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## CONTROL IN STOCHASTIC SYSTEMS AND UNDER UNCERTAINTY CONDITIONS

# Time Invariance of Optimal Control in a Stochastic Linear Controller Design with Dynamic Scaling of Coefficients

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**Abstract**—This paper considers the design problem of a stochastic linear-quadratic controller over an infinite time-horizon with dynamic scaling of the coefficients in the state equation and the cost criterion. Dynamic scaling means multiplying the coefficients by a positive time-varying function. The optimality criteria used are extensions of the long-term average cost and pathwise long-term average cost. The integral of the scaling function is applied to normalize the performance indices. It is shown that, the optimal control law is time-invariant and can be obtained through a steady-state optimal strategy known for the autonomous system.

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## INTRODUCTION

Linear controlled systems with the state vector subjected to additive disturbances are widely used in modeling dynamics in various applications, particularly mechanics and motion control; for example, see [1–5]. Note the significant contribution of Russian researchers in the development of this area, starting with the works by A.M. Letov [6–8]. Also, see the survey paper [9] written by V.A. Yakubovich for the stochastic case and the results presented in the monographs [4, 5, 10, 11]. Long-term optimization problems often involve the assumption that the parameters of the corresponding models are constant over time [12–14]. As a consequence, it is possible to use the known methods of optimal control theory developed for the case of autonomous equations and stationary processes [9]. At the same time, such a problem statement neglects several features of system specifics and decision-making. Among them, we mention the asynchronous nature of the time scales of the processes and their observations [15, 16] as well as the presence of subjective time [17]. For the linear state dynamics model considered below, this assumption leads to the scaling of parameters. The scaling is dynamic in the sense that the scaling functions depend on the time variable. Dynamic scaling of the coefficients becomes necessary when passing from the internal (subjective) time to the real (physical) time of the control system functioning. The control strategy is chosen to stabilize the system in the long run, and the cost criterion has an integral quadratic form. Long-term optimization in such problems is based on constructing the steady-state controller [18, Section 3.4] and determining an appropriate optimality over an infinite time-horizon. It is well-known that the stable control law has a linear feedback form corresponding to the limiting optimal strategies obtained for a finite horizon. In the case of constant parameters, the form of the stable feedback law contains the solution of the algebraic Riccati equation, which is an obvious advantage in its implementation. We will demonstrate below that the time invariance of the optimal strategy can also arise for time-varying coefficients. The main aim of this research is to study the design problem of a stochastic linear controller with the dynamic scaling of the coefficients. The remainder of the paper is organized as follows. Section 1 describes the control system under consideration and provides a rigorous statement of the problem. Section 2 contains the optimal control results for the system with scaling and necessary background on optimal stochastic linear controller design with constant coefficients. Section 3 gives an example of a scalar control system, including a thorough analysis of the key assumptions on the parameters. Section 4 is devoted to a possible application to dynamic stabilization in macroeconomics. The main outcomes of this paper and some lines of further research are presented in the Conclusions section.

## 1. CONTROL SYSTEM DESCRIPTION AND PROBLEM STATEMENT

Consider a complete probability space $\{\Omega, \mathsf{F}, \mathbf{P}\}$, and let $X_t$, $t \geq 0$, be the given $n$-dimensional stochastic process on this space described by the equation

$$dX_t = \alpha_t A X_t dt + \alpha_t B U_t dt + \sqrt{\alpha_t} G dW_t, \quad X_0 = x, \tag{1.1}$$

with the following notations: $x$ is a nonrandom initial state; $W_t$, $t \geq 0$, is the $d$-dimensional standard Wiener process; $U_t$, $t \geq 0$, is an admissible control or a $k$-dimensional stochastic process adapted to the filtration $\{\mathsf{F}_t\}_{t\geq 0}$, $\mathsf{F}_t = \sigma\{W_s, s \leq t\}$, where $\sigma(\cdot)$ denotes the $\sigma$-algebra, such that Eq. (1.1) has a solution; and $A$, $B$, and $G$ are constant matrices of compatible dimensions, and $G \neq 0$. The set of all admissible controls will be denoted by $\mathsf{U}$. The set $\mathsf{U}$ contains the state-feedback control (the control in a closed loop system), which depends on the values $\{X_s, 0 \leq s \leq t\}$. (In this case, $U_t$ is measurable with respect to $\sigma\{X_s, s \leq t\}$.) In Eq. (1.1), $\alpha_t > 0$ is a scaling function. The additive noises $dW_t$ are scaled using the term $\sqrt{\alpha_t}$ since $\sqrt{\alpha_t}dW_t$ has the mean-square order of $\alpha_t dt$, i.e., $E(\sqrt{\alpha_t}dW_t)^2 = \alpha_t dt$, where $E(\cdot)$ denotes the expectation operator.

