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STOCHASTIC SYSTEMS

Optimization of the Superstable Linear Stochastic System Applied to the Model with Extremely Impatient Agents

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Abstract—We consider the problem of stochastic linear regulator over an infinite time horizon with superexponentially stable matrix in the equation of state dynamics. The form of the optimal control based on the criterion taking into account the information about the parameters of disturbances and the matrix stability rate was determined. The results obtained were used to analyze the model of a system with extremely impatient agents where the objective functional includes discounting by the asymptotically unbounded rate.

Keywords: stochastic linear regulator, infinite time horizon, nonexponential stability, discounting

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1. INTRODUCTION

The optimal control of linear systems over an an infinite time horizon and achievement of asymptotic stability of the chosen strategies are classified with the problems of significant theoretical and practical value. It is usually assumed at that that the coefficients of the state dynamics equation are bounded time functions (see, for example, [1, p. 267]). At the same time, there are situations where this condition is not satisfied. For example, for the systems obtained as the result of linearizing [2, p. 101; 3, p. 124] or when considering models of certain random processes in various fields such as physical [4, 5] and cognitive [6] this property may be missed. In the aforementioned cases, the state dynamics is characterized by the strongly stable—as compared with a certain exponential type of stability—deterministic component, but at that the disturbances whose impact is measured by the norm of the diffusion matrix play a significant role. As will be discussed in what follows, such systems also arise at analysis of the models with extremely discounting. Now, we proceed to describing a control system which is the subject matter of the present paper.

Let an n-dimensional stochastic process X_t, $t \geqslant 0$, obeying the equation

$$dX_t = A_t X_t dt + B_t U_t dt + G_t dw_t, \qquad X_0 = x, \tag{1}$$

where the initial state x is nonrandom, w_t, $t \geqslant 0$, is the d-dimensional standard Wiener process, U_t, $t \geqslant 0$, is the admissible or k-dimensional stochastic process adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, $\mathcal{F}_t = \sigma\{w_s,\, s \leqslant t\}$ such that Eq. (1) has solution; A_t, B_t, G_t, $t \geqslant 0$, are time-varying matrices of dimensions under which (1) makes sense, be defined on a complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$. At that, $\int_0^\infty \|G_t\|^2\, dt > 0$, the matrix B_t is bounded, the matrix A_t is such that $\|A_t\| \to \infty$, $t \to \infty$ ($\|\cdot\|$ is the Euclidean matrix norm) and has superexponential stability property (the exact definition will be given in what follows). The set of admissible controls is denoted by $\mathcal{U}$. Introduce the following definition.

Definition 1. The matrix $\mathcal{A}_t$ is called superexponentially stable with the rate δ_t (or δ_t-superexponentially stable) if there exists a function $\delta_t > 0$, $t \geqslant 0$, $\delta_t \to \infty$, $t \to \infty$ such that

$$\limsup_{t\to\infty}(\|\mathcal{A}_t\|/\delta_t) < \infty; \quad \|\Phi(t,s)\| \leqslant \kappa \exp\left\{-\int_s^t \delta_v\, dv\right\}, \quad s \leqslant t,$$

for a constant $\kappa > 0$, at that $\Phi(t,s)$ is a fundamental matrix corresponding to $\mathcal{A}_t$.

We recall that the fundamental matrix $\Phi(t,s)$ is the solution of the problem

$$\frac{\partial \Phi(t,s)}{\partial t} = \mathcal{A}_t \Phi(t,s), \quad \Phi(s,s) = I.$$

For each $T > 0$, the random variable

$$J_T(U) = \int_0^T (X_t' Q_t X_t + U_t' R_t U_t)\, dt \tag{2}$$

is defined as an objective functional. Here, $U \in \mathcal{U}$ is the admissible control on the interval $[0,T]$; $Q_t \geqslant 0$, $R_t \geqslant \rho I$, $t \geqslant 0$ are bounded symmetric matrices with $'$ for transposition, $\rho > 0$ is some constant, $A \geqslant B$ means for matrices that the difference $A - B$ is positive semidefinite, and I is the identity matrix).

Assumption $\mathcal{A}$. The matrix A_t is superexponentially stable with the rate δ_t, at that δ_t is a nondecreasing differentiable function, $t \geqslant 0$.

We are interested in a formulation of the problem of optimal control for (1) and (2) when $T \to \infty$. In the general case, the control $U^* \in \mathcal{U}$ *is called average optimal over an infinite time horizon with respect to the criterion* $\mathcal{K}$ if it is the solution of the problem

$$\limsup_{T\to\infty} \mathrm{E}\mathcal{K}_T(U) \to \inf_{U\in\mathcal{U}}, \tag{3}$$

where $\mathrm{E}\mathcal{K}_T(U)$ is the expectation of some functional $\mathcal{K}_T(U)$ depending on the admissible control $U \in \mathcal{U}$ and length of the planning horizon T. In the theory of stochastic control, there exists a criterion of the long-run average with $\mathrm{E}\mathcal{K}_T(U) = \mathrm{E}J_T(U)/T$. A criterion with $\mathrm{E}\mathcal{K}_T(U) = \mathrm{E}J_T(U)/\int_0^T \|G_t\|^2\, dt$, the extended long-run average, see [7], also was introduced for the linear stochastic systems with bounded coefficients. In the present paper, the normalization $\int_0^T \|G_t\|^2\, dt$ will be modified in order to take into account the properties of (1) and (2) under Assumption $\mathcal{A}$. At that, the solution of U^* is determined based on the approach using the notion of optimal stable feedback control law with subsequent detailed description in Section 2.

