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ON OPTIMAL STOCHASTIC LINEAR QUADRATIC CONTROL WITH INVERSELY PROPORTIONAL TIME-WEIGHTING IN THE COST*

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(*Translated by A. R. Alimov*)

Abstract. We consider an optimal linear-quadratic control problem for a control system where the matrices corresponding to the state in the controlled process equation and the cost functional are absolutely integrable over an infinite time interval. The integral quadratic performance index includes two mutually inversely proportional time-weighting functions. It is shown that a well-known linear stable feedback law turns out to be optimal with respect to criteria from the class of the extended long-run averages. The results are applied to studying a control system under time-varying dynamic scaling of its parameters.

Key words. stochastic linear-quadratic regulator, pathwise optimality, inversely proportional time-weighting of costs, absolutely integrable state matrix

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1. Introduction. We consider the problem of a stochastic linear-quadratic regulator over an infinite time-horizon with mutually inverse accounting for the costs in the objective functional. Let us introduce the control system under consideration. Assume that, on a complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$, we are given an n-dimensional stochastic process X_t, $t \geqslant 0$, described by the equation

$$dX_t = A_t X_t\, dt + B_t U_t\, dt + G_t\, dw_t, \qquad X_0 = x, \tag{1}$$

where the initial state x is nonrandom; w_t, $t \geqslant 0$, is a d-dimensional standard Wiener process; U_t, $t \geqslant 0$, is an admissible control, or a k-dimensional stochastic process adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, $\mathcal{F}_t = \sigma\{w_s,\, s \leqslant t\}$, such that (1) has a solution (here, $\sigma(\,\cdot\,)$ is the sign of the σ-algebra); A_t, B_t are bounded matrices of corresponding dimensions, and the matrix A_t is absolutely integrable at infinity; G_t is the diffusion matrix where the assumptions on its entries are specified below (we also note that both constant and time-varying G_t are allowed, and the case $\|G_t\| \to \infty$ or $\|G_t\| \to 0$ as $t \to \infty$ is also possible). Here and in what follows, it is assumed that $\int_0^\infty \|G_t\|^2\, dt > 0$, where $\|\cdot\|$ is the Euclidean matrix norm. We denote by $\mathcal{U}$ the set of admissible controls.

Linear stochastic differential equations with a state matrix absolutely integrable at infinity (i.e., if there is an A_t such that $\int_0^\infty \|A_t\|\, dt < \infty$) have numerous applications, for example, in physics, in particular, for the description of anomalous diffusion [1], [2], [3], [4], in problems of statistic modeling [5], etc. Scalar cases with $A_t = \exp\{-rt\}$, $A_t \sim t^\alpha$ and constant $G_t = G$ were addressed in [1], [2], and a time-changed Brownian motion, i.e., with $A_t = 0$, $G_t \sim t^\gamma$, was studied in [3], [4]. Here and in what follows, the symbol $\sim$ is used to indicate the asymptotic equivalence (up to a constant)

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of two scalar nonnegative functions, i.e., $f_t \sim g_t$ if $0 < \lim_{t\to\infty}(f_t/g_t) < \infty$. Controlled external inputs and the equality $A_t = 0$ lead to known models of a controlled Brownian motion (see, for example, [6], [7]). The presence of a control naturally suggests the question of assessing the performance of different control laws. For linear systems, a common approach involves the so-called integral quadratic objective functional over the planning horizon $[0, T]$. This functional includes two types of costs. More precisely, here one considers the costs appearing due to the deviation of the process from the target value (for example, the zero value), and the costs associated with control. In addition, adjustment of losses related to different times is also possible. For example, one applies the discounting procedure (reduction) or, the other way around, the increment procedure by multiplying the corresponding quantities by functions of time. The dynamics of such factors shows the importance of various types of costs for the agent. Previous studies were concerned with situations of the same adjustment for the two types of costs (see [8]), or a time-varying multiplier was assigned to one kind of loss, while the remaining cost had a constant weighting (for the deterministic case, see [9] and also [10]). In the present paper, we study the case of inversely proportional adjustment functions under the priority of control costs. For each $T > 0$, as the objective functional we consider the random variable (r.v.)

$$J_T(U) = \int_0^T \left(\frac{1}{\beta_t} X_t' Q X_t + \beta_t U_t' R U_t\right) dt, \tag{2}$$

where $U \in \mathcal{U}$ is an admissible control on $[0, T]$; $Q \geqslant 0$ and $R > 0$ are symmetric matrices ($'$ denotes transpose; the notation $A \geqslant B$ ($A > B$) for matrices means, respectively, that the difference $A - B$ is nonnegative or positive definite); $\beta_t > 0$, $t \geqslant 0$, is the function setting the priority of various types of costs at time t. Moreover, the function β_t increases sufficiently fast. More precisely, the parameters of the control system (1), (2) satisfy the following assumption.

Assumption $\mathcal{AB}$. The matrix A_t in the state equation (1) is such that $\int_0^\infty \|A_t\|\,dt < \infty$. The function β_t in the objective functional (2) satisfies the conditions $\beta_t > 0$, $t \geqslant 0$, $\beta_t \to \infty$ as $t \to \infty$ and $\int_0^\infty (1/\beta_t)\,dt < \infty$.

An example of a deterministic scalar system with functions A_t, $1/\beta_t \sim 1/t^2$, was considered in [11, Example 3.11, p. 43]. We are interested in the existence of an optimal stochastic control (1) using (2) in the case $T \to \infty$. Since the objective functional (2) involves both unbounded in time ($\beta_t R$) and singular ($(1/\beta_t)Q$) matrices of costs, and because the diffusion matrix G_t is time-varying, the control system under consideration, for $T \to \infty$, does not belong to the standard type of stochastic linear regulators (see, for example, [12, Chap. 3]). Earlier (see, for example, [10]), for problems involving linear-quadratic regulators under nonstandard assumptions on the system parameters, it was proposed that one should use criteria for generalized adjusted long-run averages of the form

$$\limsup_{T\to\infty} \frac{\mathbf{E} J_T(U)}{\int_0^T p_t \|G_t\|^2\,dt} \to \inf_{U\in\mathcal{U}} \quad \text{and} \quad \limsup_{T\to\infty} \frac{J_T(U)}{\int_0^T p_t \|G_t\|^2\,dt} \to \inf_{U\in\mathcal{U}} \tag{3}$$

with probability 1, where p_t is the adjustment function that depends on the specifics of the control system and is obtained via estimation of the solution to the Riccati equation. In the case under study, we consider $p_t = \int_t^\infty (1/\beta_s)\,ds$. The existing criteria for long-run average and pathwise ergodics for systems with constant parameters, i.e., with the normalization T, are obtained from (3) with $G_t \equiv G \neq 0$, $p_t \equiv 1$.

