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STOCHASTIC SYSTEMS

On the Optimal Control Problem for a Linear Stochastic System with an Unstable State Matrix Unbounded at Infinity

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Abstract—We consider a control problem over an infinite time horizon with a linear stochastic system with an unstable asymptotically unbounded state matrix. We extend the notion of anti-stability of a matrix to the case of non-exponential anti-stability, and introduce an anti-stability rate function as a characteristic of the rate of growth for the norm of the corresponding fundamental matrix. We show that the linear stable feedback control law is optimal with respect to the criterion of the adjusted extended long-run average. The designed criterion explicitly includes information about the rate of anti-stability and the parameters of the disturbances. We also analyze optimality conditions.

Keywords: stochastic linear-quadratic regulator, anti-stability, instability, superexponential growth, Riccati equation

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1. INTRODUCTION

Stabilization of unstable systems is one of the major issues of control theory [1–7]. The need to establish the stability property independently of a specific planning horizon leads to the set up over an infinite time horizon. For linear systems, the possibility of stabilization and the existence of optimal stabilizing control are directly related to the specifics of the coefficients of the systems. The standard assumption here is that parameters are bounded in time; see [1, 8, p. 267]. Nevertheless, there are examples of systems (see, e.g., [9–13]) that do not satisfy the above condition, which a particular treatment of the corresponding situations. In what follows we describe the control system considered in this work.

Consider the complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$ and an n-dimensional random process on this space $X_t = X_t(\omega)$, $t \geqslant 0$, $\omega \in \Omega$, defined by the equation

$$dX_t = A_t X_t dt + B_t U_t dt + G_t dw_t, \quad X_0 = x, \tag{1}$$

where the initial state x is not random; $w_t = w_t(\omega)$, $t \geqslant 0$, is a d-dimensional standard Wiener process; $U_t = U_t(\omega)$ is the control, a k-dimensional random process; A_t, B_t, G_t, $t \geqslant 0$, are deterministic matrix functions of time of dimensions suitable for (1). Admissible controls $U_t = U_t(\omega)$ are considered as random processes adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, $\mathcal{F}_t = \sigma\{w_s, s \leqslant t\}$ ($\sigma(\cdot)$ is the sign of the σ-algebra) such that Eq. (1) has a solution and at the same time $U_t(\omega)$ is square integrable with probability one, i.e., $\int_0^t \|U_s(\omega)\|^2 ds < \infty$ almost surely for any $t \geqslant 0$ ($\|\cdot\|$ is the

Euclidean norm). We denote the set of admissible controls by $\mathcal{U}$. The variables introduced below, which are functions of time t and characterize the elements of the control system, will be understood as non-random unless otherwise indicated, as we did for processes $X_t = X_t(\omega)$, $w_t = w_t(\omega)$, and $U_t = U_t(\omega)$. At the same time, we also assume that $\int_0^\infty \|G_t\|^2\,dt > 0$, the matrix B_t is bounded; matrix A_t is unbounded at infinity, i.e., $\|A_t\| \to \infty$ as $t \to \infty$. It is important to emphasize that the main assumption about the state matrix A_t concerns the absence of its asymptotic stability property. It is also known, see [14–16], that for matrices with time-dependent entries, along with exponential stability, a more general concept of stability with variable rate δ_t is also considered. We formulate the necessary definitions for such a case.

Definition 1 (see [16]). A matrix $\bar{A}_t$ is stable with rate δ_t if there exists a function $\delta_t > 0$ for $t \geqslant 0$, $\lim_{t\to\infty} \int_0^t \delta_v\,dv \to \infty$, such that $\limsup_{t\to\infty} (\|\bar{\mathcal{A}}_t\|/\delta_t) < \infty$, and the fundamental matrix $\bar{\Phi}(t,s)$, corresponding to $\bar{A}_t$, satisfies $\|\bar{\Phi}(t,s)\| \leqslant \kappa \exp\{-\int_s^t \delta_v\,dv\}$, $0 \leqslant s \leqslant t$, with some constant $\kappa > 0$. If $\delta_t \equiv \text{const}$, then stability is exponential, with $\delta_t \to 0$ it is subexponential, and with $\delta_t \to \infty$ it is superexponential for $t \to \infty$.

The following definition is a natural generalization of the well-known concept of exponential anti-stability, see [17].

Definition 2. A matrix $\mathcal{A}_t$ is called anti-stable with rate δ_t (or δ_t-anti-stable) if the matrix $\bar{\mathcal{A}}_t = -\mathcal{A}_t'$ ($'$ denotes transposition) is stable with rate δ_t. Exponential, subexponential, or superexponential anti-stability is characterized according to Definition 1.

The following are the main assumptions regarding parameters of system (1) under which we will obtain the main results of this work.

Assumption $\mathcal{A}$. *Matrix A_t is superexponentially anti-stable with rate δ_t, where δ_t is a non-decreasing differentiable function, $t \geqslant 0$, and $\lim_{t\to\infty}(\dot{\delta}_t/\delta_t^2) = 0$ (here $\dot{}$ denotes the time derivative of a function).*

Assumption $\mathcal{B}$. *Matrix B_t such that $B_t B_t' \geqslant bI$ with $t \geqslant 0$, where $b > 0$ is some constant (notation $A \geqslant B$ for matrices means that the difference $A - B$ is non-negative definite).*

Conditions in Assumptions $\mathcal{A}$ and $\mathcal{B}$ are discussed in more detail in Section 2.

