_v\,dv\right)$$
$$+ \exp\left\{-2\int\limits_0^t \delta_v\,dv - 2(1-\lambda_0)\int\limits_0^t \sigma_v^2\,dv\right\} x^2$$

with constants λ_0, λ_1 and λ_2 chosen so that $0 < \lambda_0 < 1$, $0 < \lambda_1 < 1$, and $0 < \lambda_2 < 1$.

II. In the case of $\int_0^\infty \sigma_v^2\,dv < \infty$, it is easily seen that with probability 1 we have the convergence $\int_0^t \sigma_v\,dw_v \to \chi_\infty$, where $\chi_\infty = \int_0^\infty \sigma_v\,dw_v$ is a.s. a finite random variable. Consequently, $|\int_s^\infty \sigma_v\,dw_v| < \varepsilon$ a.s. for $s \ge \hat{t}_0(\omega)$, where $\hat{t}_0(\omega)$ is a.s. a finite moment. Then for the function $\Phi(t,s)$ with $t \ge s > \hat{t}_0(\omega)$ we have the estimate

$$\Phi^2(t,s) \le \kappa_1 \exp\left\{-2\int\limits_s^t \delta_v\,dv\right\} \exp(2\varepsilon) \le \hat{\kappa}_1 \exp\left\{-2\int\limits_s^t \delta_v\,dv\right\} \quad \text{a.s.},$$

where $\hat{\kappa}_1 = \kappa_1\exp(2\varepsilon)$. Accordingly, passing to the estimation of I_t in (20), we conclude that for $t > \max\{t_0(\omega), \hat{t}_0(\omega)\}$ with probability 1 we have the inequality

$$I_t \le c\int\limits_0^t \exp\left\{-2\int\limits_s^t \delta_v\,dv\right\} G_s^2\,ds \ln\left(\int\limits_0^t \delta_v\,dv\right),$$

and when estimating the sum $J_t^{(1)} + J_t^{(2)}$ in (19) from above, we arrive at the relation

$$J_t^{(1)} + J_t^{(2)} \le c\int\limits_0^t \exp\left\{-2(1-\lambda_2)\int\limits_s^t \delta_v\,dv\right\}\left(G_s^2 + \frac{f_s^2}{\delta_s}\right) ds \quad \text{a.s.}$$

Combining all the estimates, we obtain an analog of formula (24) with $\lambda_1 = 0$ and $\alpha = 0$. Further, setting $h_t^{(0)} = \exp\{-2\int_0^t \delta_v\,dv - 2(1-\lambda_0)\int_0^t \sigma_v^2\,dv\}(x^2+1)$, for the case of $\mathcal{N}_\infty(\varepsilon) < \infty$ we arrive at the assertion of the theorem. The proof of the theorem is complete.

Comparing the result obtained in Theorem 1 with the estimate derived in the paper [17] only for additive disturbances, one can notice that the new factors included in the dynamics of Eq. (1) impacted the form of the corresponding estimate (16). The presence of multiplicative noise increases the upper bound in proportion to the value of $(\int_0^t \sigma_v^2\,dv)^\alpha$, along with the assumption about correlation $\rho \neq 0$ and stochastic external inputs f_t. At the same time, the degree value α can be reduced by using property (4) in Assumption A and also in the case of $\sigma_t^2/\delta_t \to 0$ as $t \to \infty$. Moreover, it is worth noting (see the proof of Theorem 1) that an estimate of the form (16) remains true even for $\kappa < 0$ in (4); only an adjustment in the expression for α is required. In particular, for $a_t \equiv a < 0$ and $\sigma_t \equiv \sigma \neq 0$, one has the constant $\kappa = 1 + \sigma^2/(2a)$. Then $\alpha = 2(1 - 2a/\sigma^2)^{-1} + \beta$; i.e., $\alpha < 1$ for $2|a| > \sigma^2$, when $\kappa > 0$ in (4), and $1 < \alpha < 2$ for $2|a| \le \sigma^2$, when $\kappa \le 0$ in (4). Thus, (16) implies the result previously obtained in [15], where it was assumed that $G_t \equiv 0$, $f_t > 0$ is a deterministic function, and $\dot{f}_t/f_t \to 0$ as $t \to \infty$ (here $\dot{}$ is the time derivative of a function). At the same time, inequality (4) played an important role in the paper [16], where the time-invariant functions $a_t \equiv a$ and $\sigma_t \equiv \sigma$ were also considered for $\rho = 1$ and nonrandom f_t. The integral condition given in [16] for a.s. convergence to zero of the process X_t, which has the form