The purpose of the present paper is to study optimal control problems of the form (3) for system (1), (2) under the above assumptions. The paper is organized as follows. In section 2, we provide the stable control law U^* in the form of a state linear feedback involving the solution to the Riccati equation. In section 3, we establish the optimality of a designed feedback law over an infinite time-horizon. Section 4 is devoted to application of our results to the analysis of the control system with dynamical scaling of its parameters. In section 5, we formulate the main conclusions of the paper and indicate directions for further research.

2. Design of the stable control law. As mentioned above, derivation of the form of a stable control law U^* and optimality criteria over an infinite time-horizon is related to the study of the Riccati equation. The corresponding result is as follows.

LEMMA 1. *Let Assumption $\mathcal{AB}$ hold. Then there exists a function Π_t, $t \geqslant 0$, with values in the set of nonnegative definite symmetric matrices, which satisfy the differential Riccati equation*

$$\dot{\Pi}_t + \Pi_t A_t + A_t' \Pi_t - \Pi_t B_t (\beta_t R)^{-1} B_t' \Pi_t + \frac{1}{\beta_t} Q = 0 \tag{4}$$

and are such that $\limsup_{t\to\infty}\{\|\Pi_t\|/p_t\} < \infty$, *where* $p_t = \int_t^\infty (1/\beta_t)\, dt$ (*the dot is used to denote the time derivative*). *If, in addition* $Q > 0$, *then* $\liminf_{t\to\infty}\{\|\Pi_t\|/p_t\} > 0$.

Proof. The existence and other properties of a solution of (4) are established by making $T \to \infty$ in the matrix functions Π_t^T. Moreover, the functions Π_t^T are related to deterministic optimal control problems under finite T. Consider an arbitrary initial time $t_0 \geqslant 0$, the dynamics

$$dx_t = A_t x_t\, dt + B_t u_t\, dt,$$

and the objective functional

$$J_{t_0,T}(u) = \int_{t_0}^{T} \left(\frac{1}{\beta_t} x_t' Q x_t + \beta_t u_t' R u_t \right) dt;$$

the initial state x_{t_0} is also given. It is well known (see, for example, [12, Theorem 3.4]) that the problem

$$J_{t_0,T}(u) \to \min$$

has a solution

$$u_t^{*T} = -(\beta_t R)^{-1} B_t' \Pi_t^T x_t^{*T},$$

where Π_t^T satisfies (4) with boundary condition $\Pi_T^T = 0$, and, moreover, $J_{t_0,T}(u^{*T}) = x_{t_0}' \Pi_{t_0}^T x_{t_0}$ (here the superscript ^{T} denotes solutions with finite T). In view of Assumption $\mathcal{AB}$, an application of a competing control law $u_t^{(0)} \equiv 0$ gives the estimate

$$J_{t_0,T}(u^0) \leqslant c \int_{t_0}^{\infty} \frac{1}{\beta_t}\, dt\, \|x_{t_0}\|^2.$$

Here and in what follows, c is some positive constant such that its exact value has no importance and can be different in different contexts. Hence,

$$x_{t_0}' \Pi_{t_0}^T x_{t_0} \leqslant c \int_{t_0}^{\infty} \frac{1}{\beta_t}\, dt\, \|x_{t_0}\|^2 \quad \text{and} \quad \|\Pi_t^T\| \leqslant c \int_t^\infty \frac{1}{\beta_s}\, ds, \qquad t \geqslant 0.$$

Next, standard arguments (see [12, section 3.4.2]) easily show that $\lim_{T\to\infty} \Pi_t^T = \Pi_t$, where the function Π_t has the same properties as Π_t^T and satisfies (4). In particular,

$$\limsup_{t\to\infty} \frac{\|\Pi_t\|}{p_t} < \infty$$

if $p_t = \int_t^\infty (1/\beta_t)\, dt$. Moreover,

$$x_{t_0}'\Pi_{t_0}x_{t_0} = \int_{t_0}^\infty \left(\frac{1}{\beta_t}(x_t^*)'Qx_t^* + \beta_t(u_t^*)'Ru_t^*\right) dt$$

with control $u_t^* = -(\beta_t R)^{-1}B_t'\Pi_t x_t^*$. Hence, by using the condition $Q > 0$, the estimate $\int_0^\infty \|A_t - B_t(\beta_t R)^{-1}B_t'\Pi_t\|\, dt < \infty$ (absolute integrability), and the Lyapunov estimate (see [13, section 4.6]) we obtain

$$\exp\left\{-\int_{t_0}^t \|\overline{A}_v\|\, dv\right\}\|\overline{X}_{t_0}\| \leqslant \|\overline{X}_t\| \leqslant \exp\left\{\int_{t_0}^t \|\overline{A}_v\|\, dv\right\}\|\overline{X}_{t_0}\|, \qquad t_0 \leqslant t, \tag{5}$$

for the solution $\overline{X}_t$ to the linear system $d\overline{X}_t = \overline{A}_t\overline{X}_t\, dt$ with $\overline{A}_t = A_t - B_t(\beta_t R)^{-1}B_t'\Pi_t$ and $\overline{X}_t = x_t^*$, we get the estimate

$$x_{t_0}'\Pi_{t_0}x_{t_0} \geqslant \bar{c}\int_{t_0}^\infty \frac{1}{\beta_t}\, dt\, \|x_{t_0}\|^2,$$

which holds for some constant $\bar{c} > 0$. Hence, $\liminf_{t\to\infty}\{\|\Pi_t\|/p_t\} > 0$. This proves Lemma 1.

The next example illustrates Lemma 1.

Example 1. Consider the Riccati equation corresponding to the scalar control system from [11, Example 3.11, p. 43]:

$$\dot{\Pi}_t + \frac{2\Pi_t}{(t+1)^2} - \frac{\Pi_t^2}{(t+1)^2} + \frac{1}{(t+1)^2} = 0. \tag{6}$$

The solution of (6) has the form (see [11, Example 3.11, p. 43])

$$\Pi_t = \frac{(1+\sqrt{2})(1-\exp\{-2\sqrt{2}/(t+1)\})}{1+(1+\sqrt{2})^2\exp\{-2\sqrt{2}/(t+1)\}}, \qquad t \geqslant 0, \tag{7}$$

and can be obtained as the limit (see the proof of Lemma 1) as $T \to \infty$ of the functions Π_t^T, where Π_t^T is a solution of (6) with the boundary condition $\Pi_T^T = 0$. Indeed,

$$\Pi_t^T = \frac{(1+\sqrt{2})(1-\exp\{2\sqrt{2}(t-T)/[(t+1)(T+1)]\})}{1+(1+\sqrt{2})^2\exp\{2\sqrt{2}(t-T)/[(t+1)(T+1)]\}}, \qquad 0 \leqslant t \leqslant T$$