For every $T > 0$, as the objective functional we introduce a random variable $J_T(U)$:

$$J_T(U) = \int_0^T (X_t' Q_t X_t + U_t' R_t U_t) dt, \tag{2}$$

where $U \in \mathcal{U}$ is an admissible control on the interval $[0,T]$; $Q_t \geqslant qI$, $R_t \geqslant \rho I$, $t \geqslant 0$, are bounded symmetric matrices, q, ρ are positive constants. We note that the stabilization of a system (in a broad sense) can be understood as maintaining its trajectory close to a given level during the planning horizon by choosing control actions; see, for example, [18, Part 3]. This approach also explains the use of (2) in evaluating control performance. Indeed, (2) measures the cumulative losses arising from the deviation of X_t from the zero state and, moreover, takes into account the costs of applying the appropriate strategy.

Next, we need to formulate a control problem that includes optimization of $\mathrm{E}J_T(U)$ with $T \to \infty$ ($\mathrm{E}(\cdot)$ denotes the expectation operator), which can be done by choosing an appropriate normalization of the expected value of the objective functional. Then the corresponding optimal control is said to be optimal on average over an infinite time horizon. Based on known results [8, p. 306; 19–21], it can be expected that the resulting strategy will have the form of the optimal stable feedback law [8] involving solution of a Riccati equation, and at the same time will stabilize the system. In this case, the stabilization problem is considered in a context specific to linear stochastic control systems with additive noise, which is independent of either state or control. The optimal strategy is designed to stabilize long-term losses in the sense of minimizing the growth of the expected value of the objective functional (2); for a description of this approach, see, for example, [22, Chap. 3]. In view of the above, the purpose of this work is to find the control that provides the solution to

$$\limsup_{T\to\infty} \frac{\mathrm{E}J_T(U)}{\int\limits_0^T \delta_t \|G_t\|^2\, dt} \to \inf_{U\in\mathcal{U}}, \tag{3}$$

where δ_t is the function that defines the anti-stability rate of the matrix A_t from the equation of state dynamics (1); see assumption $\mathcal{A}$. The method of constructing the criterion in (3) and the necessary justification will be given in Section 3. Section 3 also provides additional conditions on the coefficients that guarantee the possibility of achieving stability of the trajectories in the stochastic system by using a control which represents a solution of (3). The paper is organized as follows. Section 2 discusses the above assumptions on system parameters. Section 3 states the main result on the existence of a control U^* in the form of a linear stable feedback law which is a solution to problem (3) with a corresponding criterion that relates to the type of adjusted extended long-run averages. We also show there that U^* will be the stabilizing control in the deterministic system. Section 4 is devoted to the analysis of the technical condition necessary to establish the optimality of U^*, and provides examples. We conclude with a discussion of our results and a overview of possible directions for further research.

2. ON BASIC ASSUMPTIONS REGARDING THE PARAMETERS

Let us discuss the assumptions formulated above on the parameters of the control system (1), (2). The emphasis will be on the specific features of Assumptions $\mathcal{A}$ and $\mathcal{B}$. In this case, Assumption $\mathcal{A}$ corresponds to the properties of unbounded at infinity matrix A_t in (1) and is related to Definitions 1 and 2. In Definition 1, the function δ_t sets the rate of decrease for the upper bound on the norm of the fundamental matrix (with a fixed s), acting as a characteristic of asymptotic stability, and is called the stability rate. Exponential stability holds for $\delta_t \equiv \text{const}$. If $\delta_t \to 0$ with $t \to \infty$, then a sub-exponential type of stability occurs, and with $\delta_t \to \infty$ it is superexponential; this terminology was introduced in [14]. For unbounded matrices, the definition of the concept of superexponential stability allows us to give a more complete description of the behavior of solutions of the corresponding linear equations since in this case the exponentially decreasing upper bound with an arbitrary constant rate (corresponding to exponential stability) turns out to be uninformative, as noted in [1]. In this regard, superexponentially stable matrices are also natural to call superstable, see [19]. For unstable matrices, the conditions given in Definition 1 do not hold. In particular, an asymptotically unbounded increase in the norm of the fundamental matrix is possible. In order to clarify the nature of instability, the concept of anti-stability is used, related to the theory of operators; see, for example, [17, p. 11]. Turning further to the corresponding Definition 2, it is easy to see that exponential anti-stability corresponds to $\delta_t \equiv \text{const}$ in Definition 2, if $\delta_t \to 0$ we have subexponential anti-stability, and with $\delta_t \to \infty$ matrix $\mathcal{A}_t$ is superexponentially anti-stable. Superexponential anti-stability can also be regarded as super-instability. Indeed, taking advantage of the fact that the fundamental matrix $\Phi(t,s)$ for $\mathcal{A}_t$ is the solution to the problem

$$\frac{\partial \Phi(t,s)}{\partial t} = \mathcal{A}_t \Phi(t,s), \quad \Phi(s,s) = I,$$

where I is the identity matrix, $\bar{\Phi}(t,s) = \Phi'(s,t)$, and here $\bar{\Phi}(t,s)$ is defined for $\bar{\mathcal{A}}_t = -\mathcal{A}'_t$ ($\Phi(s,t) = \Phi^{-1}(t,s)$, see also [17, p. 2]), it is easy to see that as a result of the upper bound given in Definition 1 there will be a superexponentially growing lower bound in parameter t for $\Phi(t,s)$

with a fixed $s \geqslant 0$:

$$\|\Phi(t,s)\| \geqslant (1/\kappa)\exp\left\{\int\limits_s^t \delta_v\,dv\right\}, \quad 0 \leqslant s \leqslant t.$$