An equation of the form (1.1) was previously considered in various applications with the partial scaling of the coefficients. For example, a deterministic version of (1.1) with the only variable matrix $\alpha_t A$ arose when solving the stabilization problem for a class of nonlinear nonholonomic systems [19], in which the function $\alpha_t$ is a characteristic of stability. The dynamics of system (1.1) with $A = 0$ and $G \neq 0$ were studied in the context of cognitive processes [20]; in this case, $\alpha_t$ determines the impact of an external impulse, also affecting the diffusion coefficient. In the papers [21–23], a class of dynamic processes of form (1.1) with the power-type scaling function $\alpha_t$ was introduced for the econometric modeling of signal transmission and dynamics of several economic variables.

For each $T > 0$, as the cost criterion, we choose the random variable

$$J_T(U) = \int_0^T \alpha_t (X_t^{\mathrm{T}} Q X_t + U_t^{\mathrm{T}} R U_t) dt, \tag{1.2}$$

where $U \in \mathsf{U}$ is an admissible control on a finite horizon $[0, T]$ (see the definition of all admissible controls and the set $\mathsf{U}$ for system (1.1)); $Q \geq 0$ and $R > 0$ are symmetric matrices; $^{\mathrm{T}}$ denotes the transpose operator; for matrices $A$ and $B$, the expression $A \geq B$ means that their difference is positive semidefinite. If the function $\alpha_t > 0$ in (1.2) is monotone, it can be treated as a discounting function. Positive discounting arises for a decreasing function $\alpha_t$, whereas negative discounting for an increasing function $\alpha_t$; for details, see [24]. This terminology can be explained by the well-known formula $\phi_t = -\dot{\alpha}_t/\alpha_t$ of the discounting rate $\phi_t$, where $\cdot$ denotes the time derivative of a function.

For the analysis of (1.1) and (1.2) in the scaling situation, if the function $\alpha_t$ is monotone and $\alpha_0 = 1$, then for $\alpha_t > 1$, we have the inflation of the coefficients (the growth of their absolute values), and the case $\alpha_t \to \infty$ as $t \to \infty$ is similar to "hyperinflation." The situation $\alpha_t \equiv 1$ means no scaling and the constant values of all coefficients over time, corresponding to an autonomous control system. If $\alpha_t < 1$, then the parameters are deflated; in the limiting case $\alpha_t \to 0$ as $t \to \infty$, the matrices therefore become singular.

The main results of this paper will be obtained under the following assumptions.

**Assumption A.** For $t > 0$ the scaling function $\alpha_t > 0$ is integrable and

$$\int_0^t \alpha_s ds \to \infty \quad \text{as} \quad t \to \infty.$$

This assumption means that scaling will preserve the asymptotically unbounded growth of the total variance of the cumulative disturbances affecting the system as $T \to \infty$. Indeed, defining

$$Z_T = \int_0^T \sqrt{\alpha_s} dW_t$$

and

$$E(Z_T^{\mathrm{T}} Z_T) = \|G\|^2 \int_0^T \alpha_t dt,$$

we obtain $E(Z_T^{\mathrm{T}} Z_T) \to \infty$ as $T \to \infty$, where $\| \ \|$ denotes the matrix norm.

**Assumption B.** The pair of matrices $(A, B)$ is stabilizable, and the pair of matrices $(A, \sqrt{Q})$ is detectable.

The stabilizability of the pair $(A, B)$ (the detectability of the pair $(A, \sqrt{Q})$) means the existence of a matrix $K$ (a matrix $L$, respectively) for which the matrix $A + BK$ ($A + L\sqrt{Q}$, respectively) is exponentially stable; see [25, pp. 167–168]. It is known that [18, Theorem 3.7, p. 275], under Assumption B, there exists the so-called optimal stable feedback $U^*$ of the form $U_t^* = -R^{-1}B^{\mathrm{T}}\bar{\Pi}X_t^*$, where the matrix $\bar{\Pi} \geq 0$ is the solution of the algebraic Riccati equation $\bar{\Pi}A + A\bar{\Pi} - \bar{\Pi}BR^{-1}B^{\mathrm{T}}\bar{\Pi} + Q = 0$. In the autonomous control system ($\alpha_t \equiv 1$), the strategy $U^*$ is the solution of the infinite-horizon optimal control problem with the criterion of the long-run average cost, e.g., see [25, Theorem 5.4.3, p. 169]:

$$\limsup_{T\to\infty} \frac{EJ_T(U)}{T} \to \inf_{U\in\mathrm{U}}. \tag{1.3}$$

In addition, $U^*$ acts as the optimal strategy when using a stronger probabilistic criterion of the pathwise average (pathwise ergodic) cost [26] in the problem

$$\limsup_{T\to\infty} \frac{J_T(U)}{T} \to \inf_{U\in\mathrm{U}} \quad \textit{with probability } 1. \tag{1.4}$$

As was shown in [27], these criteria properly account for the effect of uncertainty on control performance only for system (1.1) and (1.2) with bounded coefficients and a nonsingular diffusion matrix. In the case of dynamically scaled parameters, to compare control strategies as $T \to \infty$, we will employ the concepts of the extended long-term average cost and the extended stochastic (pathwise) long-run average cost introduced in [28] for time-varying $G_t$: according to these concepts, value $T$ in formulas (1.3) and (1.4) is replaced with the normalization

$$\int_0^T \|G_t\|^2 \, dt.$$

In (1.1), the diffusion matrix is $G_t = \sqrt{\alpha_t} G$. The aim of this paper is to find the optimal control $U^*$ for the problems

$$\limsup_{T\to\infty} \frac{EJ_T(U)}{\int_0^T \alpha_t dt} \to \inf_{U\in\mathrm{U}} \quad \text{and} \quad \limsup_{T\to\infty} \frac{J_T(U)}{\int_0^T \alpha_t dt} \to \inf_{U\in\mathrm{U}} \quad \textit{with probability } 1.$$

As is shown below, the form of the optimal control $U^*$ turns out time-invariant when passing from the autonomous control system to the one with dynamic scaling.

## 2. TIME INVARIANCE OF OPTIMAL CONTROL IN SYSTEM WITH DYNAMIC SCALING

As mentioned earlier, when analyzing the infinite-horizon optimal control problem for the system with dynamic scaling (1.1) and (1.2), an important role is played by the case $\alpha_t \equiv 1$ (the constant coefficients). For this case, the corresponding results are known and will be presented in this section as well. When describing the elements of such an autonomous control system, we will introduce special notations to avoid any confusion in the subsequent time change procedure. The system's state $\tilde{X}_\tau$, $\tau \geq 0$, is described by the equation

$$d\tilde{X}_\tau = A\tilde{X}_\tau d\tau + B\tilde{U}_\tau d\tau + Gd\tilde{W}_\tau, \quad \tilde{X}_0 = x. \tag{2.1}$$

The cost criterion on $[0,\tilde{T}]$ has the form

$$\tilde{J}_{\tilde{T}}(\tilde{U}) = \int_0^{\tilde{T}} (\tilde{X}_\tau^{\mathrm{T}} Q \tilde{X}_\tau + \tilde{U}_\tau^{\mathrm{T}} R \tilde{U}_\tau) d\tau. \tag{2.2}$$

The optimal stable feedback $\tilde{U}^*$ for system (2.1) and (2.2) is determined as

$$\tilde{U}_\tau^* = -R^{-1} B^{\mathrm{T}} \overline{\Pi} \tilde{X}_\tau^*, \tag{2.3}$$

where a symmetric matrix $\overline{\Pi}$ is the unique nonnegative definite solution of the algebraic Riccati equation

$$\overline{\Pi} A + A \overline{\Pi} - \overline{\Pi} B R^{-1} B^{\mathrm{T}} \overline{\Pi} + Q = 0, \tag{2.4}$$

and the process $\tilde{X}_\tau^*, \tau \geq 0$, representing the optimal path, satisfies the equation

$$d\tilde{X}_\tau^* = (A - BR^{-1} B^{\mathrm{T}} \overline{\Pi}) \tilde{X}_\tau^* dt + G d\tilde{W}_\tau, \quad \tilde{X}_0^* = x. \tag{2.5}$$

The main results on the optimality of $\tilde{U}_\tau^*$ are known; see [25, Theorem 5.4.3, p. 169; 26, Theorem 2; 18, Theorem 3.7, p. 275]. They are brought together in the theorem below.