(see [11, Example 3.11, p. 43]), and so $\lim_{T\to\infty}\Pi_t^T = \Pi_t$. From (7) one can also determine the bounds for Π_t, $t \geqslant 0$, by using $p_t = 1/(t+1)$. As a result, we get

$$\frac{1+\sqrt{2}}{1+(1+\sqrt{2})^2}\,\frac{c_1}{t+1} \leqslant \Pi_t \leqslant \frac{1+\sqrt{2}}{1+(1+\sqrt{2})^2\exp\{-2\sqrt{2}\}}\,\frac{c_2}{t+1},$$

where $c_1, c_2 > 0$ are arbitrary constants satisfying

$$c_1 \leqslant 1 - \exp\{-2\sqrt{2}\}, \quad c_2 \geqslant 2\sqrt{2}.$$

Further, we define the so-called stable optimal control law U^*. Here, the phrase "stable optimal" means that U^* is obtained by making $T \to \infty$ in the optimal controls U^{*T} that are obtained by solving the problems $\mathbf{E} J_T(U) \to \inf_{U \in \mathcal{U}}$ with finite T (see [12, section 3.6.3]). As a result,

$$U_t^* = -(\beta_t R)^{-1} B_t' \Pi_t X_t^*, \tag{8}$$

where the matrix Π_t satisfies the Riccati equation (4), and the process X_t^*, $t \geqslant 0$, is given by the equation

$$dX_t^* = (A_t - B_t(\beta_t R)^{-1} B_t' \Pi_t) X_t^* \, dt + G_t \, dw_t, \qquad X_0^* = x. \tag{9}$$

The next lemma summarizes the results of analysis of the process X_t^*, $t \geqslant 0$, corresponding to the stable control law U^*.

LEMMA 2. *Let Assumption $\mathcal{AB}$ hold. Then, for the process X_t^*, $t \geqslant 0$, defined by* (9),

$$0 < \liminf_{t\to\infty} \frac{\mathbf{E}\|X_t^*\|^2}{\int_0^t \|G_s\|^2 \, ds} \leqslant \limsup_{t\to\infty} \frac{\mathbf{E}\|X_t^*\|^2}{\int_0^t \|G_s\|^2 \, ds} < \infty. \tag{10}$$

If, in addition,

$$\int_0^\infty \left(\|A_t\| + \frac{1}{\beta_t} \int_t^\infty \frac{1}{\beta_s} \, ds \right) \sqrt{\int_0^t \|G_s\|^2 \, ds} \, dt < \infty, \tag{11}$$

then

$$X_t^* - \widehat{W}_t \to \zeta_\infty, \qquad t \to \infty, \tag{12}$$

where ζ_∞ is a Gaussian r.v. and $\widehat{W}_t = \int_0^t G_s \, dw_s$. Moreover, convergence (12) *holds in the mean square and a.s.*

Proof. It is easily checked that the process X_t^*, $t \geqslant 0$, is the solution of a stochastic linear differential equation, and hence $\mathbf{E}\|X_t^*\|^2 = \operatorname{tr}(C_t)$, where $\operatorname{tr}(\,\cdot\,)$ is the trace of the matrix, and the covariance matrix C_t is as follows (see, for example, [14, section 4.2]):

$$C_t = \Phi(t,0) x x' \Phi'(t,0) + \int_0^t \Phi(t,s) G_t G_t' \Phi'(t,s) \, ds.$$

Moreover, $\Phi(t,s)$ is the fundamental matrix corresponding to the matrix $A_t^* = A_t - B_t(\beta_t R)^{-1} B_t' \Pi_t$, and from Assumption $\mathcal{AB}$ we have $\|\Phi(t,s)\| \leqslant c$ for all $s \leqslant t$. More precisely, by virtue of the Lyapunov estimate (5) with $\overline{A}_t = A_t^*$ and with $\overline{A}_t = -(A_t^*)'$, there exist constants $\bar{c}_1, \bar{c}_2 > 0$ such that, for every number $0 \leqslant \mu \leqslant 1$,

$$\begin{aligned} &\bar{c}_1 \exp\left\{ -\int_s^t \left(\|A_v\| + \frac{1}{\beta_v} \int_v^\infty \frac{1}{\beta_\tau} \, d\tau \right) dv \right\} \\ &\qquad \leqslant \mu \|\Phi(t,s)\| + (1-\mu) \|\Phi(s,t)\| \\ &\qquad\qquad \leqslant \bar{c}_2 \exp\left\{ \int_s^t \left(\|A_v\| + \frac{1}{\beta_v} \int_v^\infty \frac{1}{\beta_\tau} \, d\tau \right) dv \right\}, \qquad s \leqslant t. \end{aligned} \tag{13}$$

For further analysis, we require some relations known for positive definite matrices. For a square matrix $\mathcal{M} > 0$ of size $n \times n$, the estimate $y'\mathcal{M}y \leqslant \widehat{c}_n \|\mathcal{M}\| \|y\|^2$ gives $\mathcal{M} \leqslant \widehat{c}_n \|\mathcal{M}\| I$ (here I is the identity matrix, y is an arbitrary vector of appropriate dimension, and $\widehat{c}_n > 0$ is some constant), and $\mathcal{M}^{-1} \geqslant (\widehat{c}_n \|\mathcal{M}\|)^{-1} I$. Setting successively $\mathcal{M} = \Phi(s,t)\Phi'(s,t)$, $\mathcal{M} = \Phi(t,s)\Phi'(t,s)$ and using (13) with $\mu = 0, 1$, we see that, for any number $0 \leqslant \widetilde{\mu} \leqslant 1$, there exist constants $\widetilde{c}_1, \widetilde{c}_2 > 0$ such that

$$
\begin{aligned}
&\widetilde{c}_1 \exp\left\{-2\int_s^t \left(\|A_v\| + \frac{1}{\beta_v}\int_v^\infty \frac{1}{\beta_\tau}\,d\tau\right) dv\right\} I \\
&\qquad \leqslant \widetilde{\mu}\Phi'(t,s)\Phi(t,s) + (1-\widetilde{\mu})\Phi'(s,t)\Phi(s,t) \\
&\qquad\qquad \leqslant \widetilde{c}_2 \exp\left\{2\int_s^t \left(\|A_v\| + \frac{1}{\beta_v}\int_v^\infty \frac{1}{\beta_\tau}\,d\tau\right) dv\right\} I, \qquad s \leqslant t.
\end{aligned} \tag{14}
$$

Next,

$$
\mathbf{E}\|X_t^*\|^2 = \operatorname{tr}(C_t) = \sum_{j=1}^d \int_0^t (G_s^{(j)})'\Phi'(t,s)\Phi(t,s)G_s^{(j)}\,ds + x'\Phi'(t,0)\Phi(t,0)x,
$$

where $G_s^{(j)}$ is the jth column of the matrix G_s. Now an appeal to (14) with $\widetilde{\mu} = 1$ gives the estimate

$$
c_1\left(\|x\|^2 + \int_0^t \|G_s\|^2\,ds\right) \leqslant \mathbf{E}\|X_t^*\|^2 \leqslant c_2\left(\|x\|^2 + \int_0^t \|G_s\|^2\,ds\right),
$$

with some constants $c_1, c_2 > 0$, from which (10) follows. Using (9), we have the following representation for the difference between the processes:

$$
X_t^* - \int_0^t G_s\,dw_s = x + \int_0^t (A_s - B_s(\beta_s R)^{-1}B_s'\Pi_s)X_s^*\,ds.
$$