Obviously, any anti-stable matrix is also unstable, but the inverse is not true. Consider the following 2×2 matrices: $\mathcal{A}_t^{(1)} = (2t\ \ 0;\ 0\ \ -2t)$, $\mathcal{A}_t^{(2)} = (2t\ \ 0;\ 0\ \ 2t)$ (; separates rows), $\|\mathcal{A}_t^{(1)}\| = \|\mathcal{A}_t^{(2)}\| = 2\sqrt{2}t$. Here $\Phi^{(1)}(t,s) = \Big(\exp\left(t^2 - s^2\right)\ \ 0;\ 0\ \ \exp\left(-t^2 + s^2\right)\Big)$, $\Phi^{(2)}(t,s) = \Big(\exp\left(t^2 - s^2\right)\ \ 0;\ 0\ \ \exp\left(t^2 - s^2\right)\Big)$ and $\|\Phi^{(1)}(t,s)\| \to \infty$, $\|\Phi^{(2)}(t,s)\| \to \infty$, if $t \to \infty$, i.e., both matrices are unstable. However, if we take $\bar{\mathcal{A}}_t^{(1)} = -(\mathcal{A}_t^{(1)})'$ and $\bar{\mathcal{A}}_t^{(2)} = -(\mathcal{A}_t^{(2)})'$, then $\bar{\mathcal{A}}_t^{(1)} = (-2t\ \ 0;\ 0\ \ 2t)$, $\bar{\mathcal{A}}_t^{(2)} = (-2t\ \ 0;\ 0\ \ -2t)$, then $\bar{\Phi}^{(1)}(t,s) = (\exp(-t^2+s^2)\ \ 0;\ 0\ \ \exp(t^2-s^2))$, which also corresponds to an unstable matrix, and $\bar{\Phi}^{(2)}(t,s) = (\exp(-t^2+s^2)\ \ 0;\ 0\ \ \exp(-t^2+s^2))$ will characterize superexponential stability. Thus, of the two unstable matrices $\mathcal{A}_t^{(1)}$ and $\mathcal{A}_t^{(2)}$ the matrix $\mathcal{A}_t^{(2)}$ is anti-stable, and $\mathcal{A}_t^{(1)}$ is not anti-stable.

Assumption $\mathcal{B}$ introduced above specifies constraints imposed on the matrix B_t, which characterizes the contribution of the control action to the dynamics of the system state. As we will show later, Assumption $\mathcal{B}$ will ensure the possibility of superexponential stabilization for a linear deterministic system, i.e., that there exists a piecewise continuous matrix K_t such that the matrix $A_t + B_tK_t$ is superexponentially stable. The design of control laws in the form of state feedback is a common approach used to stabilize not only linear [1, 3, 7, Ch. 6] but also nonlinear systems, see, e.g., [2, 23]. We note a number of properties of system (1) that do not let us use of the previously proposed methods. First, the situation when $\|A_t\| \to \infty$ as $t \to \infty$ is not covered by stabilizability cases of autonomous systems and systems with bounded coefficients; see, e.g., [1, 3]. Second, the requirement of superexponential stabilization with the possibility of $\|K_t\| \to \infty$ as $t \to \infty$ does not satisfy the key assumptions formulated for systems with $\|A_t\| \to \infty$ in [5, 6, 24]. We also note that the standard condition, which is sufficient for the stabilizability of systems with bounded coefficients, namely controllability of a pair of matrices (A_t, B_t), see [1], in case when $\|A_t\| \to \infty$ as $t \to \infty$ can only provide non-uniform in time stabilization (when in Definition 1 $\kappa = \kappa(s)$ is a function of s and $\kappa(s) \to \infty$ as $s \to \infty$), as shown in [13, 15]. As a result, in our situation we formulate Assumption $\mathcal{B}$ for system parameters.

3. MAIN RESULTS

As has been noted in [19], derivation the control U^* optimal on average over an infinite time horizon can be done by solving a problem of the form

$$\limsup_{T\to\infty} \mathrm{E}\mathcal{K}_T(U) \to \inf_{U\in\mathcal{U}}, \tag{4}$$

where $\mathrm{E}\mathcal{K}_T(U)$ is the expectation of some functional $\mathcal{K}_T(U)$ depending on the admissible control $U \in \mathcal{U}$ and the length of the planning horizon T. As an example we can cite the well-known long-run average cost criterion $\mathrm{E}\mathcal{K}_T(U) = \mathrm{E}J_T(U)/T$ for (1) and (2) with bounded coefficients, which was later extended and adjusted in [19, 20, 25], in the sense of refining the normalization of $\mathrm{E}J_T(U)$ and depicting the specific factors affecting the system dynamics. The criterion defined in this study, see (3), also belongs to the class of long-term averages. Note that in the design of the criterion (4) and its subsequent analysis we use an approach (see also [19] for a detailed description) based on finding the stable feedback control law $U_t^* = -R_t^{-1}B_t'\Pi_tX_t^*$, whose structure contains the solution of the Riccati equation (provided that it exists):

$$\dot{\Pi}_t + \Pi_tA_t + A_t'\Pi_t - \Pi_tB_tR_t^{-1}B_t'\Pi_t + Q_t = 0. \tag{5}$$

By solution of (5) we mean a function Π_t such that it yields a correct equality when substituted into (5). The works [19, 21, 26] show that a suitable normalization of the expected value of the functional $\mathrm{E}J_T(U)$ for criterion (4) includes an estimate on the change of Π_t, which is then used to adjust the variance of cumulative disturbances $\int_0^T \|G_t\|^2\,dt$. In particular, it was found in [19] that $\limsup_{t\to\infty}(\|\Pi_t\|\delta_t) < \infty$, where δ_t is the given stability rate, which contributed to introducing the adjusted extended long-run average cost criterion with $\mathrm{E}K_T(U) = \mathrm{E}J_T(U)/\int_0^T(1/\delta_t)\|G_t\|^2\,dt$. For the case of control system (1) and (2), under Assumptions $\mathcal{A}$ and $\mathcal{B}$ we further carry out the corresponding study dealing with the Riccati equation and specifying the criterion, and then prove the optimality of U^* from the point of view of the designed criterion. The next statement establishes the existence of a symmetric nonnegative definite solution of the Riccati equation (5), estimates its variations, and determines the stabilizing properties of the linear control law $u_t = -R_t^{-1}B_t'\Pi_t x_t$ in the deterministic system $dx_t = A_t x_t dt + B_t u_t dt$.