It is worthwhile to note that the study of (1) and (2) for $T \to \infty$ is important also from the view of analysis of systems with extremely discounting. Let in (1) $A_t \equiv A$, $B_t \equiv B$, and in (2) the matrices $Q_t = f_t Q$, $R_t = f_t R$, the discounting function $f_t > 0$, $f_t \to 0$ monotonically decreasing, the discount rate $\phi_t = -\dot{f}_t/f_t \to \infty$, $t \to \infty$. Appearance of the discount function with such asymptotic characteristics is due to employment of various factors that impact the decision making process. For example, as the result of reflecting the cognitive processes—nonlinear subjective time perception [8, 9] by expanded time scale, inclusion of an additional source of uncertainty of the length of perception horizon having probabilistic distribution with "light" tails, see [10, 11], as well as certain phenomena of the behavioural economics related to fastly increasing impatience of the agents [12]. For the case of discounting by bounded rate ϕ_t, it was proposed in [13] to use the criterion of long-run expected loss per unit of cumulated discount discount

$$\limsup_{T\to\infty} \frac{\mathrm{E}J_T(U)}{\int_0^T f_t\, dt} \tag{4}$$

for the problem of determination of the optimal strategy, that is, when in (3) $\mathrm{E}\mathcal{K}_T(U) = \mathrm{E}J_T(U)/\int_0^T f_t\,dt$. In the present paper we will show demonstrate that by change of the variables the system with extremely discounting can be transformed into (1) and (2) and the corresponding optimality criterion modifying (4) may be determined.

The goal of the present study consist in the design of the optimality criterion over an infinite time horizon for the stochastic linear regulator under the assumption of superexponential stability of the state matrix with subsequent determination of a minimizing strategy and analysis of systems with extremely discounting. The present paper is structured as follows. Section 2 provides an approach to the problem based on determination of the stable control law. Main results on existence of such law and its optimality with respect to the criterion using the definition of the adjusted extended long-run average are presented. Section 3 is devoted to studying efficiency of the new criterion and examples. The results obtained are then used in Section 4 to consider the model of system with extremely impatient agents. The basic findings are summarized in the conclusions where the possible lines of further research are defined.

2. MAIN RESULTS

2.1. On the Stable Control Law for the Stochastic Linear Regulators

Turning to analysis of the control system under Assumption $\mathcal{A}$, we should note that the notion of superexponential stability arises naturally when studying the asymptotic stability of unbounded matrices and allows one to characterize more precisely the possible order of decrease of the upper estimate of the norm of the corresponding fundamental matrix. It is a stronger type of stability than the well-known exponential stability. The term itself arose at analyzing the nonlinear equations (see, for example, [14]). Exponential stability occurs if one assumes the rate $\delta_t \equiv \kappa_1 > 0$ (also see Definition 1). At that, the limitations of the approach with determination of the value of the constant κ_1 for the case of the state matrix $\|\mathcal{A}_t\| \to \infty$, $t \to \infty$, was noticed in [15]. We also indicate that [16] considered the issues of nonexponential stabilization of the linear autonomous deterministic system ($A_t \equiv A$, $B_t \equiv B$) by the choice of a control $U_t = K_t X_t$ with an unbounded K_t, $t \to \infty$, without reference to the problems of optimality.

Returning to the control in (1) and (2) when $T \to \infty$, consider further an approach related with determination of the stable law U^* and review the previous results on its optimality with with respect to of some criteria (see (3)), along with assumptions on the parameters of system (1) and (2). It is well known [1, Theorem 3.5, p. 267] that in the deterministic case of (1) and (2) with bounded coefficients and in the presence of some additional properties such as controllability, observability of matrix pairs or exponential stability of A_t, there exists a so-called optimal stable feedback control law $U_t^* = -R_t^{-1} B_t' \Pi_t X_t^*$ where the matrix $\Pi_t \geqslant 0$, $t \geqslant 0$ satisfies the Riccati equation

$$\dot{\Pi}_t + \Pi_t A_t + A_t' \Pi_t - \Pi_t B_t R_t^{-1} B_t' \Pi_t + Q_t = 0. \tag{5}$$

As it was noticed in [1, p. 306], when G_t is bounded the strategy U^* is optimal with respect to the long-run average cost criterion in the problem

$$\limsup_{T\to\infty} \frac{\mathrm{E}J_T(U)}{T} \to \inf_{U\in\mathcal{U}}. \tag{6}$$

It was proposed in [7] to modify (6) taking into account the impact of disturbances, see [17]. More precisely, the control U^* also proved to be a solution of the problem when using a new average optimality criterion over an an infinite time horizon, the extended long-run average when

$$\limsup_{T\to\infty} \frac{\mathrm{E}J_T(U)}{\int\limits_0^T \|G_t\|^2\,dt} \to \inf_{U\in\mathcal{U}}. \tag{7}$$

Consideration of (7) makes sense also in the system with $\|G_t\| \to \infty$, $t \to \infty$ if $\frac{d(\ln \|G_t\|)}{dt} \to 0$, $t \to \infty$, see [18]. Under the conditions of Assumption $\mathcal{A}$ and with no restrictions on the diffusion matrix G_t, the problem of constructing the optimality criterion in (1) and (2) involving the additional factor of superexponential stability arises in an obvious manner. We emphasize that in such case this strategy U^*, if any, is a natural candidate for being optimal because it has a form obtained taking the limit ($T \to \infty$) in the control laws determined as the solutions of the problem $\mathrm{E}J_T(U) \to \inf\limits_{U\in\mathcal{U}}$ for a finite T.