$$\int_0^\infty \int_0^t \exp\big\{-\gamma(t-s)m\big\}\left[\Big(\sup_{s\le v\le t}|G_v|\Big)^m + \Big(\sup_{s\le v\le t}|f_v - \sigma^2 G_v|\Big)^m\right] ds\,dt < \infty$$

with some constants γ and m such that $0 < \gamma < \kappa$ and $0 < m < 1 + 2|a|\sigma^{-2}$, can be attributed to analogs of the requirement $\int_0^\infty (G_t^2 + f_t^2)\,dt < \infty$ in Remark 1.

In the following examples, we consider an application of the assertion in Theorem 1 as well as Lemmas 2 and 3 to the analysis of the convergence to zero of solutions of particular linear SDEs. In what follows, writing $g_t \sim \tilde{g}_t$ for two scalar functions $g_t, \tilde{g}_t \ge 0$ means that $0 < \lim\limits_{t\to\infty}(g_t/\tilde{g}_t) < \infty$.

Example 1. Let there be no external inputs in Eq. (1), i.e., $f_t \equiv 0$, and assume that we consider a power-law family of coefficients in (1), i.e., $\delta_t \sim kt^p$, $G_t \sim t^\mu$, and $\sigma_t \sim t^\gamma$, with real numbers p, μ, and γ and some constant $k > 0$. By Assumption A we have $2\mu \le p$ and $2\gamma \le p$ with $p \ge -1$. It follows from Lemma 2 that if $\bar{h}_t \to 0$, $t \to \infty$, then also $EX_t^2 \to 0$, where the function $\bar{h}_t$ is given by formula (8). Note that $\bar{h}_t \sim t^{2\mu - p}$ for $p > -1$. For the case of $p = -1$, we have $\bar{h}_t \sim t^{2\mu+1}$ if $2k(1-\lambda)\kappa + 2\mu > -1$, $\bar{h}_t \sim t^{-2k(1-\lambda)\kappa}\ln t$ if $2k(1-\lambda)\kappa + 2\mu = -1$, and $\bar{h}_t \sim t^{-2k(1-\lambda)\kappa}$ if $2k(1-\lambda)\kappa + 2\mu < -1$. Thus, the condition $2\mu < p$ ensures that $\bar{h}_t \to 0$ and hence the mean square convergence to zero for the process X_t as $t \to \infty$. When studying the convergence with probability 1, the estimates in Theorem 1 and Lemma 3 are used. For $2\gamma < p$, from (16) we conclude that $h_t \le \bar{c}t^{2\mu-p}t^\alpha \ln t$; here $\alpha > 0$ is an arbitrarily small number and $\bar{c} > 0$ is some constant. Therefore, for $2\mu < p$ it is also guaranteed that $X_t^2 \to 0$ a.s. as $t \to \infty$. Further, let $2\gamma = p$. In the case of $p = -1$, one has $(\int_0^t \sigma_v^2\,dv)^\alpha \sim \ln^\alpha t$, and the condition $2\mu < p$ is preserved if it is necessary to ensure that $X_t^2 \to 0$ a.s. as $t \to \infty$. If $p > -1$, then $h_t \sim t^{(2\mu-p)}t^{(p+1)\alpha}\ln t$, where α is defined in (15). Accordingly, for $2\mu < p - (p+1)\alpha$ we have the convergence $X_t^2 \to 0$ with probability 1 as $t \to \infty$. At the same time, the condition $\int_0^\infty \sigma_t^2 \bar{h}_t\,dt < \infty$ in Lemma 3 is satisfied for $2\mu < -1$, and then $X_t^2 \to 0$ a.s. Thus, for $p > -1$ a general condition of the form $2\mu < \max\{-1, p - (p+1)\alpha\}$ is stated, which ensures the convergence $X_t^2 \to 0$ a.s. as $t \to \infty$. This condition also coincides with the condition in the example in [16], where the case of $p = \gamma = 0$ is considered and the convergence $X_t \to 0$ with probability 1 is analyzed.