Lemma. *Suppose that Assumptions $\mathcal{A}$ and $\mathcal{B}$ hold. Then there exists an absolutely continuous function Π_t, $t \geqslant 0$, with values in the set of nonnegative definite symmetric matrices, that satisfies the Riccati equation (5), while $c_1\delta_t I \leqslant \Pi_t \leqslant c_2\delta_t I$, where $c_1, c_2 > 0$ are some constants. The matrix $A_t - B_tR_t^{-1}B_t'\Pi_t$ is $\tilde{\delta}_t$-superexponentially stable with $\tilde{\delta}_t = \lambda\delta_t$, where δ_t is the anti-stability rate of the matrix A_t, and λ is some positive constant.*

Proofs of the lemma and subsequent results are given in the Appendix.

Remark 1. Under the assumptions of the lemma, the function $\Pi_t \geqslant 0$ with $t \geqslant 0$ that satisfies (5) can be obtained as the limit for $T \to \infty$ of the solutions Π_t^T of Eq. (5) with boundary condition $\Pi_T^T = 0$ (here the superscript ^{T} denotes the solution of the equation with the boundary condition), i.e., $\lim_{T\to\infty} \Pi_t^T = \Pi_t$. For systems with bounded coefficients, this fact is well known, see [8, Theorem 3.5, p. 267], and in our case, the feasibility of passing to this limit is established in the proof of the lemma.

In order to study the optimality of the stable feedback control law U^* in the stochastic system, we will need the following technical condition that relates the admissible diffusion matrix G_t and anti-stability rate δ_t:

Assumption $\mathcal{G}$.

$$\lim_{T\to\infty} \frac{\delta_T^2\|G_T\|^2}{\int\limits_0^T \delta_t\|G_t\|^2\,dt} = 0. \tag{6}$$

To characterize the optimality of U^*, we will also use an approach that compares unnormalized values of objective functionals under different controls, based on the concept of the so-called overtaking optimality, see, e.g., [20].

Definition 3 (cf. [20]). A control $U^* \in \mathcal{U}$ has the overtaking optimality property on average (is overtaking optimal on the average) over an infinite time horizon if for any number $\epsilon > 0$ there exists $T_0 > 0$ such that for any admissible control $U \in \mathcal{U}$ the inequality

$$\mathrm{E}J_T(U^*) < \mathrm{E}J_T(U) + \epsilon \ \textit{holds for every}\ T > T_0. \tag{7}$$

The main result of this work is the following statement.

Theorem. *Suppose that Assumptions $\mathcal{A}$ and $\mathcal{B}$ hold. Then the control law of the form*

$$U_t^* = -R_t^{-1}B_t'\Pi_t X_t^*, \tag{8}$$

where process X_t^*, $t \geqslant 0$, is defined by equation

$$dX_t^* = (A_t - B_tR_t^{-1}B_t'\Pi_t)X_t^*dt + G_t dw_t, \quad X_0^* = x, \tag{9}$$

is the solution to the problem

$$\limsup_{T\to\infty} \frac{EJ_T(U)}{\int\limits_0^T \delta_t \|G_t\|^2\, dt} \to \inf_{U\in\mathcal{U}}. \tag{10}$$

Moreover, the matrix function $\Pi_t \geqslant 0$ *with* $t \geqslant 0$ *satisfies the Riccati equation* (5) *and has the properties formulated in the lemma. The value of the criterion on the optimal control* $J^* = \limsup_{T\to\infty} \frac{\mathrm{E}J_T(U^*)}{\int_0^T \delta_t\|G_t\|^2\,dt}$ *is a finite positive number*:

$$0 < J^* = \limsup_{T\to\infty} \frac{\int\limits_0^T \mathrm{tr}(G_t'\Pi_t G_t)\,dt}{\int\limits_0^T \delta_t\|G_t\|^2\,dt} < \infty,$$

where δ_t *is the anti-stability rate of the matrix* A_t, $\mathrm{tr}(\cdot)$ *denotes the trace of a matrix. In addition, if* $\|G_t\|\delta_t \to 0$, $t\to\infty$, *then control* U^* *is also overtaking optimal on average over an infinite time horizon.*

Remark 2. For a deterministic control system ($G_t \equiv 0$), subject to Assumptions $\mathcal{A}$ and $\mathcal{B}$, the strategy U^* will be the solution to problem $\limsup_{T\to\infty} J_T(U) \to \inf_{U\in\mathcal{U}}$, and the value is $\limsup_{T\to\infty} J_T(U^*) = x'\Pi_0 x$.

The criterion in (10) can also be viewed as an adjusted extended long-run average cost criterion; see [19, 21]. In contrast to the case of control systems with subexponentially [21] and superexponentially [19] stable state matrices, the adjustment is carried out in the direction of increasing the normalization of the expected value of the objective functional (multiplying $\|G_t\|^2$ by δ_t in the integrand of (10)).