2.2. Main Results on the Stable Feedback Law

As was already mentioned, in the stochastic system (1) and (2) the form of the stable feedback law U_t^* includes the matrix Π_t satisfying the Riccati equation (5). The following lemma establishes existence of the solution (5) and its properties.

Lemma. *Let Assumption $\mathcal{A}$ be satisfied. Then, there exists an absolutely continuous function Π_t, $t \geqslant 0$, with values in the set of positive semidefinite symmetric matrices satisfying the Riccati equation* (5) *and such that the matrix $A_t - B_t R_t^{-1} B_t' \Pi_t$ is $\tilde{\delta}_t$-superexponentially stable with $\tilde{\delta}_t = \lambda\delta_t$, where δ_t is the stability rate of the matrix A_t and λ is a positive constant. The value of $\lambda < 1$ if $\int_0^t (1/\delta_v)\,dv \to \infty$, $t \to \infty$, and $\lambda = 1$ for $\int_0^\infty (1/\delta_v)\,dv < \infty$. Moreover, the relation $\limsup\limits_{t\to\infty}(\|\Pi_t\|\delta_t) < \infty$ is valid as well.*

The lemma and Theorem 1 are proved in the Appendix.

Along with the use of normalization in the optimality criteria design over an an infinite time horizon as in (6) or (7), it is also possible to study the properties of the so-called average overtaking optimality of the control laws.

Definition 2 [7]. The control $U^* \in \mathcal{U}$ is called the average overtaking optimal if for any number $\epsilon > 0$ there exists $T_0 > 0$ such that for an arbitrary admissible control $U \in \mathcal{U}$ the inequality holds

$$\mathrm{E}J_T(U^*) < \mathrm{E}J_T(U) + \epsilon \text{ for any } \quad T > T_0. \tag{8}$$

The following theorem establishes the average optimality of the stable feedback law U^* over an an infinite time horizon.

Theorem 1. *Let Assumption $\mathcal{A}$ be satisfied. Then, the control law*

$$U_t^* = -R_t^{-1} B_t' \Pi_t X_t^*, \tag{9}$$

where the process X_t^*, $t \geqslant 0$, is governed by the equation

$$dX_t^* = (A_t - B_t R_t^{-1} B_t' \Pi_t) X_t^* dt + G_t dw_t, \quad X_0^* = x, \tag{10}$$

is the solution to the problem

$$\limsup_{T\to\infty} \frac{\mathrm{E}J_T(U)}{\int\limits_0^T (\|G_t\|^2/\delta_t)\,dt} \to \inf_{U\in\mathcal{U}}. \tag{11}$$

At that, the matrix function $\Pi_t \geqslant 0$, $t \geqslant 0$, satisfies the Riccati equation (5) *and has properties provided in the lemma. The value of the criterion on the optimal control*

$$\limsup_{T\to\infty} \frac{\mathrm{E}J_T(U^*)}{\int\limits_0^T (\|G_t\|^2/\delta_t)\,dt} < \infty, \tag{12}$$

where δ_t is the stability rate of the matrix A_t. In addition if $\|G_t\|/\delta_t^2 \to 0$, $t \to \infty$, then control U^* is also average overtaking optimal over an infinite time horizon.

Remark. Let Assumption $\mathcal{A}$ be satisfied. Consider a deterministic control system (1) and (2), that is, for $G_t \equiv 0$. Then, a a control law (9) $U_t^* = -R_t^{-1} B_t' \Pi_t X_t^*$, where the trajectory $X_t^* = \Phi(t,0)x$ and $\Phi(t,s)$ is the fundamental matrix for the function $A_t - B_t R_t^{-1} B_t' \Pi_t$, is the solution to the problem $\limsup\limits_{T\to\infty} J_T(U) \to \inf\limits_{U\in\mathcal{U}}$. The value $\limsup\limits_{T\to\infty} J_T(U^*) = x'\Pi_0 x$.

It is natural to call the criterion in (11) the adjusted extended long-run average cost criterion. Adjustment is made towards the reduction of the denominator. If in the extended long-run average of (7) the value $\mathrm{E}(\mathcal{Z}_T'\mathcal{Z}_T)$ was used, where $\mathcal{Z}_T = \int_0^T G_t dw_t$ is the vector of disturbances the system, then in the present case it is taken that $\mathrm{E}(\hat{\mathcal{Z}}_T'\hat{\mathcal{Z}}_T)$, $\hat{\mathcal{Z}}_T = \int_0^T (G_t/\sqrt{\delta_t}) dw_t$, that is, the cumulative adjusted perturbations with the diffusion matrix $\hat{G}_t = G_t/\sqrt{\delta_t}$ which are smaller (in the norm) than the original ones in a proportion inverse to the stability rate δ_t.