Example 2. The paper [13] considered a pharmacokinetic model of the concentration of a chemical substance in the form of a stochastic process with two components $(\tilde{Y}_t, \hat{Y}_t)$, $t \ge 0$. Here the processes $\tilde{Y}_t$ and $\hat{Y}_t$, $t \ge 0$, are deviations of the current concentrations from the planned values, and the underlying SDEs have the form

$$d\tilde{Y}_t = -k_0\tilde{Y}_t\,dt + \tilde{\sigma}d\tilde{Y}_t d\tilde{w}_t, \tag{25}$$

$$d\hat{Y}_t = k_0\tilde{Y}_t\,dt - k_1\hat{Y}_t\,dt + \hat{\sigma}\hat{Y}_t d\hat{w}_t \tag{26}$$

with nonrandom initial conditions $\tilde{Y}_0 = \tilde{y}$ and $\hat{Y}_0 = \hat{y}$; $k_0, k_1 > 0$ are the absorption rates (in (26), the constant k_0 also sets the rate of substance inflow); $(\tilde{w}_t, \hat{w}_t)$, $t \geq 0$, is a two-dimensional Wiener process; $\tilde{\sigma}, \hat{\sigma} > 0$ are constants characterizing the magnitude of impact of random factors. The process $(\tilde{Y}_t, \hat{Y}_t)$, $t \geq 0$, was obtained after applying the control strategy chosen by the authors in [13], and the question arose of how much this enables the system to reach the zero state in the long run. It is easily seen that the process in Eq. (25) is a geometric Brownian motion, and (26) defines the so-called inhomogeneous geometric Brownian motion. Thus, $\tilde{Y}_t = \tilde{y}\exp\{-k_0 t - (1/2)\tilde{\sigma}^2 t + \tilde{\sigma}\tilde{w}_t\}$, and obviously, $\tilde{Y}_t \to 0$ a.s. as $t \to \infty$. The assertion in Theorem 1 is used to study the convergence of $\hat{Y}_t$. Setting $X_t = \hat{Y}_t$, $x = \hat{y}$, $f_t = k_0\tilde{Y}_t$, $a_t \equiv -k_1$, and $\sigma_t \equiv \hat{\sigma}$, we obtain the form of Eq. (1). Moreover, conditions (2) and (3) in Assumption A are satisfied for $\delta_t = -k_1$. Since no assumptions are made in [13] about the relationship between k_1 and $\tilde{\sigma}^2$, we have $\kappa = 1 - \hat{\sigma}^2/(2k_1)$ in condition (4). From (16), we have

$$\begin{aligned} h_t \sim{}& \exp\big\{-2k_1 t - (1-\lambda_0)\hat{\sigma}^2 t\big\}\hat{y}^2 \\ &+ t^\alpha k_0^2 \tilde{y}^2 \int_0^t \exp\big\{-2k_1(1-\lambda_1)(t-s)\big\}\exp\{-2k_0 s - \tilde{\sigma}^2 s + 2\tilde{\sigma}\tilde{w}_s\}\,ds, \end{aligned}$$

where λ_0 and λ_1 are constants, $0 < \lambda_0 < 1$, $0 < \lambda_1 < 1$; $\alpha = 2 + \beta$, where $\beta > 0$ is an arbitrarily small number. For $2k_1 - 2k_0 - \tilde{\sigma}^2 < 0$, there exists a random variable

$$\chi_\infty = \int_0^\infty \exp\big\{2k_1(1-\lambda_1)s\big\}\exp\{-2k_0 s - \tilde{\sigma}^2 s + 2\tilde{\sigma}\tilde{w}_s\}\,ds$$