Next, consider the Gaussian processes

$$
\widehat{W}_t = \int_0^t G_s\,dw_s \quad \text{and} \quad \nu_t = \int_0^t (A_s - B_s(\beta_s R)^{-1}B_s'\Pi_s)X_s^*\,ds.
$$

For a process $\nu_t = \int_0^t \xi_s\,ds$, there is a sufficient condition for the convergence $\nu_t \to \nu_\infty$, $t \to \infty$, in the mean square and a.s. This condition reads as $\int_0^\infty \sqrt{\mathbf{E}\|\xi_t\|^2}\,dt < \infty$ (see [15, section 5.4]). In our setting,

$$
\xi_t = (A_t - B_t(\beta_t R)^{-1}B_t'\Pi_t)X_t^*,
$$

and so this requirement is satisfied by assumption (11) and in view of the above estimates for the functions $\mathbf{E}\|X_t^*\|^2$ and $\|\Pi_t\|$. Indeed,

$$
\mathbf{E}\|\xi_t\|^2 \leqslant c\left(\|A_t\| + \frac{1}{\beta_t}\int_t^\infty \frac{1}{\beta_s}\,ds\right)^2 \left(\|x\|^2 + \int_0^t \|G_s\|^2\,ds\right).
$$

This completes the proof of Lemma 2.

Next, we note that the solution of (9) has the form

$$
X_t^* = \Phi(t,0)x + \int_0^t \Phi(t,s)G_s\,dw_s, \tag{15}
$$

where $\Phi(t,s)$ is the fundamental matrix corresponding to the matrix $A_t^* = A_t - B_t(\beta_t R)^{-1}B_t'\Pi_t$. Recall that $\Phi(t,s)$ is defined as the solution of the problem

$$\frac{\partial \Phi(t,s)}{\partial t} = A_t^*\Phi(t,s), \qquad \frac{\partial \Phi(t,s)}{\partial s} = -\Phi(t,s)A_s^*, \qquad \Phi(t,t) = I,$$

where I is the identity matrix. Hence, by using the time-change in stochastic integrals (see, for example, [16, section 8.5, Corollary 8.5.4]), we get the following representation for the ith component of the process X_t^* (we denote this component by $X_t^{(i)*}$, $i = 1,\dots,n$):

$$X_t^{(i)*} = \sum_{j=1}^{n} \Phi_{ij}(t,0)x^{(i)} + \sum_{j=1}^{n} \Phi_{ij}(t,0)\widehat{w}^{(j)}_{\tau_{jj}(t)}; \tag{16}$$

here $x^{(i)}$ is the ith component of the vector x; $\Phi_{ij}(t,0)$ is an element of the matrix $\Phi(t,0)$, $i,j = 1,\dots,n$; and $\widehat{w}^{(j)}$ is the new jth Wiener process with time-change $\tau_{jj}(t)$. Note that $\tau_{jj}(t)$ is the jth diagonal element of the matrix

$$\mathcal{T}(t) = \int_0^t \Phi(0,s)G_sG_s'\Phi'(0,s)\,ds.$$

From Assumption $\mathcal{AB}$ and inequalities (14) for $\widetilde{\mu} = 0$, it follows that there exist positive constants $\widetilde{c}_1$ and $\widetilde{c}_2$ such that

$$\widetilde{c}_1 \exp\biggl\{-2\int_0^t \|A_v^*\|\,dv\biggr\}I \leqslant \Phi'(0,t)\Phi(0,t) \leqslant \widetilde{c}_2 \exp\biggl\{2\int_0^t \|A_v^*\|\,dv\biggr\}I.$$

As a result, we have the estimate

$$\widehat{\kappa}_1 \int_0^t \|G_s\|^2\,ds \leqslant \operatorname{tr}(\mathcal{T}(t)) \leqslant \widehat{\kappa}_2 \int_0^t \|G_s\|^2\,ds,$$

where $\widehat{\kappa}_1$, $\widehat{\kappa}_2 > 0$ are some constants. Thus, each component of X_t^* can be written as a linear combination (with bounded coefficients) of Wiener processes with time-changes $\tau_{jj}(t)$ and components of the vector of initial conditions. The boundaries $\tau_{jj}(t)$ are defined via the function $\int_0^t \|G_s\|^2\,ds$, which is a sum of all time-changes performed in the common disturbance process $\widehat{W}_t = \int_0^t G_s\,dw_s$. Regarding the properties of solution representations, here one can observe the difference from the case of linear stochastic differential equations with stable matrices A_t^*, i.e., where $\Phi(t,s)$ can be estimated as $\|\Phi(t,s)\| \leqslant \kappa_0 \exp\bigl\{-\int_s^t \delta_v\,dv\bigr\}$ and $\int_0^t \delta_v\,dv \to \infty$, $t \to \infty$, κ_0 is a constant, and $\delta_t > 0$ is the stability rate. In particular, for the standard Ornstein–Uhlenbeck process with $A_t^* = -\lambda I$, $G_t = G$, $\lambda > 0$, the coefficients in (16) are known to exponentially decay:

$$\Phi_{ii}(t,0) = \exp\{-\lambda t\}$$

(here, $\Phi_{ij}(t,0) = 0$, $i \neq j$); simultaneously, the time-change grows exponentially:

$$\tau_{jj}(t) = \exp\{2\lambda t\}\sum_{i=1}^{d} G_{ji}^2.$$

3. Optimal control over an infinite time-horizon. In this section, we study optimality of the stable control law U^* over an infinite time-horizon. In cases where the optimality criterion is based on the expected value of the objective functional (see the first criterion in (3)), the average optimality is considered.