3. STUDY ON THE EFFICIENCY OF THE LONG-RUN AVERAGES AND EXAMPLES

Now we demonstrate that a a given criterion of the extended long-run average, see (7), in the case of the control system (1) and (2) is inefficient because its structure ignores the superexponential stability feature of the matrix A_t.

Definition 3 [17]. Let U^* be a stable feedback control law that is average optimal over an infinite time horizon with respect to the criterion $\mathcal{K}$ in the system (1) and (2). Then, call the criterion $\mathcal{K}$

(a) efficient if $0 < \limsup\limits_{T\to\infty} \mathrm{E}\mathcal{K}_T(U^*) < \infty$ for $\limsup\limits_{T\to\infty} \mathrm{E}J_T(U^*) > 0$;

(b) inefficient if there exists a set $\mathcal{U}^{\mathcal{E}} \subseteq \mathcal{U}$ such that $\limsup\limits_{T\to\infty} \mathrm{E}\mathcal{K}_T(U^*) = \limsup\limits_{T\to\infty} \mathrm{E}\mathcal{K}_T(U^\epsilon) = 0$ for any $U^\epsilon \in \mathcal{U}^{\mathcal{E}}$.

Assume that $\int_0^T \|G_t\|^2\,dt \to \infty$, $T\to\infty$ (if $\int_0^\infty \|G_t\|^2\,dt < \infty$, then the value of $\mathrm{E}J_T(U^*)$ is a finite number for $T\to\infty$, and the differences in normalizations (7) or (11) are of no importance). Then it follows from the representation

$$\mathrm{E}J_T(U^*) = x'\Pi_0 x - \mathrm{E}[(X_T^*)'\Pi_T X_T^*] + \int_0^T \mathrm{tr}\,(G_t'\Pi_t G_t)\,dt$$

(here, $\mathrm{tr}(\cdot)$ is the trace of the matrix, see, for example, [17]) that $\limsup\limits_{T\to\infty} \left(\mathrm{E}J_T(U^*)/\int_0^T \|G_t\|^2 dt\right) = 0$. Take the set $\mathcal{U}^{\mathcal{E}} = \left\{U_t^{(0)}\right\}$ where $U_t^{(0)} \equiv 0$. Determine the matrix $P_t = \int_t^\infty \Phi'(s,t) Q_s \Phi(s,t) ds$ and notice that $\|P_t\| \leqslant \frac{\tilde{c}}{\delta_t}$, $t \geqslant 0$ ($\tilde{c} > 0$ is a constant, see the proof of lemma), here $\Phi(t,s)$ is obtained for the superexponentially stable matrix A_t. Find the expression

$$\mathrm{E}J_T(U^{(0)}) = \int_0^T \mathrm{E}\left[(X_t^{(0)})' Q_t X_t^{(0)}\right] dt = x'P_0 x - \mathrm{E}\left[(X_T^{(0)})' P_T X_T^{(0)}\right] + \int_0^T \mathrm{tr}(G_t' P_t G_t)\,dt.$$

At that, the process $X_t^{(0)}$, $t \geqslant 0$ obeys Eq. (1). Obviously, $\delta_t^{-1}\mathrm{E}\|X_T^{(0)}\|^2/\int_0^T \|G_t\|^2\,dt \to 0$, $T\to\infty$. Therefore, $\limsup\limits_{T\to\infty} \left(\mathrm{E}J_T(U^{(0)})/\int_0^T \|G_t\|^2\,dt\right) = 0$, and according to Definition 3 the extended long-run average is inefficient. As follows from the aforementioned representation for $\mathrm{E}J_T(U^*)$, under some additional assumptions the suggested criterion of the adjusted extended long-run average will be efficient with respect to the exponential stability feature of the state matrix. First, it is required that the rest of the parameters of the standard deterministic system provide the lower bound $\Pi_t \geqslant (\alpha/\delta_t)I$, $t \geqslant 0$ ($\alpha > 0$ is some constant). In particular, this requirement is satisfied if in the objective functional $Q_t \geqslant qI$ under some constant $q > 0$, and for the stability rate δ_t the condition $\dot{\delta}_t/\delta_t^2 \to 0$ is valid asymptotically for $t \to \infty$. Secondly, the contribution of $\int_0^T (\|G_t\|^2/\delta_t\, dt)$ (the variance of the adjusted integrated perturbations) must overlap the ratio $\mathrm{E}\|X_T^*\|^2/\delta_T$ (variance of the process to a unit of stability rate). Stated differently, $\delta_T^{-1}\mathrm{E}\|X_T^*\|^2/\int_0^T (\|G_t\|^2/\delta_t)\, dt \to 0$, $T \to \infty$. In particular, we get the desired property if $\delta_T^{-2}\|G_T\|^2/\int_0^T (\|G_t\|^2/\delta_t)\, dt \to 0$, $T \to \infty$; for example, for $\limsup\limits_{t\to\infty}(\|G_t\|^2/\delta_t) < \infty$.