(see [27]), and then the estimate $\limsup_{t\to\infty}\{h_t/\hat{h}_t\} < \infty$ holds true, where $\hat{h}_t = \exp\{-2k_1(1-\lambda)t\}$, for each constant λ, $0 < \lambda < 1$. If, however, $2k_1 - 2k_0 - \tilde{\sigma}^2 \geq 0$, then from the law of the iterated logarithm we have $\limsup_{t\to\infty}\{h_t/\hat{h}_t\} < \infty$ for $\hat{h}_t = \exp\{-2k_0 t - (1-\lambda)\tilde{\sigma}^2 t\}$ with any constant λ, $0 < \lambda < 1$. Thus, $\hat{Y}_t \to 0$ with probability 1 as $t \to \infty$, and one has $\limsup_{t\to\infty}\{\hat{Y}_t^2/\exp(-\gamma t)\} < \infty$ with $\gamma = \min\{2k_1(1-\lambda), 2k_0 + (1-\lambda)\tilde{\sigma}^2\}$ for any constant $0 < \lambda < 1$.

2. APPLICATION TO MODELING ANOMALOUS DIFFUSION

Physical processes known as anomalous diffusion are often observed in the real world. The term "anomalous" is adopted because some of the known properties of these processes differ significantly from the "normal" diffusion modeled by Brownian motion. The basic characteristic attributed to any diffusion is the mean-square displacement (MSD). Then anomalous diffusion is defined as a process with a nonlinear increase of the mean-square displacement in time.

Definition 1. Let X_t, $t \geq 0$, define a velocity process, and let $T > 0$ be the observation horizon length. Then $Y_T = \int_0^T X_t\,dt$ is the corresponding process of displacement under the initial condition $Y_0 = 0$. The mean-square displacement is determined by the formula

$$D_T = EY_T^2 = E\left(\int_0^T X_t\,dt\right)^2 = \int_0^T\int_0^T E(X_t X_s)\,ds\,dt.$$

Note that if X_t is a "white noise" process, then $Y_T = B_T$, where B_t, $t \geq 0$, is a Brownian motion, and $D_T = T$.

Definition 2 [6]. Let $\hat{d}_1 = \liminf_{T\to\infty}(D_T/T)$ and $\hat{d}_2 = \limsup_{T\to\infty}(D_T/T)$. Then the diffusion is said to be normal for $0 < d_1 \leq d_2 < \infty$ and anomalous otherwise (subdiffusion for $d_2 = 0$ and superdiffusion for $d_1 = \infty$).

The mean-square displacement is one important statistical characteristic, but it does not give an answer to the question about possible fluctuations of an individual path of the stochastic process Y_T as $T \to \infty$. It is well known that this goal is served by the concept of an upper function of a process, when the relation $\limsup_{T\to\infty}\{Z_T/\tilde{h}_T\} < \infty$ is satisfied a.s. for a stochastic process $Z_T \geq 0$, $T \geq 0$, and a function $\tilde{h}_T \geq 0$, which is usually chosen to be deterministic. In particular, it follows from the law of the iterated logarithm that $\tilde{h}_T = \sqrt{T \ln\ln T}$ for $Z_T = |B_T|$. The following definition is based on the idea of comparing the process Y_T with the function $\sqrt{T\ln\ln T}$ as $T \to \infty$.

Definition 3. If $\limsup_{T\to\infty}\{|Y_T|/\sqrt{T\ln\ln T}\} = 0$, then the process is called subdiffusion with respect to the upper function.

Note that if $Y_T \to Y_\infty$ a.s. as $T \to \infty$, where Y_∞ is a random variable, then the "degenerate" case of subdiffusion occurs, which is also a consequence of the boundedness of D_T, $T \geq 0$.

An equation of the form (1) generalizes previously known specifications of diffusion by allowing for time-dependent coefficients, correlated noise of various types, and stochastic external inputs. In this part of the paper, the results of Sec. 1 are used to reveal subdiffusion. Based on the results of Lemma 2 and Theorem 1, we formulate the following statement.