**Theorem 1.** *Let Assumption* B *hold. Then the control law* $\tilde{U}^*$ *given by* (2.3)–(2.5) *is the solution of the problems*

$$\limsup_{\tilde{T}\to\infty} \frac{E\tilde{J}_{\tilde{T}}(\tilde{U})}{\tilde{T}} \to \inf_{\tilde{U}\in \mathrm{U}},$$

$$\limsup_{\tilde{T}\to\infty} \frac{\tilde{J}_{\tilde{T}}(\tilde{U})}{\tilde{T}} \to \inf_{\tilde{U}\in \mathrm{U}} \quad \text{with probability } 1.$$

The optimal values of both criteria coincide:

$$\limsup_{\tilde{T}\to\infty} \frac{E\tilde{J}_{\tilde{T}}(\tilde{U}^*)}{\tilde{T}} = \limsup_{\tilde{T}\to\infty} \frac{\tilde{J}_{\tilde{T}}(\tilde{U}^*)}{\tilde{T}} = \mathrm{tr}(G^{\mathrm{T}} \overline{\Pi} G),$$

where $\mathrm{tr}(\cdot)$ denotes the matrix trace (the sum of all diagonal elements of a matrix). Moreover, the matrix $\tilde{A}^* = A - BR^{-1}B^{\mathrm{T}}\overline{\Pi}$ is exponentially stable.

In addition to the optimality characteristics of $\tilde{U}_\tau^*$, it is important to estimate the behavior of the process sample paths $\tilde{X}_\tau^*, \tau \geq 0$, given by (2.5). The lemma below is based on [24, Theorem 2; 29, Lemma A.2].

**Lemma 1.** *Let the hypotheses of Theorem* 1 *hold. Then the paths of the process* $\tilde{X}_\tau^*, \tau \geq 0$, (2.5) *have the following properties*:

(1) *For* $\tau \geq 0$, $c_1 \leq E\|\tilde{X}_\tau^*\|^2 \leq c_2$, *where* $c_1, c_2 > 0$ *are some constants* [24].

(2) *There exists a nonrandom constant* $\overline{c} > 0$ *such that the inequality*

$$\limsup_{\tau\to\infty} \frac{\|\tilde{X}_\tau^*\|^2}{\ln \tau} < \overline{c} < \infty$$

*is satisfied with probability* 1 [29].

A well-known general method to eliminate the time inhomogeneity of linear nonstationary systems is the change of variables [30, 31]. In the case under consideration, this technique turns out to be inapplicable due to the presence of control in the dynamics equation and the related quadratic cost criterion (1.2). However, using the time change

$$\tau = \int_0^t \alpha_s ds,$$

we can transform the control system with dynamic scaling (1.1) and (1.2) into the autonomous system (2.1) and (2.2).

**Lemma 2.** *Let*

$$\tau = \int_0^t \alpha_s ds. \tag{2.6}$$

*Then the control systems* (1.1), (1.2) *and* (2.1), (2.2) *are related by*

$$X_t = \tilde{X}_\tau, \quad U_t = \tilde{U}_\tau, \quad J_T(U) = \tilde{J}_{\tilde{T}}(\tilde{U}),$$

*where* $\tilde{T} = \int_0^T \alpha_t dt$.

**Proof of Lemma 2.** Consider $\tau$ given by (2.6). The integral representation corresponding to (2.1) has the form

$$\tilde{X}_\tau = x + \int_0^\tau A\tilde{X}_s ds + \int_0^\tau B\tilde{U}_s ds + \int_0^\tau G d\tilde{W}_s.$$

Changing the time variable $t$, we define the differentials

$$d\left(\int_0^\tau A\tilde{X}_s ds\right) = A\tilde{X}_\tau d\tau = \alpha_t A\tilde{X}_\tau dt, \quad d\left(\int_0^\tau B\tilde{U}_s ds\right) = B\tilde{U}_\tau d\tau = \alpha_t B\tilde{U}_\tau dt.$$