The following examples consider special cases of the adjusted extended long-run average cost criterion and verify whether the condition of Theorem 1 regarding the overtaking optimality is satisfied. We note that by analysis of the ratio $\mathrm{E}\|X_T^*\|^2/\delta_T^3$ (see the proof of Theorem 1) one can elicit the order of variation of the upper estimate for the expected value of the deficiency process $\Delta_T = J_T(U^*) - J_T(U)$, see [7]. If one succeeds in determining the function $h_T > 0$ such that $\mathrm{E}\Delta_T(U) \leqslant h_T$, $T > T_0$, for any $U \in \mathcal{U}$, then h_T is called the upper function (see [7]).

Example 1. Under increasing perturbations for which $\|G_t\|^2 \sim \delta_t$, the criterion in (11) takes the form of the well-known long-run average cost criterion from (6). Since $\|G_t\|/\delta_t^2 = \sqrt{\delta_t}/\delta_t^2 \to 0$, $t \to \infty$, the overtaking optimality takes place and $h_T \sim 1/\delta_T^3$.

Example 2. The constant disturbances $G_t \equiv G \neq 0$ give rise to the normalization $\int_0^T (1/\delta_t)\, dt$ in (11) and obvious overtaking optimality with $h_T \sim 1/\delta_T^4$.

Example 3. If $\|G_t\|^2 \sim \delta_t^3$ (the disturbance force grows faster than the rate of stability, see the diffusion model in [4]), then normalization of the criterion $\int_0^T \delta_t^2\, dt$ is always increasing and, on the contrary, $h_T = 1/\delta_t$ decreases slower under overtaking optimality (condition of Theorem 1 is satisfied because $\|G_t\|/\delta_t^2 = \delta_t^{3/2}/\delta_t^2 \to 0$, $t \to \infty$).

4. OPTIMAL CONTROL OF A SYSTEM WITH EXTREMELY IMPATIENT AGENTS

Let us assume that the system state is defined by a random process $\tilde{X}_t$, $t \geqslant 0$ with dynamics

$$d\tilde{X}_t = A\tilde{X}_t dt + B\tilde{U}_t dt + \tilde{G}_t dw_t, \qquad \tilde{X}_0 = x, \tag{13}$$

where A, B are constant matrices, the remaining characteristics (13) are defined similar to (1) with replacement of $\tilde{G}_t$s for G_t and the control $\tilde{U}_t$ by U_t.

The objective functional of the total loss is given by

$$J_T^{(d)}(\tilde{U}) = \int_0^T f_t(\tilde{X}_t' Q \tilde{X}_t + \tilde{U}_t' R \tilde{U}_t)\, dt, \tag{14}$$

where $Q \geqslant 0$, $R > 0$ are constant matrices and f_t is a discount function used by the agents to estimate the losses at different point in time.

Assumption $\mathcal{D}$. The discount function $f_t > 0$, $t \geqslant 0$, $f_0 = 1$, is twice differentiable, decreases monotonically, and is logarithmically concave ($(\ln f_t)'' < 0$). The discount rate $\phi_t = -\dot{f}_t/f_t$ is such that $\phi_t \to \infty$, $t \to \infty$.

The agents are extremely impatient because in time the discounting rate becomes unbounded.

Example 4. The Weibull discount function $f_t = \exp(-rt^q)$ ($q > 1, r > 0$), related to strongly nonlinear time perception according to the Stevens' power law with the expanded time scale $\tau_t = t^q$ [9], as well as to possible derivation in terms of the survival function with "fast" Weibull distribution [8].

Example 5. The discount function using a double exponent like $f_t = \exp(-r(\exp t))$ $(r > 0)$ and considered in [12] with rapidly increasing rate $\phi_t = r\exp(t)$ which characterizes the probabilistic Gompertz distribution, see also [11].

The discount functions from Examples 4 and 5 were brought to light on the basis of empirical analysis of the return of the long-term bonds, see [19, 20], and can be used to reflect the time preferences of the subjects in decision-making, for example, in the actuarial models, which also was noted in [19, 20]. At that, the origin of the unbounded rates was accounted for in [21] to the high level of uncertainty, in particular, to the financial markets of the developing countries. Example 6 considers a control model in the field of insurance which is a special case of the control system described in [22].

Example 6. Consider the process of dynamics of the excess of capital of an insurance company in the presence of dividend payments and in the absence of the expected inflow of premiums. Then, the scalar process of excess $\tilde{X}_t$, $t \geqslant 0$ can be approximated using the equation, see [22],

$$d\tilde{X}_t = a\tilde{X}_t dt - \tilde{U}_t dt + G_t dw_t, \qquad \tilde{X}_0 = x,$$

where the constant $a > 0$, w_t, $t \geqslant 0$ is a one-dimensional Wiener process. The scalar process $\tilde{U}_t$, $t \geqslant 0$ representing the deviation of the rate of dividend payments from the nominal (desired, objective) deterministic trajectory plays the role of control. The total loss ("negative utility") due to the excess deviation from zero (priority of the company management) and the dividend payments from the desirable trajectory (priority of the shareholders) with regard for the weight coefficients $q_t, p_t > 0$ also reflecting the agents preferences $J_T^{(d)}(\tilde{U}) = \int_0^T (q_t\tilde{X}_t^2 + p_t\tilde{U}_t^2)\,dt$ is considered in [22] as the objective functional. If one assumes that the main impact here is attributed to the time preferences and using the aforementioned approach on the intertemporal evaluation for the actuarial models $q_t = qf_t$ $(q > 0)$, $p_t = f_t$, we take the discount functions from Examples 4 or 5, then we get a control system like (13) and (14).