Assertion. *Let Assumption* A *be true. Then the following assertions hold*:

(a) *If $\bar{h}_t t \to 0$ as $t \to \infty$ for the function $\bar{h}_t$ in* (8), *then the process X_t defines subdiffusion in mean square.*

(b) *If $\{\bar{h}_t(t/\ln\ln t)\} \to 0$ as $t \to \infty$ for the function h_t in* (16), *then the process X_t describes subdiffusion with respect to the upper function.*

Proof. It follows from the estimate $D_T = \int_0^T\int_0^T E(X_tX_s)\,ds\,dt \leq (\int_0^T \sqrt{EX_t^2}\,dt)^2$ and the result in Lemma 2 that the validity of the condition $\bar{h}_t t \to 0$, $t \to \infty$, also guarantees the relation $D_T/T \to 0$, $T \to \infty$, i.e., statement (a). In (b), we use the form of h_t obtained in Theorem 1, the estimate $|X_t| \leq c\sqrt{h_t}$ holds a.s. for $t > t_0(\omega)$, $t_0(\omega)$ is a.s. a finite moment, and one can also readily establish that the presence of the condition $\{h_t(t/\ln\ln t)\} \to 0$, $t \to \infty$, implies the convergence $|Y_T|/\sqrt{T\ln\ln T} \to 0$ as $T \to \infty$ with probability 1. The proof of the assertion is complete.

The following example shows that the conditions in the Assertion can be relaxed.

Example 3. Let the coefficients $a_t \equiv -1$ and $f_t \equiv 0$ and the initial condition $X_0 = 0$ be given in Eq. (1), and suppose also that inequality (4) holds true. Then (1) becomes

$$dX_t = -X_t\,dt + G_t\,dW_t + \sigma_t X_t\,dw_t,$$

whence the displacement process is determined by the formula $Y_T = M_T^{(1)} + M_T^{(2)} - X_T$, where $M_T^{(1)} = \int_0^T G_t\,dW_t$ and $M_T^{(2)} = \int_0^T \sigma_t X_t\,dw_t$. In this case, $D_T \leq 2(E[M_T^{(1)}]^2 + E[M_T^{(2)}]^2 + EX_T^2)$, where $E[M_T^{(1)}]^2 = \int_0^T G_s^2\,ds$ and $E[M_T^{(2)}]^2 = \int_0^T \sigma_s^2 EX_s^2\,ds$; an estimate for EX_t^2 is given in (2). Note that the condition $\int_0^\infty G_t^2\,dt < \infty$ implies, as $T \to \infty$, the convergences $EX_T^2 \to 0$ and $E[M_T^{(1)}]^2 + E[M_T^{(1)}]^2 \to E[M_\infty^{(1)}]^2 + E[M_\infty^{(1)}]^2$ and the boundedness of D_T, i.e., subdiffusion. If, however, $G_t \to 0$, $t \to \infty$, then also $EX_T^2 \to 0$ and $\{E[M_T^{(1)}]^2 + E[M_T^{(1)}]^2\}/T \to 0$ as $T \to \infty$, and we again have the case of subdiffusion in mean square. Passing to the pathwise analysis of the displacement process Y_T, $T \to \infty$, we consider two situations. For $\int_0^\infty G_t^2\,dt < \infty$, the convergences $M_T^{(1)} \to M_\infty^{(1)}$ and $M_T^{(2)} \to M_\infty^{(2)}$ as $T \to \infty$ are observed. Besides, according to Lemma 3, $X_T \to 0$ a.s., and then $Y_T \to Y_\infty$ with probability 1 as $T \to \infty$, and we obtain the case of "degenerate" subdiffusion. Now let $\int_0^T G_t^2\,dt \to \infty$ as $T \to \infty$. By the law of the iterated logarithm for martingales, with probability 1 we have the relations

$$\limsup_{T\to\infty}\left\{M_T^{(i)}/\sqrt{\langle M_T^{(i)}\rangle \ln\ln\langle M_T^{(i)}\rangle}\right\} < \infty, \quad i = 1, 2;$$