For the stochastic integral, we apply the well-known result on the time change [32, Corollary 8.5.4, p. 188]:

$$\int_0^\tau d\tilde{W}_s = \tilde{W}_\tau = \int_0^t \sqrt{\alpha_s} dW_s.$$

Hence,

$$d\left(\int_0^\tau d\tilde{W}_s\right) = \sqrt{\alpha_t} dW_t.$$

As a result,

$$d\tilde{X}_\tau = \alpha_t A\tilde{X}_\tau dt + \alpha_t B\tilde{U}_\tau dt + \sqrt{\alpha_t} G dW_t, \quad \tilde{X}_0 = x.$$

Comparing this equation with the dynamics equation (1.1) for $X_t$, we establish that $X_t = \tilde{X}_\tau$ and $U_t = \tilde{U}_\tau$. Using these relations and (2.6), we transform the cost criterion (1.2) by an appropriate change of the variables in the integrand as follows:

$$J_T(U) = \int_0^T \alpha_t (X_t^{\mathrm{T}} Q X_t + U_t^{\mathrm{T}} R U_t) dt = \int_0^{\tilde{T}} (\tilde{X}_\tau^{\mathrm{T}} Q \tilde{X}_\tau + \tilde{U}_\tau^{\mathrm{T}} R \tilde{U}_\tau) d\tau,$$

where $\tilde{T} = \int_0^T \alpha_t dt$. Thus, $J_T(U) = \tilde{J}_{\tilde{T}}(\tilde{U})$, and the proof of Lemma 2 is complete.

The relation obtained in Lemma 2 may clarify the origin of the control system with dynamic scaling. If the original control system (2.1) and (2.2) is autonomous but only the process $X_t = \tilde{X}_\tau$ with (2.6) is directly accessible (e.g., due to the asynchronous time scales of the ongoing process and the observer that implements the control) [16], then system (2.1) and (2.2) is controlled by changing (1.1) with the cost criterion (1.2).

By Assumption A,

$$\int_0^T \alpha_t dt \to \infty$$

as $T \to \infty$, and hence $\tilde{T} \to \infty$ in the autonomous system (2.1) and (2.2) with the time change (2.6). Then Theorem 1 on the optimal control can be applied to the system with dynamic scaling.

Under Assumption B, we define the optimal stable feedback law

$$U_t^* = -R^{-1}B^{\mathrm{T}}\bar{\Pi}X_t^*, \tag{2.7}$$

where the optimal process $X_t^*$, $t \geq 0$, satisfies the equation

$$dX_t^* = \alpha_t(A - BR^{-1}B^{\mathrm{T}}\bar{\Pi})X_t^*dt + \sqrt{\alpha_t}GdW_t, \quad X_0^* = x, \tag{2.8}$$

and the matrix $\bar{\Pi} \geq 0$ is the solution of the algebraic Riccati equation (2.4).

The next result follows from Lemma 2 and Theorem 1.

**Theorem 2.** *Let Assumptions* A *and* B *hold. Then the control law $U^*$ given by* (2.7) *and* (2.8) *is optimal with respect to extended long-run and pathwise long-run average cost criteria in the system with dynamic scaling; i.e., it is the solution to the problems*

$$\limsup_{T\to\infty} \frac{EJ_T(U)}{\int_0^T \alpha_t dt} \to \inf_{U\in\mathrm{U}}, \tag{2.9}$$

$$\limsup_{T\to\infty} \frac{J_T(U)}{\int_0^T \alpha_t dt} \to \inf_{U\in\mathrm{U}} \quad \textit{with probability } 1. \tag{2.10}$$

The values of the criteria on the optimal control $U^*$ are

$$\limsup_{T\to\infty} \frac{EJ_T(U^*)}{T} = \limsup_{T\to\infty} \frac{J_T(U^*)}{T} = \mathrm{tr}(G^{\mathrm{T}}\bar{\Pi}G).$$

According to Theorem 2, the optimal control has a time-invariant form when considering optimization problems in the system with dynamic scaling of the coefficients. Indeed, $U^*$ in the form $U_t^* = -R^{-1}B^{\mathrm{T}}\bar{\Pi}X_t^*$ is known as the optimal stable feedback law for an autonomous stochastic linear system over an infinite time-horizon; see (2.3) and Theorem 1. At the same time, note the difference in the optimality criteria used: dynamic scaling of the parameters by the function $\alpha_t$ implies the extension of the long-term average cost criteria (1.3) and (1.4) by applying the normalization $\int_0^T \alpha_t dt$; see (2.9) and (2.10).