3.1. Average optimality.

THEOREM 1. *Let Assumption* $\mathcal{AB}$ *be met. Then the control law* U^*, *as defined by* (8), (9), *is the solution of the problem*

$$\limsup_{T\to\infty} \frac{\mathbf{E}J_T(U)}{\int_0^T p_t\|G_t\|^2\,dt} \to \inf_{U\in\mathcal{U}} \tag{17}$$

with $p_t = \int_t^\infty (1/\beta_s)\,ds$ *and without restrictions on the diffusion matrix* G_t. *Moreover,*

$$\limsup_{T\to\infty} \frac{\mathbf{E}J_T(U^*)}{\int_0^T p_t\|G_t\|^2\,dt} < \infty. \tag{18}$$

Proof. By Assumption $\mathcal{AB}$, Lemma 1 applies and there exists a stable optimal control law U^* defined by (8) and (9). We fix an arbitrary admissible control $U \in \mathcal{U}$ and define the corresponding process X_t via (1). Setting $x_t = X_t - X_t^*$, $u_t = U_t - U_t^*$, we have the representation

$$J_T(U^*) - J_T(U) = 2x_T'\Pi_T X_T^* - \int_0^T \left(\frac{1}{\beta_t}x_t'Qx_t + \beta_t u_t'Ru_t\right)dt - 2\int_0^T x_t'\Pi_t G_t\,dw_t.$$

Using the elementary inequality $2ab \leqslant a^2/\widetilde{c} + \widetilde{c}b^2$, which holds for arbitrary numbers a and b and a constant $\widetilde{c} \neq 0$, and employing the estimate $\|\Pi_t\| \leqslant cp_t$ for Π_t (see Lemma 1), we get, for some $c_1, c_2 > 0$, the relation

$$J_T(U^*) - J_T(U) \leqslant c_1 p_T^2\|X_T^*\|^2 - c_2\int_0^T \frac{1}{\beta_t}\|x_t\|^2\,dt - 2\int_0^T x_t'\Pi_t G_t\,dw_t; \tag{19}$$

here, we also use the inequality

$$\|x_T\|^2 + \int_0^T \frac{1}{\beta_t}\|x_t\|^2\,dt \leqslant c_0\int_0^T\left(\frac{1}{\beta_t}x_t'Qx_t + \beta_t u_t'Ru_t\right)dt,$$

which holds for some constant c_0 and can be derived from the dynamics $dx_t = (A_tx_t + B_tu_t)\,dt$, $x_0 = 0$. Indeed, $x_t = \int_0^t \overline{\Phi}(t,s)B_su_s\,ds$, where the fundamental matrix $\overline{\Phi}(t,s)$ corresponds to A_t, and $\overline{\Phi}(t,s) \leqslant c$, $s \leqslant t$. As a result,

$$\|x_t\| \leqslant c\int_0^t \frac{1}{\sqrt{\beta_s}}\sqrt{\beta_s}\,\|\sqrt{R}\,u_s\|\,ds,$$

and hence, by applying the Cauchy–Bunyakovskiĭ–Schwarz inequality and using the condition $\int_0^\infty (1/\beta_t)\,dt < \infty$, we get $\|x_t\|^2 \leqslant c\int_0^t \beta_s u_s'Ru_s\,ds$. The required estimate follows by integrating the resulting inequality multiplied by $1/\beta_t$. We also note that $p_t \to 0$, $t\to\infty$. Since p_t is nonincreasing, an appeal to (10) produces the estimate

$p_T\mathbf{E}\|X_T^*\|^2 \leqslant c\int_0^T p_t\|G_t\|^2\,dt$. Using this estimate, taking the expectation in (19), applying the normalization $\int_0^T p_t\|G_t\|^2\,dt$, and making $T\to\infty$, we find that

$$\limsup_{T\to\infty}\frac{\mathbf{E}J_T(U^*)}{\int_0^T p_t\|G_t\|^2\,dt}\leqslant\limsup_{T\to\infty}\frac{\mathbf{E}J_T(U)}{\int_0^T p_t\|G_t\|^2\,dt}.$$

Thus, U^* is a solution to problem (17). Employing the Itô formula, we obtain

$$\begin{aligned}J_T(U^*) &= x'\Pi_0x-(X_T^*)'\Pi_TX_T^*+\int_0^T\operatorname{tr}(G_t'\Pi_tG_t)\,dt\\ &\quad+2\int_0^T(X_t^*)'\Pi_tG_t\,dw_t.\end{aligned}\tag{20}$$

Now (18) follows easily from representation (20) and the above estimate for the dynamics of $\mathbf{E}\|X_T^*\|^2$. Theorem 1 is proved.

Remark 1. For the deterministic ($G_t\equiv 0$) control system (1), (2), the control law U^* (8), (9) is the solution of the problem $\limsup_{T\to\infty}J_T(U)\to\inf_{U\in\mathcal{U}}$ under Assumption $\mathcal{AB}$. Moreover, $\limsup_{T\to\infty}J_T(U^*)=x'\Pi_0x$.

It is worth pointing out that when dealing with control systems involving a process of the form (1) and an increasing planning horizon, it is usually assumed that various conditions on $\|G_t\|$ are met (see, for example, [17]). Hence, the average optimality property of the stable control law (see Theorem 1) over an infinite time-horizon under an arbitrary diffusion matrix G_t occurs quite rarely for stochastic linear regulators (for an earlier example, see [18]). In [18], the corresponding result was implied by the assumption on superexponential stability of the state matrix in the process dynamics and bounded matrices in the cost functional. However, in our setting, such specifics in optimality are related to absolute integrability at infinity of the matrices A_t in (1) and $(1/\beta_t)Q$ in (2).

3.2. Pathwise optimality and examples. Probabilistically, a stronger optimality property is the so-called pathwise optimality (stochastic optimality, optimality a.s. — these terms were introduced and used, in particular, in [19] and [20]). In this case, the corresponding control law optimizes, with probability 1, the criterion that properly includes the objective functional itself, rather than its expected value (see the criterion in the right-hand side of (3)).

In the study of the pathwise optimality of the control U^*, we require an additional assumption about the relationships between the diffusion coefficients and the cost multiplier from the objective functional.

Assumption $\mathcal{G}$. The following relations hold for $p_t=\int_t^\infty(1/\beta_s)\,ds$ and the diffusion matrix G_t:

(1) $\int_0^T p_t\|G_t\|^2\,dt\to\infty$, $T\to\infty$;

(2) $\limsup_{t\to\infty}\{\beta_t\|G_t\|^2p_t^2\}<\infty$.

THEOREM 2. *Let Assumptions* $\mathcal{AB}$ *and* $\mathcal{G}$ *be met. Then the control law* U^*, *as defined by* (8), (9), *is the solution of the problem*

$$\limsup_{T\to\infty}\frac{J_T(U)}{\int_0^T p_t\|G_t\|^2\,dt}\to\inf_{U\in\mathcal{U}}\quad\textit{with probability } 1,\tag{21}$$

where $p_t = \int_t^{\infty} (1/\beta_s)\, ds$. *Moreover,*

$$(22) \qquad \limsup_{T\to\infty} \frac{J_T(U^*)}{\int_0^T p_t \|G_t\|^2\, dt} = \limsup_{T\to\infty} \frac{\mathbf{E} J_T(U^*)}{\int_0^T p_t \|G_t\|^2\, dt} = \limsup_{T\to\infty} \frac{\int_0^T \mathrm{tr}(G_t' \Pi_t G_t)\, dt}{\int_0^T p_t \|G_t\|^2\, dt},$$

where the matrix Π_t *satisfies the Riccati equation* (4) *and has the properties from Lemma* 1 ($\mathrm{tr}(\,\cdot\,)$ *is the trace of the matrix*).