Passing to analysis of (13) and (14), one can readily see that in virtue of $f_t R \to 0$, $t \to \infty$, and $A_t \equiv A$ the given control system does not belong to the considered above type of systems (1) and (2). As it was said before (see Section 1), in the case of f_t with bounded rate and constant $\tilde{G}_t \equiv G$ a criterion of the long-run expected loss per unit of cumulated discount was proposed to compare different strategies when $T \to \infty$, see [13] (also including the case of constant rate $\phi_t \equiv r > 0$ which corresponds to the well-known exponential discounting function of form $f_t = \exp(-rt)$). Such control system was examined by also including an equivalent system with a standard objective functional and nonautonomous equation of state dynamics. Define new variables

$$X_t = \sqrt{f_t}\tilde{X}_t, \quad U_t = \sqrt{f_t}\tilde{U}_t \tag{15}$$

and obtain

$$dX_t = (A - (1/2)\phi_t I)X_t dt + BU_t dt + \sqrt{f_t}\tilde{G}_t dw_t, \qquad X_0 = x, \tag{16}$$

$$J_T(U) = \int_0^T (X_t'QX_t + U_t'RU_t)\,dt, \qquad J_T(U) = J_T^{(d)}(\tilde{U}). \tag{17}$$

We notice that in virtue of Assumption $\mathcal{D}$ the matrix $A_t = A - (1/2)\phi_t I$ is superexponentially stable with rate $\delta_t = (1/2)\phi_t$. Then, (16) and (17) is a special case of (1) and (2) with $B_t \equiv B$, $Q_t \equiv Q$, $R_t \equiv R$, and $G_t = \sqrt{f_t}\tilde{G}_t$. According to Theorem 1, the control law U_t^* of form (9) is the solution of problem (11). The inverse change of variables in (15) confirms validity of Theorem 2.

Theorem 2. *Let Assumption $\mathcal{D}$ be satisfied. Then, the control strategy*

$$\tilde{U}_t^* = -R^{-1}B'\Pi_t\tilde{X}_t^*,$$

where the process $\tilde{X}_t^*,\ t \geqslant 0$, *is given by the equation*

$$d\tilde{X}_t^* = (A - BR^{-1}B'\Pi_t)\tilde{X}_t^* dt + \tilde{G}_t dw_t, \quad \tilde{X}_0^* = x,$$

is the solution to the problem

$$\limsup_{T\to\infty} \frac{\mathrm{E}J_T^{(d)}(U)}{\int\limits_0^T (f_t/\phi_t)\|\tilde{G}_t\|^2\,dt} \to \inf_{U\in\mathcal{U}}. \tag{18}$$

At that, the matrix function $\Pi_t \geqslant 0$, $t \geqslant 0$, *satisfies the Riccati equation* (5) *with* $A_t = A - (1/2)\phi_t$, $B_t \equiv B$, $Q_t \equiv Q$, $R_t \equiv R$ *and* $\limsup\limits_{t\to\infty}(\|\Pi_t\|/\phi_t) < \infty$.

In virtue of Assumption $\mathcal{D}$, the function $g_t = f_t/\phi_t$ decreases and, consequently, can be regarded as a discount one. At that, $g_t < f_t$, $t \to \infty$, that is, the denominator of the optimality criterion in (18) represents a variance of the accumulated superdiscounted disturbances. Therefore, it is natural to regard the limit of the ratio in (18) as a modification of the long-run expected loss per unit of cumulated discount. If $\tilde{G}_t \equiv \tilde{G} \neq 0$, then $\int_0^\infty (f_t/\phi_t)\,dt < \infty$, and problem (18) takes form similar to the one posed for rapidly decaying f_t with bounded rate, for example, $f_t = \exp(-rt)$, $r > 0$, where normalization is not necessary.

5. CONCLUSIONS

The present paper considered the problem of stochastic linear regulator over an infinite time horizon for the system with superexponentially stable matrix in the equation of state dynamics. It was shown that the control law having the form of the linear feedback law and involving solution of the Riccati equation minimizes the adjusted extended long-run average cost criterion. The derived optimality criterion is defined on the basis of the upper limit (for $T \to \infty$, T is the length of the planning horizon) of the ratio of expectation of the quadratic objective functional to the integral over the interval $[0, T]$ of the function $(1/\delta_t)\|G_t\|^2$ (square of the norm of the diffusion matrix G_t divided by the rate of stability δ_t of the matrix A_t from the state equation). Interestingly, the result of Theorem 1 on the form of the optimal control is valid without any assumptions on boundedness of possible order of growth of the diffusion matrix G_t. This fact distinguishes significantly the situation at hand from the previously considered cases of linear stochastic control systems with bounded A_t. In [7] the optimality of the control law U^* was established for $\|G_t\| \leqslant \hat{c}$ ($\hat{c} > 0$ is a constant) and in [18] the relation $\|G_t\| \to \infty$, $t \to \infty$, took place, but the rate of variation $\|G_t\|$ was finite. Therefore, in the presence of superexponentially stability A_t in the state Eq. (1) the power of disturbances does not affect the possibility derive the optimal control in (1) and (2) when $T \to \infty$. The proposed criterion of the adjusted extended long-run average containing not only the information about the parameters of the diffusion matrix, but also the rate of stability of the state matrix enabled one to analyze also the models with extremely impatient agents and has led to modification of the a known performance index of the long-run expected loss per unit of cumulated discount by adding an additional discounting multiplier. As the line of future research, we can mention analysis of the problem of stochastic linear regulator with unbounded matrix A_t which is not stable, for example, anti-stable, where the lower bound of the norm corresponding to the fundamental matrix grows exponentially with the rate δ_t.