The remark below characterizes the stabilizing properties of the control $U^*$ and its optimality in the deterministic system (1.1) and (1.2) with $G = 0$. The hypotheses of Theorem 1 are assumed to hold.

**Remark 1.** The matrix $A^* = \alpha_t(A - BR^{-1}B^{\mathrm{T}}\bar{\Pi})$ in (2.7) is asymptotically stable with the rate $\delta_t = \lambda\alpha_t$, where $\lambda > 0$ is some constant. In other words, the fundamental matrix $\Phi(t,s)$ corresponding to $A_t^*$ has an upper bound of the form

$$\|\Phi(t,s\| \leq \kappa \exp\left(-\lambda\int_s^t \alpha_v dv\right), \quad s \leq t,$$

with some constant $\kappa > 0$. This fact follows from the exponential stability of the matrix $\tilde{A}^* = A - BR^{-1}B^{\mathrm{T}}\bar{\Pi}$ (Theorem 1) and the relations

$$A_t^* = \alpha_t\tilde{A}^* \quad \text{and} \quad \Phi(t,s) = \exp\left\{\tilde{A}^*\int_s^t \alpha_v dv\right\},$$

which were derived in [30]. For the deterministic control system (1.1) and (1.2) the strategy $U^*$ is the solution to the problem $\limsup_{T\to\infty} J_T(U) \to \inf_{U\in\mathrm{U}}$, and $\limsup_{T\to\infty} J_T(U^*) = x^{\mathrm{T}}\bar{\Pi}x$.

The results below on the asymptotic behavior of the optimal process $X_t^*$, in the mean-square sense and almost surely, are based on Lemma 1 with an appropriate modification due to the time change.

**Remark 2.** There exist constants $c_1, c_2 > 0$ such that $c_1 \le E\|X_t^*\|^2 \le c_2$ for $t \ge 0$. This uniform boundedness of the process in the mean-square sense is similar to the result obtained earlier for the optimal path in the autonomous stochastic linear controller design [24].

**Remark 3.** There exists a nonrandom constant $\bar{c} > 0$ such that

$$\limsup_{t\to\infty} \frac{\|X_t^*\|^2}{\ln\left(\int_0^t \alpha_s ds\right)} < \bar{c} < \infty \quad \text{with probability 1.}$$

Such an upper bound is a generalization of the logarithmic upper function, which arises in the case $\alpha_t \equiv 1$ when analyzing the asymptotic behavior of the squared norm of the optimal process path in the autonomous system [29].

Among the important characteristics of the optimal control $U^*$ (2.6) and (2.7), note its relation to the solutions of the optimization problems under a finite value $T$. It is known that [18, Theorem 3.9, p. 301], the strategy $U_t^{*T} = -R^{-1}B^{\mathrm{T}}\Pi_t^T X_t^*$ is optimal in the problem $EJ_T(U) \to \inf_{U\in \mathrm{U}}$, the function $\Pi_t^T \ge 0$ satisfies the Riccati equation $\dot{\Pi}_t^T + \alpha_t\Pi_t^T A + \alpha_t A^{\mathrm{T}}\Pi_t^T - \alpha_t\Pi_t^T BR^{-1}B^{\mathrm{T}}\Pi_t^T + \alpha_t Q = 0$ with the terminal condition $\Pi_T^T = 0$, and $X_t^{*T}$ is the corresponding process given by (1.1) with $t \le T$. (The index $^T$ means the solution for a finite value $T$.) Under the assumptions formulated above, there exists $\lim_{T\to\infty}\Pi_t^T = \bar{\Pi}$, where the matrix $\bar{\Pi} \ge 0$ is the solution of the algebraic Riccati equation (2.4). Here, a key condition is

$$\int_0^t \alpha_s ds \to \infty$$

as $t \to \infty$ (Assumption A). This conclusion follows from an example of the scalar control problem presented in the next section.