Remark 3. According to the results from the statements of the lemma and the theorem, it can be concluded that the optimal control U^* is stabilizing for a deterministic system (see the lemma), and in a stochastic system such control U^* stabilizes the growth of the expected value of the objective functional at $T\to\infty$, which does not exceed the value of $\int_0^T \delta_t\|G_t\|^2\,dt$ up to a multiplicative constant. At the same time, the possibility of stabilization by control U^* of the corresponding optimal trajectory X_t^*, $t\geqslant 0$, will depend on the behavior of the diffusion matrix G_t. Based on the results of [16] on the stability of the process in the mean square, i.e., $\mathrm{E}\|X_t^*\|^2 \to 0$ for $t\to\infty$, see [27, p. 171], it suffices to require $\|G_t\|^2/\delta_t \to 0$, $t\to\infty$, which, in particular, holds when using an overtaking optimal strategy U^* (see the condition in the theorem) or a bounded diffusion matrix G_t. As part of the stronger condition $(\|G_t\|^2/\delta_t)\ln(\int_0^t \delta_v\,dv) \to 0$, $t\to\infty$, also see [16], the trajectory also satisfies stochastic stability with probability one according to the definition from [28, p. 111] when $\|X_t^*(\omega)\| \to 0$ almost surely as $t\to\infty$, i.e., for almost all $\omega\in\Omega$. Such a behavior of X_t^* represents a significant difference compared to the behavior of the optimal trajectory for stochastic linear controllers where the noise depends on the state or on the control (i.e., when in (1) instead of $G_t dw_t$ the noise terms have the form $X_t'G_t dw_t$ or $U_t'G_t dw_t$). In such systems, the multiplicative nature of random disturbances naturally gives rise to stabilizing properties of the corresponding optimal control strategy, see [27, Ch. 8].

4. ANALYSIS OF OPTIMALITY CONDITIONS AND EXAMPLES

Let us analyze the technical condition (6), under which the main statement of the theorem holds. Introducing the notation $\Gamma_T = \int_0^T \delta_t\|G_t\|^2\,dt$, we can rewrite (6) as

$$\lim_{T\to\infty} \frac{d\Gamma_T/dT}{\Gamma_T}\delta_T = 0. \tag{11}$$

Thus, (6) means that the growth of the norm of the criterion should be rather slow in relation to the function δ_t that characterizes the anti-stability of the state matrix. As can be seen from (11), a necessary condition here is that the rate of change of Γ_T tends to zero. On the other hand, if we let $\tilde{G}_t = \delta_t G_t$ ("strengthen" the perturbation matrix), then (6) will take the form

$$\lim_{T\to\infty} \frac{\|\tilde{G}_T\|^2}{\int\limits_0^T (1/\delta_t)\|\tilde{G}_t\|^2\,dt} = 0. \tag{12}$$

The denominator in (12) coincides with the norm of the criterion for systems with superexponentially stable state matrix and diffusion matrix $\tilde{G}_t$, see [19]. Next, we consider a system with bounded A_t and assume that for such a system there exists an optimal stable feedback control law $\bar{U}_t^*$. It is known (see [20]) that the optimality of $\bar{U}^*$ in the corresponding stochastic system with diffusion matrix $\tilde{G}_t$ can be investigated using the concept of g-optimality on average over an infinite time horizon, which holds in case when $\limsup_{T\to\infty} g_T(\mathrm{E}J_T(\bar{U}^*) - \mathrm{E}J_T(U)) \leqslant 0$ for every $U \in \mathcal{U}$ for a given function $g_T > 0$, $T > 0$. This approach allows us to estimate the order of change for the difference of the expectations of objective functionals, in contrast to the long-term averages, comparing the limit (as $T\to\infty$) normalized values of $\mathrm{E}J_T(U)$. In particular, the function $g_T = 1/\int_0^T \|\tilde{G}_t\|^2\,dt$ is the norm of the extended long-run average cost criterion, see [20, 29]. Then, if (12) holds it means that the standard stochastic system with matrix $\tilde{G}_t$ satisfies g-optimality with a more slowly increasing normalizing function $g_T = 1/\int_0^T (1/\delta_t)\|\tilde{G}_t\|^2\,dt$, taken from the criterion for superstable systems.

The following example shows that (6) allows to analyze control systems with various types of changes in the parameters of disturbances in time.

Example 1. Consider the case $\|G_t\|^2 \sim 1/\delta_t^m$, where m is a real number (the $\sim$ sign indicates that two functions are asymptotically of the same order: $f_t \sim g_t$ if $\lim_{t\to\infty}(f_t/g_t) = c \neq 0$). The case when $m = 0$ corresponds to a constant diffusion matrix, for $m > 0$ we have the so-called damped perturbations, for $m < 0$, increasing disturbances.

(a) $m > 2$: in this case $\|G_t\|^2\delta_t^2 \to 0$, $t\to\infty$, i.e., a stronger condition is satisfied than (6), leading to overtaking optimality on average over an infinite time horizon;

(b) $m = 2$: condition (6) is fulfilled with $\lim_{t\to\infty}\int_0^t (1/\delta_s)\,ds \to \infty$, i.e., it is possible to consider only rather slowly growing functions of the anti-stability rate, for example $\delta_t \sim t^k$, $0 < k \leqslant 1$;

(c) $m < 2$, $m \neq 1$: relation (6) holds if $\dot{\delta}_t \to 0$, $t\to\infty$, i.e., for slowly growing functions δ_t, for example $\delta_t \sim \ln t$;

(d) $m = 1$: validity of (6) is ensured if $\delta_t/t \to 0$, $t\to\infty$, when the anti-stability rate grows slower than the linear function, in particular for $\delta_t \sim t^k$, $0 < k < 1$.