here we use the quadratic variations $\langle M_T^{(1)}\rangle = \int_0^T G_t^2\,dt$ and $\langle M_T^{(2)}\rangle = \int_0^T \sigma_t^2 X_t^2\,dt$. Assume that $X_t \to 0$ and $G_t \to 0$ as $t \to \infty$; then for $i = 1, 2$ we have $\langle M_T^{(i)}\rangle/T \to 0$ as $T \to \infty$, and hence also $|Y_T|/\sqrt{T \ln\ln T} \to 0$ with probability 1 as $T \to \infty$. Using Theorem 1 and the boundedness of σ_t^2, we conclude that the validity of the condition $\hat{h}_t = G_t^2 \ln t(\int_0^t \sigma_v^2\,dv)^\alpha \to 0$, $t \to \infty$, implies the convergence $X_t \to 0$ a.s. as $t \to \infty$ with the constant α given in (15). Thus, the requirement $\hat{h}_t \to 0$ will be sufficient for revealing subdiffusion in the case of $\int_0^T G_t^2\,dt \to \infty$ as $T \to \infty$.

Assumption D. The stability rate δ_t is a monotone differentiable function, $t \ge 0$, and for the function $\phi_t = \dot{\delta}_t/\delta_t^2$ ($\dot{}$ stands for the time derivative) at least one of the following two relations holds:

$$\lim_{t\to\infty} \phi_t = \hat{\kappa}_1, \qquad \lim_{t\to\infty} (1/\phi_t) = \hat{\kappa}_2, \tag{27}$$

where $\hat{\kappa}_1$ and $\hat{\kappa}_2$ are nonpositive constants.

Assumption D was previously introduced in the paper [28] when studying the displacement Y_T given by a time-varying Ornstein–Uhlenbeck process, i.e., for $\sigma_t \equiv 0$ and $f_t \equiv 0$ in Eq. (1). It should be noted that the cases of $\hat{\kappa}_1, \hat{\kappa}_2 > 0$ correspond to $\delta_t < 0$, which does not satisfy Assumption A, so they are not considered here. The main result of this section is the following assertion.

Theorem 2. *Let Assumptions* A *and* D *hold true with a value* $\hat{\kappa}_1 > -2$. *If the function* $\bar{h}_t$ *defined in* (8) *and the coefficients of Eq.* (1) *satisfy*

(a) $\dfrac{G_t^2 + \sigma_t^2 \bar{h}_t + tEf_t^2}{\delta_t^2} \to 0$ *as* $t \to \infty$, *then the process* X_t *defines subdiffusion in mean square.*

(b) *The condition that the function* $\dfrac{G_t^2 + \sigma_t^2 \bar{h}_t}{\delta_t^2}\sqrt{\dfrac{t}{\ln\ln t}}$ *is bounded as* $t \to \infty$ *and also one of the following two conditions*:

$$\frac{|f_t|}{\delta_t}\sqrt{\frac{t}{\ln\ln t}} \to 0 \quad \text{a.s. as} \quad t \to \infty$$

or otherwise the expression

$$t\frac{Ef_t^2}{\delta_t^2}\sqrt{\frac{t}{\ln\ln t}} \quad \text{as} \quad t \to \infty$$

is bounded, then the process X_t *defines subdiffusion with respect to the upper function.*

Proof. We use the representation (10) to find the estimate

$$D_T \le 2E\big[I_T^{(1)}\big]^2 + 2E\big[I_T^{(2)}\big]^2 + 2E\big[I_T^{(3)}\big]^2 + 2c\int_0^T \exp\left\{-2\int_0^t \delta_v\,dv\right\} dt x^2,$$

where $E[I_T^{(1)}]^2 = \int_0^T \int_0^T E(Y_t^{(0)} Y_s^{(0)})\,ds\,dt$ and, moreover,

$$E\big[I_T^{(1)}\big]^2 \le 2c\int_0^T\int_0^t \exp\left\{-\int_s^t \delta_v\,dv\right\}\sigma_s^2 E X_s^2\,ds\,dt;$$

see the proof of Lemma 3. In addition (see [6]),

$$E\big[I_T^{(2)}\big]^2 = \int_0^T\int_0^T E\big(X_t^{(0)} X_s^{(0)}\big)\,ds\,dt,$$

$$E\big[I_T^{(2)}\big]^2 \le 2c \int\limits_0^T \int\limits_0^t \exp\left\{-\int\limits_s^t \delta_v\, dv\right\} G_s^2\, ds\, dt.$$