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APPENDIX

Proof of Lemma. Consider (5) with the boundary condition $\Pi_T^T = 0$. As it is known from the optimal control theory (see, for example, [1, Theorem 3.4, p. 253] that the solution Π_t^T, $0 \leqslant t \leqslant T$, exists and is a symmetrical positive semidefinite definite matrix. Let $0 \leqslant t_0 < T$ be a fixed time instant. In the control system

$$dx_t = (A_t x_t + B_t u_t)dt, \quad x_{t_0} = \bar{x},$$

where $\bar{x}$ is an arbitrary vector of the initial state with the objective functional

$$J_{T,t_0} = \int\limits_{t_0}^{T} (x_t' Q_t x_t + u_t' R_t u_t)\, dt,$$

the law of the form $u_t^{*T} = -R_t^{-1} B_t' \Pi_t^T x_t^{*T}$ is a solution of the problem $J_{T,t_0}(u) \to \min$ and $J_{T,t_0}(u^{*T}) = \bar{x}' \Pi_{t_0}^T \bar{x}$. Take a competing control $u_t^{(0)} \equiv 0$. In virtue of superexponential stability of the matrix A_t and boundedness of the matrix Q_t, we may define the function $P_t = \int_t^\infty \Phi'(s,t) Q_s \Phi(s,t)\, ds$ which is the solution to the equation $\dot{P}_t + A_t' P_t + P_t A_t + Q_t = 0$ where $\Phi(t,s)$ is the fundamental matrix for A_t and $\|P_t\| \leqslant c \int_t^\infty \exp\{-\int_t^s \delta_v\, dv\}\, ds$. Here and below c denotes some positive constant whose particular importance is of no significance and can vary from formula to formula. As a result of nondecreasing δ_t, we have the estimate $\|P_t\| \leqslant c\delta_t$, $t \geqslant 0$. One can readily show that $J_{T,t_0}(u^{(0)}) = \bar{x}' P_{t_0} \bar{x} - x_T^{(0)'} P_T x_T^{(0)} \leqslant c\|\bar{x}\|^2/\delta_{t_0}$, and optimality of control u_t^{*T} provides $\|\Pi_{t_0}^T\| \leqslant \bar{c}/\delta_{t_0}$ ($\bar{c} > 0$ is some constant). Since the instant t_0 was selected arbitrarily, $\|\Pi_t^T\| \leqslant \bar{c}/\delta_t$, $0 \leqslant t \leqslant T$. The further reasoning follows the logic of [1, Theorem 3.5, p. 267]. The function Π_t^T does not decrease in T, is bounded and, consequently, has the limit $\lim\limits_{T\to\infty} \Pi_t^T = \Pi_t$ satisfying (5) and featuring the same properties as Π_t^T, at that $\limsup\limits_{t\to\infty}(\|\Pi_t\|\delta_t) < \infty$.

To prove superexponential stability of the matrix $A_t - B_t R_t^{-1} B_t' \Pi_t$, consider the linear system

$$dz_t = (A_t - B_t R_t^{-1} B_t' \Pi_t) z_t dt, \quad z_{t_0} = \bar{z},$$

where $\bar{z}$ is an arbitrary vector of the initial state. The solution $z_t = \Phi(t,t_0)\bar{z}$ for the fundamental matrix for $\Phi(t,s)$ is $A_t - B_t R_t^{-1} B_t' \Pi_t$. Also z_t is representable as

$$z_t = \tilde{\Phi}(t,t_0)\bar{z} + \int\limits_{t_0}^{t} \tilde{\Phi}(t,s) B_s R_s^{-1} B_s' \Pi_s z_s\, ds,$$

where $\tilde{\Phi}(t,s)$ corresponds to δ_t-superexponentially stable matrix A_t. Taking in consideration this fact and the resulting relation $\|\Pi_t\| \leqslant \bar{c}/\delta_t$, one can put down the estimate

$$\|z_t\| \leqslant \kappa \exp\left\{-\int\limits_{t_0}^{t} \delta_v\, dv\right\} \|\bar{z}\| + c\int\limits_{t_0}^{t} \exp\left\{-\int\limits_{s}^{t} \delta_v\, dv\right\} (\|z_s\|/\delta_s)\, ds$$

and using it with the help of the Gronwall–Bellman inequality obtain that

$$\|z_t\| \leqslant \bar{\kappa} \exp\left\{-\int_{t_0}^{t} \delta_v \, dv\right\} \exp\left\{\bar{\kappa}_1 \int_{t_0}^{t} (1/\delta_v) \, dv\right\} \|\bar{z}\|$$

with some constants $\bar{\kappa}$, $\bar{\kappa}_1 > 0$; whence it follows for $\int_0^t (1/\delta_v) \, dv \to \infty$, $t \to \infty$, that the matrix $A_t - B_t R_t^{-1} B_t' \Pi_t$ is superexponentially stable with the rate $\tilde{\delta}_t = \lambda \delta_t$ with some constant $\lambda < 1$ or in the case of $\int_0^\infty (1/\delta_v) \, dv < \infty$ we have the rate of stability $\tilde{\delta}_t = \delta_t$, which proves the lemma.