Proof. By setting

$$L_T = c_1 p_T^2 \|X_T^*\|^2, \qquad \mathcal{R}_T = -c_2 \int_0^T \frac{1}{\beta_t} \|x_t\|^2\, dt - 2\int_0^T x_t' \Pi_t G_t\, dw_t,$$

we transform (19) into the inequality

$$(23) \qquad J_T(U^*) \leqslant J_T(U) + L_T + \mathcal{R}_T.$$

By condition (2) of Assumption $\mathcal{G}$, the process $\mathcal{R}_T$ as $T \to \infty$ can be written as

$$\mathcal{R}_T = M_T - \widetilde{c}_2 \langle M_T \rangle,$$

where $M_T = -2\int_0^T x_t' \Pi_t G_t\, dw_t$ is a martingale with quadratic variation $\langle M_T\rangle = \int_0^T \|x_t' \Pi_t G_t\|^2\, dt$, and $\widetilde{c}_2 > 0$ is some constant. Hence, according to one well-known result (see, for example, [20, Lemma A.1]),

$$\limsup_{T\to\infty} (g_T \mathcal{R}_T) \leqslant 0$$

for any deterministic monotone function $g_T > 0$ satisfying $g_T \to \infty$, $T \to \infty$. By condition (1) of Assumption $\mathcal{G}$, one can take $g_T = \big(\int_0^T p_t \|G_t\|^2\, dt\big)^{-1}$. Next, let us analyze the asymptotic behavior of the process L_T as $T \to \infty$. We have

$$X_T^* = \Phi(T,0) \int_0^T \Phi(0,t) G_t\, dw_t,$$

where $\Phi(t,s)$ is the fundamental matrix corresponding to the matrix $A_t^* = A_t - B_t(\beta_t R)^{-1} B_t' \Pi_t$, and since

$$\bar{c}_1 \leqslant \|\Phi(t,s)\| \leqslant \bar{c}_2 \quad \text{for} \quad s \leqslant t,$$

where $\bar{c}_1, \bar{c}_2 > 0$ are some constants (see (13)), by applying the law of the iterated logarithm for a Wiener process, we have the estimate

$$L_T \leqslant c p_T \int_0^T p_t \|G_t\|^2\, dt \, \ln\ln\bigg(\int_0^T p_t \|G_t\|^2\, dt\bigg) \quad \text{a.s. for} \quad T > t_0(\omega).$$

Hence, in view of condition (2) of Assumption $\mathcal{G}$, the normalized process satisfies the inequality

$$L_T \bigg(\int_0^T p_t \|G_t\|^2\, dt\bigg)^{-1} \leqslant c p_T \ln\ln\ln \frac{1}{p_T} \to 0, \qquad T \to \infty.$$

Taking into account the above and using normalization (23), we have, in the limit as $T \to \infty$,

$$\limsup_{T\to\infty} \frac{J_T(U^*)}{\int_0^T p_t\|G_t\|^2\,dt} \leqslant \limsup_{T\to\infty} \frac{J_T(U)}{\int_0^T p_t\|G_t\|^2\,dt} \quad \text{with probability 1.}$$

This relation shows that U^* is a solution of problem (21). Next, from condition (2) of Assumption $\mathcal{G}$, it follows that $p_T\mathbf{E}\|X_T^*\|^2 \leqslant c$, $T \to \infty$. Now an appeal to (20) shows that the optimal value of the criterion in (17) is

$$\limsup_{T\to\infty} \frac{\mathbf{E}J_T(U^*)}{\int_0^T p_t\|G_t\|^2\,dt} = \limsup_{T\to\infty} \frac{\int_0^T \operatorname{tr}(G_t'\Pi_t G_t)\,dt}{\int_0^T p_t\|G_t\|^2\,dt}.$$

Using (20) and setting

$$Z_T = \int_0^T (X_t^*)'\Pi_t G_t\,dw_t, \qquad \Gamma_T = \int_0^T p_t\|G_t\|^2\,dt,$$

we get the estimate

$$\mathbf{E}Z_T^2 \leqslant c\int_0^T p_t^2\mathbf{E}\|X_t^*\|^2\|G_t\|^2\,dt \leqslant \overline{c}\Gamma_T$$

with some constant $\overline{c} > 0$. Now using Lemma 1 from [21], we have the convergence

$$\frac{Z_T}{\Gamma_T} \to 0 \quad \text{a.s. as} \quad T \to \infty.$$

The term $\widetilde{L}_T = (X_T^*)'\Pi_T X_T^*$ is estimated as follows:

$$\|\widetilde{L}_T\| \leqslant c\|\widetilde{X}_T\|^2,$$

where the process $\widetilde{X}_t = \sqrt{p_T}X_T^*$ satisfies the equation

$$d\widetilde{X}_t = \widetilde{A}_t\widetilde{X}_t\,dt + \widetilde{G}_t\,dw_t$$

with the matrices $\widetilde{A}_t = A_t - B_t(\beta_t R)^{-1}B_t'\Pi_t + (1/2)(\dot{p}_t/p_t)$, $\widetilde{G}_t = \sqrt{p_t}\,G_t$. Setting $\delta_t = -(1/2)(\dot{p}_t/p_t)$, we note that

$$\int_0^t \delta_v\,dv \sim -\int_0^t \frac{\dot{p}_v}{p_v}\,dv = \ln\frac{p_0}{p_t}.$$

So, the matrix $\widetilde{A}_t$ is stable with rate $\delta_t = -(1/2)(\dot{p}_t/p_t)$, because the corresponding fundamental matrix can be estimated as $\|\widetilde{\Phi}(t,s)\| \leqslant \kappa_0\exp\bigl\{-\int_s^t \delta_v\,dv\bigr\}$, $s \leqslant t$ ($\kappa_0 > 0$ is some constant) and $\int_0^t \delta_v\,dv \to \infty$, $t \to \infty$. Moreover, for the diffusion matrix, we have $\|\widetilde{G}_t\|^2 = p_t\|G_t\|^2 \leqslant c(-\dot{p}_t/p_t)$ by condition (2) of Assumption $\mathcal{G}$, and so $\mathbf{E}\|\widetilde{X}_T\|^2 \leqslant \widetilde{c}$ ($\widetilde{c} > 0$ is some constant). For processes with these properties (the stability of the matrix in the dynamic equation and boundedness of the second moment), we have

$$\frac{\|\widetilde{X}_T\|^2}{\int_0^T \|\widetilde{G}_t\|^2\,dt \to 0} \quad \text{a.s. as} \quad T \to \infty.$$

The corresponding result for the constant δ_t was proved in [21]; for $\delta_t \to 0$, $t \to \infty$, the required result was established in [10]; the case $\delta_t \to \infty$ is dealt with as in [10], [21]. The above convergence implies that

$$\frac{\|\widetilde{L}_T\|}{\int_0^T p_t \|G_t\|^2 \, dt \to 0} \quad \text{a.s. as } T \to \infty.$$

In view of the above, for the terms in (20), we have

$$\limsup_{T\to\infty} \frac{J_T(U^*)}{\int_0^T p_t \|G_t\|^2 \, dt} = \limsup_{T\to\infty} \frac{\int_0^T \operatorname{tr}(G_t' \Pi_t G_t) \, dt}{\int_0^T p_t \|G_t\|^2 \, dt} = \limsup_{T\to\infty} \frac{\mathbf{E} J_T(U^*)}{\int_0^T p_t \|G_t\|^2 \, dt}.$$

Theorem 2 is proved.