## 3. EXAMPLE OF SCALAR CONTROL PROBLEM AND ANALYSIS OF OPTIMALITY CONDITIONS

Consider the control system (1.1) and (1.2) in the scalar case:

$$dX_t = \alpha_t a X_t dt + \alpha_t b U_t dt + \sqrt{\alpha_t} g dW_t, \quad X_0 = x, \quad g \ne 0, \tag{3.1}$$

$$J_T(U) = \int_0^T \alpha_t(qX_t^2 + rU_t^2)dt, \quad q \ge 0, \quad r > 0. \tag{3.2}$$

As is easily observed, Assumption B holds in each of the three cases below: (1) $a$ is any value, $b \ne 0$, and $q > 0$; (2) $a < 0$, $b = 0$, and $q > 0$; (3) $a < 0$, $b = 0$, and $q = 0$. Case (3) is trivial since it implies $\bar{\Pi} = 0$ in (2.4). Therefore, this case will not be considered below. The algebraic Riccati equation (2.4) takes the form $2a\bar{\Pi} - (b^2/r)\bar{\Pi}^2 + q = 0$. The solutions are $\bar{\Pi}^{(1)} = (a + \sqrt{a^2 + qb^2/r})(r/b^2)$ and $\bar{\Pi}^{(2)} = -q/(2a)$ for cases (1) and (2), respectively. The solutions of the differential Riccati equation

$$\dot{\Pi}_t^T + 2\alpha_t a\Pi_t^T - \alpha_t(b^2/r)(\Pi_t^T)^2 + q\alpha_t = 0$$

with the boundary condition $\Pi_T^T = 0$ can be obtained using the time change (Lemma 2) and the results for the constant-gain controller [33, p. 147]:

$$\Pi_t^{(1)T} = \bar{\Pi}^{(1)} - \frac{2\beta(r/b^2)}{[(\beta - a)/(\beta + a)]\exp\left\{2\beta\int_t^T \alpha_s ds\right\} + 1}, \tag{3.3}$$

$$\beta = \bar{\Pi}^{(1)}(b^2/r) - a > 0,$$

$$\Pi_t^{(2)T} = \bar{\Pi}^{(2)} - \bar{\Pi}^{(2)} \exp\left\{2a\int_t^T \alpha_s ds\right\}. \tag{3.4}$$

From the expressions (3.3) and (3.4) it follows that $\lim_{T\to\infty} \Pi_t^{(1)T} = \bar{\Pi}^{(1)}$ and $\lim_{T\to\infty} \Pi_t^{(2)T} = \bar{\Pi}^{(2)}$ only if

$$\int_0^t \alpha_s ds \to \infty, \quad t \to \infty.$$

Otherwise, when

$$\int_0^\infty \alpha_t dt < \infty,$$

the value $\lim_{T\to\infty} \Pi_t^T$ is not the solution of the algebraic Riccati equation, and the time invariance of the control $U^*$ does not hold: there is no transition to the infinite-horizon autonomous stochastic linear controller for the system with dynamic scaling.

## 4. ANALYSIS OF DYNAMIC MACROECONOMIC STABILIZATION PROBLEM

This section presents an example of a macroeconomic stabilization problem, which involves Theorem 2, Remarks 2 and 3, and the illustrative example from Section 3. The basic model for this example was described in [34], and the class of power-type functions $\alpha_t$ was determined in [23]. Note that the dynamic macroeconomic stabilization problem is understood as maintaining the system's path (the set of economic variables) near the given level [35, Part III] on the entire planning horizon, taking into account the associated control costs. Control is implemented by selecting some tools (also economic variables) and is often formulated as a linear controller design problem over an infinite time-horizon [36, 37]. In this example, unemployment rate management is considered. The emphasis is on the frictional and structural components of unemployment. (The former is associated with a voluntary change of the place of work due to relocation, etc., whereas the latter with the structural change of the economy in the areas of production and consumption.) As a management tool, government expenditures on the so-called active policy on the labor market are used (retraining, the infrastructure of employment centers, information support, increasing the mobility of the population, etc.), in contrast to the passive policy of changing the minimum wage and benefits on unemployment. Thus, we consider a scalar process of form (1.1) (also, see (3.1)). Assume that the state $X_t$ and control $U_t$ describe the deviation of the corresponding economic variables from their target levels:

$$dX_t = (1+t)^p(-\gamma)X_t dt - (1+t)^p \beta U_t dt + (1+t)^{p/2} g dW_t, \quad X_0 = x, \quad g \neq 0, \tag{4.1}$$

where a constant $\gamma > 0$ is the rate of convergence of the unemployment rate to the target natural level in the long term without any control and exogenous shocks; a constant $\beta > 0$ characterizes the multiplicative impact of government expenditures on unemployment dynamics; $g > 0$ is the degree of uncertainty; and, $\alpha_t = (t+1)^p$ with $p \geq -1$. A model of the form (4.1) with $U_t \equiv 0$ was previously considered in [34], and the scaling function $\alpha_t$ was due to the operational (internal) time of the system's evolution. The power-type function $\alpha_t$ was used in [23] for the econometric modeling of unemployment (in particular, the estimate $p = 10$ was obtained for the U.S. data). In the stabilization problem, the cost criterion has an integral quadratic form (3.2) and describes the losses due to the deviation of the state and control variables from their target values, taking into account the time preference and the priority of the costs. More precisely,

$$J_T(U) = \int_0^T (1+t)^p(\lambda X_t^2 + (1-\lambda)U_t^2)dt, \quad 0 < \lambda < 1. \tag{4.2}$$