To illustrate the application of the main statements obtained in this work (the theorem), we consider the following Example 2.

Example 2. The control system for a scalar process, see (1), (2) for $n = 1$, has the following form: $dX_t = (t+1)X_t dt + \sqrt{2}U_t dt + (t+1)^{-1}dw_t$, $X_0 = 1$, $J_T(U) = \int_0^T [X_t^2 + (t+1)^2((t+1)^2+1)^{-1}U_t^2]dt$. Here $A_t = t+1$, $B_t = \sqrt{2}$, $G_t = 1/(t+1)$, $x = 1$, $Q_t = 1$, $R_t = (t+1)^2((t+1)^2+1)^{-1}$ (also $1/2 \leqslant R_t \leqslant 1$). It is easy to see that the coefficients of the system satisfy previous assumptions: $A_t = t+1$ is superexponentially anti-stable with rate $\delta_t = t+1$, $B_t'B_t = 2 > 0$. In this case, the Riccati equation (5) takes the form

$$\dot{\Pi}_t + 2(t+1)\Pi_t - 2(1+(t+1)^{-2})\Pi_t^2 + 1 = 0 \tag{13}$$

and has a solution with the properties defined in the lemma. Indeed, $\Pi_t = t+1$ will be the solution of (13). The function Π_t can also be obtained (see Remark 1) as a limit $\lim_{T\to\infty}\Pi_t^T = \Pi_t$, where

$$\Pi_t^T = (t+1)\left[1 + 0.5(Z(T,t)-2)^{-1}\right],$$

$$Z(T,t) = \exp\left\{-(t+1)^2\right\}\frac{[\Psi_T - \Psi_t]}{(t+1)^3} + \exp\left\{(T+1)^2 - (t+1)^2\right\}\frac{[1-(T+1)(t+1)^{-1}]}{(t+1)^2}$$

with $\Psi_x = \int \exp\{-(x+1)^2\}\,dx$. At the same time, $A_t - B_t R_t^{-1} B_t' = -(t+1) - 2(t+1)^{-1}$ is superexponentially stable with rate $\tilde{\delta}_t = t+1$. Further, the function $G_t = 1/(t+1)$ satisfies Assumption $\mathcal{G}$ (see also item 6 of Example 1), $\int_0^T \delta_t G_t^2\,dt = \ln(T+1)$, therefore, by Theorem 1 the control law $U_t^* = -\sqrt{2}[(t+1)+(t+1)^{-1}]X_t^*$, with process dynamics $dX_t^* = [-(t+1)-2(t+1)^{-1}]X_t^*dt + (t+1)^{-1}dw_t$, $X_0^* = 1$, is the solution of problem

$$\limsup_{T\to\infty}\{EJ_T(U)/\ln(T+1)\} \to \inf$$

and the value $J^* = 1$, since $\int_0^T \Pi_t G_t^2\,dt = \ln(T+1)$.

5. CONCLUSION

In this work, we have considered the control problem over an infinite time horizon for a linear stochastic system with a superexponentially anti-stable (that is, extremely unstable) matrix A_t in the state equation. Such extra instability means that the lower bound for the norm of the corresponding fundamental matrix grows exponentially at an unbounded rate of δ_t, $\delta_t \to \infty$, as $t \to \infty$. We have shown that the control law (8) and (9) in the form of linear state feedback is a solution to the problem (10) with an adjusted extended long-run average cost criterion (see the theorem). The designed criterion contains the normalized expected value of the quadratic objective functional. The normalizing function $\Gamma_T = \int_0^T \delta_t \|G_t\|^2\,dt$ is the sum of the variances of the components of the vector $\mathcal{Z}_T = \int_0^T \sqrt{\delta_t} G_t\,dw_t$ of the cumulative amplified disturbances on the system. In contrast to the previously considered case of a δ_t-superexponentially stable matrix A_t, see [19], where the normalization $\int_0^T (1/\delta_t)\|G_t\|^2\,dt$ was defined, in this situation the anti-stability rate δ_t increases the value of Γ_T.

As a direction for further research, we note problems with stronger (in a stochastic sense) optimality criteria, when in (4) we minimize not the expected values but rather the normalized objective functionals as random variables in systems with superstable or super-unstable state matrices.

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APPENDIX

Proof of Lemma. Let $\tilde{\Pi}_t = \Pi_t/\delta_t$, and consider the Riccati equation for the function $\tilde{\Pi}_t$:

$$\dot{\tilde{\Pi}}_t + \tilde{\Pi}_t\tilde{A}_t + \tilde{A}_t'\tilde{\Pi}_t - \tilde{\Pi}_t B_t (R_t/\delta_t)^{-1} B_t'\tilde{\Pi}_t + Q_t/\delta_t = 0, \tag{A.1}$$

where $\tilde{A}_t = A_t + (1/2)(\dot{\delta}_t/\delta_t)I$. Equation of the form (A.1) arises in the control system $dx_t = \tilde{A}_t x_t dt + B_t u_t dt$, $x_{t_0} = \tilde{x}$, $J_{T,t_0}(u) = \int_{t_0}^T (1/\delta_t)(x_t' Q_t x_t + u_t' R_t u_t)\,dt$ with a superexponentially anti-stable matrix $\tilde{A}_t$, $\tilde{x}$ is an arbitrary initial state vector, $t_0 \geqslant 0$ is a fixed point in time. At the same time, the anti-stability rate is $\delta_t^{(0)} = \tilde{\lambda}\delta_t$, $0 < \tilde{\lambda} < 1$, which follows from the condition $\dot{\delta}_t/\delta_t^2 \to 0$, $t \to \infty$, in Assumption $\mathcal{A}$. It is well known [8, Theorem 3.4, p. 253] that for a finite T problem $J_{T,t_0}(u) \to \min$ has a solution $u_t^{*T} = -(R_t/\delta_t)^{-1}B_t'\Pi_t^T x_t^{*T}$, with the value of the functional on the optimal control $J_{T,t_0}(u^{*T}) = \tilde{x}\tilde{\Pi}_{t_0}^T\tilde{x}$, where the symmetric matrix $\tilde{\Pi}_t^T \geqslant 0$ is the solution of Eq. (A.1) with boundary condition $\tilde{\Pi}_T^T = 0$. We construct an alternative control that stabilizes the system and does not depend on T:

$$u_t^{(0)} = K_t x_t^{(0)} = -k\delta_t B_t'(B_t B_t')^{-1}x_t^{(0)}.$$

Note that, by Assumption $\mathcal{B}$, this control exists, and the constant $k > 0$ can be chosen in such a way as to ensure δ_t-superexponential stability of the matrix $\tilde{A}_t + B_t K_t = \tilde{A}_t - k\delta_t I$. Therefore,

$$\tilde{x}\tilde{\Pi}_{t_0}^T\tilde{x} \leqslant J_{T,t_0}(u^{(0)}) \leqslant c\|\tilde{x}\|^2 \int\limits_{t_0}^{T} \exp\left\{-\int\limits_{t_0}^{t} 2\delta_v\, dv\right\}(1/\delta_t + \delta_t)\, dt \leqslant \tilde{c}\|\tilde{x}\|^2.$$

Hereafter we denote by c and $\tilde{c}$ some positive constants whose specific values do not matter and can vary from formula to formula. Thus, $\tilde{\Pi}_{t_0}^T$ is a non-decreasing (in T) and bounded function. Standard considerations (see [8, p. 268]) lead to the fact that there exists a limit $\lim_{T\to\infty}\tilde{\Pi}_t^T = \tilde{\Pi}_t$ satisfying (A.1) and possessing the same properties as $\tilde{\Pi}_t^T$. Since $\Pi_t = \delta_t\tilde{\Pi}_t$, the upper boundedness of $\tilde{\Pi}_t$ implies the relation $\Pi_t \leqslant c_2\delta_t I$ with some constant $c_2 > 0$. To establish a lower bound on the change Π_t, consider the function $\bar{\Pi}_t = \Pi_t^{-1}$, which is a solution to the equation

$$\dot{\bar{\Pi}}_t - \bar{\Pi}_t'A_t' - A_t\bar{\Pi}_t - \bar{\Pi}_t Q_t\bar{\Pi}_t + B_t R_t^{-1}B_t' = 0. \tag{A.2}$$

Equation (A.2) also belongs to the class of Riccati equations and corresponds to a control system with a δ_t-superexponentially stable matrix $-A_t'$. For such systems, it is known [19] that $\bar{\Pi}_t \leqslant \bar{c}_1(1/\delta_t)$, which implies $\Pi_t \geqslant c_1\delta_t I$ for some constant $c_1 > 0$.

Turning to the study of the stability of the matrix $A_t - B_t R_t^{-1}B_t'\Pi_t$, we consider the linear equation $dz_t = (A_t - B_t R_t^{-1}B_t'\Pi_t)z_t dt$, $z_{t_0} = z$, and write $d(z_t'\Pi_t z_t) = (-z_t'Q_t z_t - z_t'\Pi_t B_t R_t^{-1}B_t'\Pi_t z_t)dt$. Assumption $\mathcal{B}$ and the resulting double inequality $c_1\delta_t I \leqslant \Pi_t \leqslant c_2\delta_t I$ lead to the following sequence of bounds:

$$\begin{gathered} d(z_t'\Pi_t z_t) \leqslant -\lambda\delta_t(z_t'\Pi_t z_t)dt, \\ z_t\Pi_t z_t \leqslant z_{t_0}\Pi_{t_0}z_{t_0}\exp\left\{-\int\limits_{t_0}^{t}\lambda\delta_v\, dv\right\}, \\ \|z_t\|^2 \leqslant \kappa\exp\left\{-\int\limits_{t_0}^{t}\lambda\delta_v\, dv\right\}\|z\|^2 \end{gathered} \tag{A.3}$$

with some positive constants λ and κ. Relation (A.3) implies the $\tilde{\delta}_t$-superexponential stability of the matrix $A_t - B_t R_t^{-1}B_t'\Pi_t$ with $\tilde{\delta}_t = \lambda\delta_t$. This completes the proof of the lemma.

Proof of Theorem. Due to Assumptions $\mathcal{A}$ and $\mathcal{B}$, the statement of the lemma on the existence and properties of the solution of the Riccati equation (5) holds, and we can also define the control law U^* in the form (8) and (9). Fixing an arbitrary competing control $U \in \mathcal{U}$ and the corresponding process X_t, we set $x_t = X_t - X_t^*$ and $u_t = U_t - U_t^*$. The pair $(x_t, u_t)_{t\leqslant T}$ satisfies the equation

$$dx_t = (A_t x_t + B_t u_t)dt, \quad x_0 = 0. \tag{A.4}$$

Since $Q_t \geqslant qI$, there exists a number $k > 0$ such that the matrix $A_t - k\delta_t\sqrt{Q_t}$ is δ_t-stable. Then, transforming (A.4), we get that

$$dx_t = (A_t - k\delta_t\sqrt{Q_t})x_t dt + k\delta_t\sqrt{Q_t}x_t dt + B_t u_t dt, \quad x_0 = 0,$$

and

$$\|x_t\| \leqslant c\int_0^t \exp\left\{-\int_s^t \delta_v\,dv\right\}\left(\delta_s\|\sqrt{Q_s}x_s\| + \|\sqrt{R_s}u_s\|\right) ds.$$