Note that from condition (2) of Assumption $\mathcal{G}$ and assertion (22) of Theorem 2 one can make a conclusion about the largest possible growth of $\mathbf{E} J_T(U^*)$ and $J_T(U^*)$ as $T \to \infty$. The upper boundary for both functionals can be described in terms of $\widehat{c} \ln(1/p_T)$, where $p_t = \int_t^\infty (1/\beta_s) \, ds$ and $\widehat{c} > 0$ is some constant. Next, with the aim of finding the admissible order of growth of the diffusion matrix G_t we consider, under Assumption $\mathcal{G}$, examples of various functions β_t that govern the dynamics of the multipliers in (2). In the examples that follow, $\bar{c}$ denotes some constant.

Example 2. The power-law family of factors $\beta_t \sim t^\alpha$, $\alpha > 1$. In this case, criteria (17) and (21) are adjusted with the help of the function $p_t \sim t^{1-\alpha}$. Hence $\|G_t\|^2 \leqslant \bar{c} t^{\alpha-2}$. For $\alpha > 2$, increasing perturbations $\|G_t\| \to \infty$, $t \to \infty$, can be included. For $\alpha = 2$, condition (2) of Assumption $\mathcal{G}$ is met only for bounded diffusion matrices G_t. If $1 < \alpha < 2$, then it proves possible to consider only decaying perturbations $\|G_t\| \to 0$, $t \to \infty$. The objective functionals are of logarithmic order, i.e., $\mathbf{E} J_T(U^*)$, $J_T(U^*) \sim \ln T$.

Example 3. The family of exponentials of power-law functions for the multipliers $\beta_t \sim \exp\{\gamma t^q\}$, $\gamma, q > 0$. In this case,

$$p_t \sim t^{1-q} \exp\{-\gamma t^q\} \quad \text{and} \quad \|G_t\|^2 \leqslant \bar{c} t^{2q-2} \exp\{\gamma t^q\}.$$

Moreover,

(a) if the factor is subexponential, i.e., $q \in (0,1)$, then the order of maximum admissible perturbations is t^{2-2q} times smaller than that of the factor;

(b) if the factor is exponential, i.e., $q = 1$, then the order of maximum admissible perturbations is the same as that of the factor;

(c) if the factor is superexponential, i.e., $q > 1$, then the order of maximum admissible perturbations is t^{2q-2} times higher.

The dynamics of the objective functionals as $T \to \infty$ turns out to be T^q. It is clear that a necessary requirement for condition (1) in Assumption $\mathcal{G}$ to hold is the presence of increasing perturbations. More precisely, $\|G_t\|^2 \sim \phi_t \exp\{\gamma t^q\}$ with the function $\phi_t > 0$ such that $\int_0^T t^{1-q} \phi_t \, dt \to \infty$, $T \to \infty$; moreover, it is also required that the inequality $\|\phi_t\| \leqslant t^{-(2-2q)}$ is satisfied.

Example 4. The family of doubly exponential functions for the multipliers $\beta_t \sim \exp\{\gamma \exp(rt)\}$, $\gamma, r > 0$. In this case,

$$p_t \sim \exp(-rt) \exp\{-\gamma \exp(rt)\}$$

and

$$\|G_t\|^2 \leqslant \bar{c}\exp(2rt)\exp\{\gamma\exp(rt)\}.$$

The maximal disturbances are allowed to exceed the multiplier (β) by no more than $\exp\{2rt\}$ times. The objective functionals on the optimal control grow exponentially (like $\exp\{rT\}$) at infinity. Condition (1) of Assumption $\mathcal{G}$ is met if

$$\|G_t\|^2 \sim \phi_t \exp\big\{\gamma\exp\{rt\}\big\},$$

where

$$\int_0^T \exp\{-rt\}\phi_t\,dt \to \infty, \qquad T\to\infty,$$

and simultaneously, $\|\phi_t\| \leqslant \exp\{2rt\}$.

Remark 2. The structure of the objective functional (2) involving an asymptotically singular matrix $(1/\beta_t)Q$ and an unbounded matrix $\beta_t R$, $t\to\infty$, leads to the task of comparing the performance of the optimal law U_t^* thus obtained (see (8), (9)) and the law $U_t^{(0)} \equiv 0$, meaning that there is no control. In order to find $J_T(U^{(0)})$, we use below a representation from [22], according to which

$$J_T(U^{(0)}) = x'P_0x - (X_T^{(0)})'P_TX_T^{(0)} + \int_0^T \operatorname{tr}(G_t'P_tG_t)\,dt + 2\int_0^T (X_t^{(0)})'P_tG_t\,dw_t,$$

where $X_t^{(0)} = \overline{\Phi}(t,0)x + \int_0^t \overline{\Phi}(t,s)G_s dw_s$, $\overline{\Phi}(t,s)$ is the fundamental matrix corresponding to the matrix A_t, and the matrix

$$P_t = \int_t^\infty \overline{\Phi}'(s,t)\beta_s^{-1}Q\overline{\Phi}(s,t)\,ds$$

exists by Assumption $\mathcal{AB}$. We also note that the solution of the Riccati equation (4) can be written as

$$\Pi_t = P_t - \int_t^\infty \overline{\Phi}'(s,t)\beta_s^{-1}\Pi_sBR^{-1}B'\Pi_s\overline{\Phi}(s,t)\,ds \tag{24}$$

and that $\Pi_t \leqslant P_t$, $t\geqslant 0$. Estimating the integrand in (24) via Lemma 1, we get that $\|P_t - \Pi_t\| \leqslant c_p p_t^3$, where $c_p > 0$ is some constant and $p_t = \int_t^\infty (1/\beta_v)\,dv$. Now, arguing as in the proof of (20) in Theorems 1 and 2, we conclude that for

$$\Gamma_T = \int_0^T p_t\|G_t\|^2\,dt \to \infty, \qquad T\to\infty,$$

we have

$$\limsup_{T\to\infty}\frac{J_T(U^{(0)})}{\Gamma_T} = \limsup_{T\to\infty}\frac{\mathbf{E}J_T(U^{(0)})}{\Gamma_T} = J^* \quad \text{a.s.},$$

where $J^* = \limsup_{T\to\infty}\{J_T(U^*)/\Gamma_T\} = \limsup_{T\to\infty}\{\mathbf{E}J_T(U^*)/\Gamma_T\}$. So, from the point of view of the optimality criteria in problems (3), the control $U_t^{(0)} \equiv 0$ gives the same result as the optimal law U^*. This happens because the normalization Γ_T offsets the deviation between $J_T(U^{(0)})$ and $J_T(U^*)$ as $T\to\infty$, inasmuch as $\Gamma_T\to\infty$. Certainly, this effect is not observed for $\Gamma_\infty < \infty$ or in the deterministic problem.