For $I_T^{(3)} = \int_0^T \xi_t\, dt$, owing to the covariance function $E(f_t f_s)$ occurring in $E(\xi_t\xi_s)$ being unknown, we use the inequality

$$E\big[I_T^{(3)}\big]^2 \le \left(\int\limits_0^T \int\limits_0^t \exp\left\{-\int\limits_s^t \delta_v\, dv\right\} \sqrt{Ef_s^2}\, ds\, dt\right)^2.$$

Further, for $\hat\kappa_1 > -2$ we have the inequality $(\dot\delta_t/\delta_t^2) + 2 > \tilde\varepsilon$, where $\tilde\varepsilon > 0$ is an arbitrarily small number and $t > t_0(\tilde\varepsilon)$; based on this, under condition (a), saying that $G_t^2 + \sigma_t^2 EX_t^2 < \varepsilon\delta_t^2(\dot\delta_t/\delta_t^2 + 2)$, $\varepsilon > 0$ is also an arbitrarily small number, $t > t_1(\varepsilon)$. Multiplying this inequality by $\exp\{-2\int_0^t \delta_v\, dv\}$, after integration we conclude that

$$E\big[Y_t^{(0)}\big]^2 + E\big[X_t^{(0)}\big]^2 \le c\exp\left\{-2\int\limits_0^t \delta_v\, dv\right\} \int\limits_0^t \exp\left\{2\int\limits_0^s \delta_v\, dv\right\}(G_s^2 + \sigma_s^2 EX_s^2)\, ds$$
$$\le \varepsilon\delta_t + c_0 \exp\left\{-2\int\limits_0^t \delta_v\, dv\right\} ds$$

with some constant $c_0 > 0$. Further operations of multiplication and integration lead to the relation

$$E\big[I_T^{(1)}\big]^2 + E\big[I_T^{(2)}\big]^2 \le \varepsilon T + (c_0/2)\left(\int\limits_0^T \exp\left\{-\int\limits_0^t \delta_v\, dv\right\} dt\right)^2 + c_1 \int\limits_0^T \exp\left\{-\int\limits_0^t \delta_v\, dv\right\} dt + c_2,$$

where $c_1, c_2 > 0$ are some constants and $T > T_0(\varepsilon, \tilde\varepsilon)$. Then, obviously, $\{E[I_T^{(1)}]^2 + E[I_T^{(2)}]^2\}/T \to 0$, $T \to \infty$. For the above estimate of $E[I_T^{(3)}]^2$, one can show in a similar way that, in view of the requirement $\hat\kappa_1 > -2$, the condition $\{(Ef_t^2/\delta_t^2)t\} \to 0$, $t \to \infty$, implies the convergence $E[I_T^{(3)}]^2/T \to 0$. In addition, for $\hat\kappa_1 > -2$ one has $\{\int_0^T \exp\{-2\int_0^t \delta_v\, dv\}\, dt x^2/T\} \to 0$, $T \to \infty$. Thus, we conclude that $D_T/T \to 0$ as $T \to \infty$ and obtain subdiffusion in mean square.

To prove assertion (b), we need to consider the ratio $|Y_T|/\sqrt{T \ln\ln T}$. We use the estimate

$$|Y_T| \le I_T^{(1)} + I_T^{(2)} + I_T^{(3)} + \int\limits_0^T \exp\left\{-\int\limits_0^t \delta_v\, dv\right\} dt|x|,$$

where $I_T^{(1)} = \int_0^T Y_t^{(0)}\, dt$, $I_T^{(2)} = \int_0^T X_t^{(0)}\, dt$, and $I_T^{(3)} = \int_0^T \xi_t\, dt$. First, note that

$$\int\limits_0^T \exp\left\{-\int\limits_0^t \delta_v\, dv\right\} dt|x| \Big/ \sqrt{T\ln\ln T} \to 0, \quad T \to \infty,$$