According to the Cauchy–Bunyakovsky inequality,

$$\|x_t\|^2 \leqslant \tilde{c}\int_0^t \exp\left\{-\int_s^t \delta_v\,dv\right\}\left(\delta_s x_s' Q_s x_s + u_s' R_s u_s\right)\,ds,$$

which implies the bound

$$\frac{1}{\delta_t}\|x_t\|^2 \leqslant \tilde{c}\int_0^t (x_s' Q_s x_s + u_s' R_s u_s)\,ds. \tag{A.5}$$

Further, the difference $J_T(U^*) - J_T(U)$ can be represented as

$$J_T(U^*) - J_T(U) = 2x_T'\Pi_T X_T^* - \int_{t_0}^T (x_t' Q_t x_t + u_t' R_t u_t)\,dt - 2\int_0^T x_t'\Pi_t G_t dw_t. \tag{A.6}$$

Taking into account (A.5), properties of the function Π_t (see the lemma), and applying the elementary inequality $2ab \leqslant a^2/c + cb^2$, for an arbitrary $c > 0$ and any numbers a and b (A.6) can be estimated as

$$J_T(U^*) - J_T(U) \leqslant \tilde{c}_1\delta_T^3\|X_T^*\|^2 - 2\int_0^T x_t'\Pi_t G_t dw_t \tag{A.7}$$

with some constant $\tilde{c}_1 > 0$. Taking the expectation in (A.7), we get

$$\mathrm{E}J_T(U^*) \leqslant \mathrm{E}J_T(U) + \tilde{c}_1\delta_T^3\mathrm{E}\|X_T^*\|^2. \tag{A.8}$$

Since the matrix $A_t - B_t R_t^{-1} B_t'\Pi_t$ in Eq. (9) is $\tilde{\delta}_t$-superexponentially stable,

$$\delta_T^3\mathrm{E}\|X_T^*\|^2 \leqslant c\delta_T^3\left(\exp\left\{-\int_0^T 2\tilde{\delta}_v\,dv\right\}\|x\|^2 + \int_0^T \exp\left\{-\int_t^T 2\tilde{\delta}_v\,dv\right\}\|G_t\|^2\,dt\right). \tag{A.9}$$

We note that the condition $\dot{\delta}_t/\delta_t^2 \to 0$, $t \to \infty$, from Assumption $\mathcal{A}$ implies the convergence $\delta_T^3\exp\{-\int_0^T 2\tilde{\delta}_v\,dv\} \to 0$ as $T \to \infty$. The second term in (A.9) can be rewritten as $L_T = \delta_T^3\int_0^T \exp\{-\int_t^T 2\tilde{\delta}_v\,dv\}\|G_t\|^2 dt = \int_0^T \exp\{-\int_t^T 2\bar{\delta}_v\,dv\}\|\bar{G}_t\|^2\,dt$, $\bar{\delta}_t = \tilde{\delta}_t + (3/2)(\dot{\delta}_t/\delta_t)$, $\bar{G}_t = \delta_t^{3/2}G_t$. Using the L'Hôpital rule, it is easy to show that condition $\|G_T\|\delta_T \to 0$, $T \to \infty$, will be sufficient for $L_T \to 0$ with $T \to \infty$, and Assumption $\mathcal{G}$ guarantees $L_T/\left(\int_0^T \delta_t\|G_t\|^2\,dt\right) \to 0$ as $T \to \infty$. Taking into account the reasoning above, we get the overtaking optimality on average for U^* if

$\|G_T\|\delta_T \to 0$, $T \to \infty$, or optimality on average over an infinite time horizon according to the criterion with normalization $\int_0^T \delta_t \|G_t\|^2\, dt$:

$$\limsup_{T\to\infty} \frac{\mathrm{E}J_T(U^*)}{\int\limits_0^T \delta_t \|G_t\|^2\, dt} \leqslant \limsup_{T\to\infty} \frac{\mathrm{E}J_T(U)}{\int\limits_0^T \delta_t \|G_t\|^2\, dt}.$$

The expected value of the objective functional on the optimal control is given by the formula $\mathrm{E}J_T(U^*) = x'\Pi_0 x - \mathrm{E}[(X_T^*)'\Pi_T X_T^*] + \int_0^T \mathrm{tr}(G_t'\Pi_t G_t)\, dt$. Since $\mathrm{E}[(X_T^*)'\Pi_T X_T^*] \leqslant c\delta_T \mathrm{E}\|X_T^*\|^2$ and $\delta_T \mathrm{E}\|X_T^*\|^2 / \left(\int_0^T \delta_t \|G_t\|^2\, dt\right) \to 0$, $T \to \infty$, and $\int_0^T \mathrm{tr}(G_t'\Pi_t G_t)\, dt \geqslant c_1 \int_0^T \delta_t \|G_t\|^2\, dt$ (see the lemma), the limit value is

$$0 < J^* = \limsup_{T\to\infty} \frac{\mathrm{E}J_T(U^*)}{\int\limits_0^T \delta_t \|G_t\|^2\, dt} = \frac{x'\Pi_0 x}{\int\limits_0^\infty \delta_t \|G_t\|^2\, dt} + \limsup_{T\to\infty} \frac{\int\limits_0^T \mathrm{tr}(G_t'\Pi_t G_t)\, dt}{\int\limits_0^T \delta_t \|G_t\|^2\, dt} < \infty.$$

This completes the proof of the theorem.

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