If $p < 0$ in (4.2), then the significance of the timing of losses decreases as $t \to \infty$, and the so-called hyperbolic discounting occurs. (This term is widespread in economics and cognitive sciences [38].) In the case $p > 0$, on the contrary, a higher weight is assigned to future costs. As is well known from control theory, this feature enhances the stabilizing properties of the optimal strategy; see [39, Section 3.5] and [24]. Such criteria are called time-weighted integral quadratic cost criteria and are used in engineering applications [40, 41]. For $p = 0$, there is a neutral approach to the timing factor of losses, and the standard autonomous control system arises accordingly. The constant $0 < \lambda < 1$ specifies the priority of two types of costs (losses due to state deviations or control costs). According to Theorem 2 and the example from Section 3, we obtain the optimal control $U_t^* = \beta(1-\lambda)^{-1}\bar{\Pi}X_t^*$, where

$$\bar{\Pi} = -\gamma\beta^{-2}(1-\lambda) + \beta^{-2}\sqrt{\gamma^2(1-\lambda)^2 + \beta^2\lambda(1-\lambda)}.$$

The optimal path has the dynamics

$$dX_t^* = (1+t)^p\left(-\sqrt{\gamma^2 + \beta^2\lambda(1-\lambda)^{-1}}\right)X_t^*dt + (1+t)^{p/2}\,gdW_t, \quad X_0 = x. \tag{4.3}$$

We denote $\mu(\lambda) = \sqrt{\gamma^2 + \beta^2\lambda(1-\lambda)^{-1}}$. The gain $k(\lambda)$ of the controller $U_t^* = k(\lambda)X_t^*$, where $k(\lambda) = -\gamma\beta^{-1} + \beta^{-1}\sqrt{\gamma^2 + \beta^2\lambda(1-\lambda)^{-1}}$ is increasing in $\lambda$. Thus, the more significant the stabilization of the economic variable in the cost criterion the higher stability index (in terms of factor $\mu(\lambda)$ for (4.3)) that will be provided by the corresponding optimal control strategy. According to Remark 2, the deviation of the unemployment rate from its planned value will be maintained within fixed limits (in the mean-square metric). The long-term fluctuations of the unemployment process (see Remark 3) can be dynamically estimated by a logarithmic function of time if $p > -1$ and a double logarithmic function if $p = -1$.

## CONCLUSIONS

In this paper, the time invariance of the solution of the control problem for the system with dynamic scaling over an infinite time-horizon has been established; see Theorem 2. The form of control $U^*$ coincides with the optimal strategy derived for the autonomous system. Also, the corresponding optimality criteria under the control $U^*$ have the same values. The form of the criteria changes: the integrated scaling function $\int_0^T \alpha_t dt$ is used as the normalization of the cost criteria instead of the horizon length $T$. Note that such invariance in stochastic linear controller design problems with time-varying parameters may arise under other assumptions. For example, see the paper [42], where a system with asymptotically constant matrices $A_t \to A$, $B_t \to B$, $G_t \to G$, $Q_t \to Q$, and $R_t \to R$ was considered under the assumption of sufficiently fast convergence, or more precisely,

$$\int_0^\infty \|A_t - A\|dt < \infty.$$

With this condition, the solution of the algebraic rather than differential Riccati equation was used to construct the optimal strategy by the standard long-term average cost criterion. As an application, this paper has considered a dynamic macroeconomic stabilization problem with the power-type scaling function. A possible line of further research is studying the case of the scaling function

$$\int_0^\infty \alpha_t dt < \infty.$$

A simple scalar example (see Section 3) indicates that under such an assumption, the time invariance of the optimal control no longer holds, and other analysis methods based on the asymptotic representations of the solutions of the differential Riccati equations should be employed.

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