Proof of Theorem 1. Fix the control $U \in \mathcal{U}$ and its corresponding process X_t in (1). By assuming that $x_t = X_t - X_t^*$ and $u_t = U_t - U_t^*$, we get the representation

$$J_T(U^*) - J_T(U) = 2x_T' \Pi_T X_T^* - \int_{t_0}^{T} (x_t' Q_t x_t + u_t' R_t u_t) \, dt - 2 \int_{0}^{T} x_t' \Pi_t G_t dw_t,$$

where the pair $(x_t, u_t)_{t \leqslant T}$ satisfies the equation

$$dx_t = (A_t x_t + B_t u_t) dt, \quad x_0 = 0, \tag{A.1}$$

with the solution $x_T = \int_0^t \Phi(T,t) B_t u_t \, dt$. At that, $\Phi(t,s)$ corresponds to A_t. By using the property of superexponential stability of A_t and the Cauchy–Buniakowsky inequality one can estimate the solution of (A.1) as

$$\|x_T\|^2 \leqslant c \exp\left\{-2\int_{0}^{T} \delta_v \, dv\right\} \int_{0}^{T} \exp\left\{\int_{0}^{t} \delta_v \, dv\right\} dt \int_{0}^{T} \exp\left\{\int_{0}^{t} \delta_v \, dv\right\} \|u_t\|^2 \, dt.$$

The condition $R_t \geqslant \rho I$, $t \geqslant 0$ enables one to write down the relation

$$\|x_T\|^2 \leqslant c \int_{0}^{T} \exp\left\{-\int_{t}^{T} \delta_v \, dv\right\} dt \int_{0}^{T} u_t' R_t u_t \, dt,$$

which with regard for nondecreasing δ_t provides the inequality

$$\delta_T \|x_T\|^2 \leqslant c_0 \int_{0}^{T} u_t' R_t u_t \, dt \tag{A.2}$$

under certain constant $c_0 > 0$. Using the elementary inequality $2ab \leqslant a^2/c + cb^2$ which is valid for an arbitrary $c > 0$ for any numbers a, b and the relation $\|\Pi_T\| \leqslant \bar{c}/\delta_T$ ($\bar{c} > 0$ is some constant), obtain with regard for (A.2) the estimate

$$\mathrm{E} J_T(U^*) \leqslant \mathrm{E} J_T(U) + c_1 \frac{\mathrm{E}\|X_T^*\|^2}{\delta_T^3}, \tag{A.3}$$

where $c_1 > 0$ is some constant.

Solution of (10) is given as $X_T^* = \Phi(T,0)x + \int_0^T \Phi(T,t) G_t dw_t$ where $\Phi(t,s)$ is defined for the matrix $A_t - B_t R_t^{-1} B_t' \Pi_t$ with the stability rate $\tilde{\delta}_t = \lambda \delta_t$ (see Lemma). Then,

$$\mathrm{E}\|X_T^*\|^2 \leqslant c \left(\exp\left\{-2\int_{0}^{T} \tilde{\delta}_v \, dv\right\} \|x\|^2 + \int_{0}^{T} \exp\left\{-2\int_{t}^{T} \tilde{\delta}_v \, dv\right\} \|G_t\|^2 dt \right).$$

Considering the relation $L_T = \frac{\delta_T^{-1}\mathrm{E}\|X_T^*\|^2}{\int_0^T(\|G_t\|^2/\delta_t)dt}$ one can readily establish its boundedness and, consequently, $L_T/\delta_T^2 \to 0$, $T \to \infty$. Therefore, by normalizing (A.3) with $\int_0^T(\|G_t\|^2/\delta_t)\,dt$ we come at the limit to the inequality

$$\limsup_{T\to\infty} \frac{\mathrm{E}J_T(U^*)}{\int\limits_0^T(\|G_t\|^2/\delta_t)\,dt} \leqslant \limsup_{T\to\infty} \frac{\mathrm{E}J_T(U)}{\int\limits_0^T(\|G_t\|^2/\delta_t)\,dt},$$

which gives U^* as the solution of the problem (11). Then, with the use of the Ito formula (see also [17]) we show that

$$\mathrm{E}J_T(U^*) = x'\Pi_0 x - \mathrm{E}[(X_T^*)'\Pi_T X_T^*] + \int\limits_0^T \mathrm{tr}(G_t'\Pi_t G_t)\,dt, \tag{A.4}$$

whence it follows with regard for $\|\Pi_T\| \leqslant \bar{c}/\delta_T$ ($\bar{c} > 0$ is some constant) that (12). It also follows from the resulting estimate for $\mathrm{E}\|X_T^*\|^2$ that fulfillment of the condition $\|G_t\|/\delta_t^2 \to 0$, $t \to \infty$, leads to $\mathrm{E}\|X_T^*\|^2/\delta_T^3 \to 0$, $T \to \infty$, and in this the overtaking optimality of the control U^* takes place owing to inequality (A.3) (see (8)), which proves Theorem 1.

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