if $\hat\kappa_1 > -2$. Then for each of the integrals $I_T^{(i)}$, $i = 1, 2, 3$, we apply the sufficient condition for the convergence $I_T^{(i)}/\Gamma_T \to 0$ with probability 1 as $T \to \infty$, where $\Gamma_T = \sqrt{T \ln\ln T}$. This condition (see [29]) has the form $E[I_T^{(i)}]^2 \le \tilde c\Gamma_T$. Based on the above estimates for $E[I_T^{(i)}]^2$ and condition (b), after reasoning similar to (a) we find that the requirement $\{(G_t^2 + \sigma_t^2 \bar h_t)/\delta_t^2\}\sqrt{t/\ln\ln t} + (tEf_t^2/\delta_t^2)\sqrt{t/\ln\ln t}$

also implies that $D_T/\sqrt{T\ln\ln T}$ is bounded; i.e., $|Y_T|/\sqrt{T\ln\ln T} \to 0$ a.s. as $T \to \infty$. If instead of the condition for Ef_t^2 in (b), $(|f_t|/\delta_t)\sqrt{t/\ln\ln t} \to 0$, $t \to \infty$, is satisfied a.s., then the estimate $I_T^{(3)} \le F_T$ a.s., where

$$F_T = \int\limits_0^T \exp\left\{-\int\limits_0^t \delta_v\,dv\right\} \int\limits_0^t \exp\left\{\int\limits_0^s \delta_v\,dv\right\}|f_s|\,ds\,dt,$$

will imply that $\{I_T^{(3)}/\sqrt{T\ln\ln T}\} \to 0$ a.s. as $T \to \infty$. Indeed, if $F_\infty < \infty$, then the desired relation is obvious. Otherwise, to find $\lim\limits_{T\to\infty}\{F_T/\sqrt{T\ln\ln T}\}$, l'Hôpital's rule is applied twice. Consequently, $I_T^{(3)}/\sqrt{T\ln\ln T} \to 0$ a.s., and then also $|Y_T|/\sqrt{T\ln\ln T} \to 0$ with probability 1 as $T \to \infty$. The proof of the theorem is complete.

Remark 2. For $\hat{\kappa}_1/\bar{\kappa} \le -2$, where $\hat{\kappa}_1$ and $\bar{\kappa}$ are the constants in (27) of Assumption D and inequality (3), respectively, even in the simplest situation of the deterministic equation (1) with nontrivial initial condition, we will have $D_T/T \ge \bar{c} > 0$ as $T \to \infty$, where $\bar{c} > 0$ is some constant. Thus, no subdiffusion is observed in this case. Indeed, here $D_T \ge x^2\kappa_2^2(\int_0^T \exp\{\int_0^t -\bar{\kappa}\delta_v\,dv\}\,dt)^2$. In this case, owing to the condition $\hat{\kappa}_1/\bar{\kappa} \le -2$, for some constant $\hat{c} > 0$ we have the estimate

$$\exp\left\{\int\limits_0^t -\bar{\kappa}\delta_v\,dv\right\}\sqrt{t} \ge \hat{c} > 0, \quad t \to \infty,$$

and consequently, D_T/T is bounded away from zero as $T \to \infty$.

CONCLUSIONS

In this paper, we study the asymptotic behavior of solutions X_t, $t \ge 0$, of the linear inhomogeneous SDEs (1) (the form of the processes X_t is defined in Lemma 1). The corresponding dynamics (1) with time-varying coefficients contains correlated additive and multiplicative disturbances as well as external stochastic inputs f_t. Upper bounds $\bar{h}_t$ and h_t are found for X_t^2 as $t \to \infty$ as functions of the parameters of Eq. (1) (see Lemma 2 and Theorem 1). Referring to (8) and (16), one can see the obvious dependence of $\bar{h}_t$ and h_t on the above factors impacting the evolution of X_t in time. In particular, the presence of multiplicative disturbances increases h_t in proportion to $(E\mathcal{Z}_t^2)^\alpha = (\int_0^t \sigma_s^2\,ds)^\alpha$, i.e., the variance of the process $\mathcal{Z}_t = \int_0^t \sigma_s\,dw_s$ raised to the power of α. The results obtained are used to identify conditions on the coefficients in (1) under which X_t defines a subdiffusion process (see the Assertion, Example 3, and Theorem 2).

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