In regard to the convergence of U_t^* to $U_t^{(0)} \equiv 0$ as $t \to \infty$, one can easily derive from (8), (15), and Lemma 2 that, for $k_t = p_t^2/\beta_t^2$ and $\overline{\Gamma}_t = \int_0^t \|G_t\|^2\,dt$, the condition $k_t\overline{\Gamma}_t \to 0$, $t \to \infty$, implies the convergence $U_t^* \to 0$ in the mean square. If $k_t(\overline{\Gamma}_t \ln\ln \overline{\Gamma}_t) \to 0$, then $U_t^* \to 0$ a.s. as $t \to \infty$. In particular, the condition (2) of Assumption $\mathcal{G}$ guarantees the above convergence with probability 1.

4. Application to the problem of optimal control of a system with dynamic scaling of parameters. Consider a controlled stochastic process $\widetilde{X}_t$, $t \geqslant 0$, with dynamics

$$d\widetilde{X}_t = \alpha_t A\widetilde{X}_t\,dt + \alpha_t B\widetilde{U}_t\,dt + \sqrt{\alpha_t}\,G\,dw_t, \qquad X_0 = x, \tag{25}$$

where A, B, G are constant matrices of appropriate dimensions, $G \neq 0$, and the admissible control $\widetilde{U} \in U$ is defined in analogy with the case of (1). In (25), $\alpha_t > 0$ is the scaling function. The objective functional has the form

$$J_T^{(\alpha)}(\widetilde{U}) = \int_0^T \alpha_t(\widetilde{X}_t'Q\widetilde{X}_t + \widetilde{U}_t'R\widetilde{U}_t)\,dt \tag{26}$$

with matrices $Q \geqslant 0$, $R > 0$. Such a control system may appear, for example, if the time scales of the underlying process and the subject implementing the control law are not synchronous. It is assumed that $\int_0^\infty \alpha_t\,dt < \infty$ (the case $\int_0^T \alpha_t\,dt \to \infty$, $T \to \infty$, was considered in [23]). This leads to the optimal control problem for the system (25), (26) as $T \to \infty$. It can be easily seen that the change of variables $X_t = \widetilde{X}_t$, $U_t = \alpha_t\widetilde{U}_t$ reduces system (25), (26) to the form (1), (2), where $A_t = \alpha_t A$, $B_t = B$, $G_t = \sqrt{\alpha_t}\,G$, and $\beta_t = 1/\alpha_t$. It is also clear that Assumption $\mathcal{AB}$ is met for the system in new variables. Indeed,

$$\int_0^\infty \|A_t\|\,dt = \|A\|\int_0^\infty \alpha_t\,dt < \infty, \qquad \int_0^\infty \frac{1}{\beta_t}\,dt = \int_0^\infty \alpha_t\,dt < \infty,$$

and therefore we have the claim of Theorem 1 about a control law, which is optimal in the mean over an infinite time-horizon. In the original variables, this control has the form

$$\widetilde{U}_t^* = -R^{-1}B'\Pi_t\widetilde{X}_t^*,$$

where the matrix Π_t satisfies the Riccati equation

$$\dot{\Pi}_t + \alpha_t\Pi_t A + \alpha_t A'\Pi_t - \alpha_t\Pi_t BR^{-1}B'\Pi_t + \alpha_t Q = 0,$$

and the process $\widetilde{X}_t^*$, $t \geqslant 0$, is given by the equation

$$d\widetilde{X}_t^* = \alpha_t(A - BR^{-1}B'\Pi_t)\widetilde{X}_t^*\,dt + \sqrt{\alpha_t}\,G\,dw_t, \qquad \widetilde{X}_0^* = x.$$

In this case, the normalization of criterion (17) contains a function integrable at infinity, i.e.,

$$\int_0^\infty p_t\|G_t\|^2\,dt = \|G\|^2\int_0^\infty \alpha_t\left(\int_t^\infty \alpha_s\,ds\right)dt < \infty,$$

the control $\widetilde{U}^*$ is the solution of the problem

$$\limsup_{T\to\infty} \mathbf{E}J_T^{(\alpha)}(\widetilde{U}) \to \inf_{\widetilde{U}\in\mathcal{U}},$$

and, moreover, $J_T^{(\alpha)}(\widetilde{U}^*) \to J_\infty^{(\alpha)}(\widetilde{U}^*)$ with probability 1.

5. Conclusion. We consider above optimal control problems over an infinite time-horizon of the linear system (1) under the assumption of absolute integrability of the state matrix in the equation of the process and absolute integrability of the multiplier $1/\beta_t$ in the integral quadratic loss functional. The objective functional (2) also includes the inversely proportional factor β_t (i.e., an asymptotically unbounded time function) found in control costs. Such specifics of the functional correspond to the situation of prioritized losses due to control. We show that, under the above assumptions, there exists a stable linear control law U^* defined in terms of the solution of the Riccati equation. This control law (8), (9), in the form of a linear state feedback X^*, is optimal over an infinite time-horizon with respect to the extended adjusted long-run average criterion (see Theorem 1). The criterion involves normalization of the expected value of the objective functional by using the integral $\int_0^T p_t \|G_t\|^2\,dt$. This normalization is the sum of the variances of the components of the vector $\int_0^T p_t G_t\,dw_t$ of accumulated adjusted disturbances on the system over the planning horizon. The adjustment is performed towards reduction of the disturbance variance with the help of the function $p_t = \int_t^\infty (1/\beta_s)\,ds$ and of $p_t \to 0$, $t \to \infty$. It is shown that the properties of the corresponding optimal process of X^* are close to the characteristics of the original disturbance process $\widehat{W}_t = \int_0^T G_t\,dw_t$ (see Lemma 2). We also put forward conditions on the coefficients of the diffusion matrix G_t and the multiplier β_t under which the designed control law U^* possesses stronger a.s. optimality property, i.e., U^* is a solution of a problem with a pathwise extended adjusted long-run average (see Theorem 2 and Assumption $\mathcal{G}$). Moreover, we find a linear transformation that reduces a control system with dynamic scaling (i.e., in the case when all matrices are multiplied by a function of time) to the already studied system (1), (2), and determine an optimal control law. A topic for further research is the study of optimal control problems (1) with reverse priority of costs, i.e., if (2) involves the matrices $\beta_t Q$ and $(1/\beta_t)R$.

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