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STOCHASTIC SYSTEMS

An Analytic Study of the Ornstein–Uhlenbeck Process with Time-Varying Coefficients in the Modeling of Anomalous Diffusions

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Abstract—We consider the problem of modeling anomalous diffusions with the Ornstein–Uhlenbeck process with time-varying coefficients. An anomalous diffusion is defined as a process whose mean-squared displacement non-linearly grows in time which is nonlinearly growing in time. We classify diffusions into types (subdiffusion, normal diffusion, or superdiffusion) depending on the parameters of the underlying process. We solve the problem of finding the coefficients of dynamics equations for the Ornstein–Uhlenbeck process to reproduce a given mean-squared displacement function.

Keywords: process Ornstein–Uhlenbeck process, anomalous diffusion, Riccati equation, filtering, linear controller

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1. INTRODUCTION

The Ornstein–Uhlenbeck process is widely used in dynamic modeling problems under the influence of random factors (see, e.g., [1, 2]) and also plays an important role in optimal control theory (in particular, it arises in the solution of the stochastic linear controller problem, see [3]). At first, the process with this name (see [4]) was used to describe the velocity of a particle moving in a random environment, which can be implemented in the form of a linear stochastic differential equation:

$$dX_t = -\kappa_1 X_t dt + \sigma dw_t, \qquad X_0 = x, \tag{1}$$

where X_t, $t \geqslant 0$, is the velocity process; $\kappa_1, \sigma > 0$ are constants reflecting respectively the rate of convergence of X_t to its asymptotic average value $\mathrm{E}X_\infty = 0$ and the contribution of the random component in the process dynamics; w_t, $t \geqslant 0$, is the standard one-dimensional Wiener process; x is the initial value of the velocity. One of the main characteristics in the study of such a model is the displacement process $Y_T = \int_0^T X_t \, dt$, where $T \geqslant 0$ is the time of motion.

In what follows we will need a modified definition of asymptotic equivalence of two functions.

Definition 1. Functions f_t and g_t are called asymptotically equivalent with respect to the order of their changes, denoted $f_t \sim g_t$, if

$$\lim_{t\to\infty} \frac{f_t}{g_t} = c > 0, \text{ where } c \text{ is some constant.}$$

An important feature of (1) is that the mean-squared displacement D_T equal to $\mathrm{E}Y_T^2$ has order T ($D_T \sim T$), i.e., it is close to the displacement $Y_T^{(w)}$ defined by the Wiener process $Y_T^{(w)} = w_T$, where also $\mathrm{E}[Y_T^{(w)}]^2 = T$. Such processes are known as normal diffusions [2]. Advantages of the displacement model $Y_T = \int_0^T X_t\,dt$ compared to $Y_T^{(w)} = w_T$ include the existence of derivative with respect to time: $dY_t = X_t dt$, which is known to be lacking for $Y_t^{(w)}$ (trajectories w_t with probability one are not differentiable [5]). At the same time, processes where $D_T \not\sim T$ have been known for a long time; they should be called anomalous diffusions (see surveys in [6, 7]). A fundamental property of an anomalous diffusion is a nonlinear change in time of the corresponding mean-squared displacement. As we have noted above, the linear case corresponds to the standard Ornstein–Uhlenbeck process (1) and to the Wiener process w_t if as the model of the particle's position one takes the Brownian motion itself. For anomalous diffusions one often assumes a functional dependence of the mean-squared displacement D_T on time T, which has a power law: $D_T = T^\alpha$ ($\alpha > 0$). Here researchers distinguish subdiffusion ($0 < \alpha < 1$) and superdiffusion when $\alpha > 1$, also considering "superslow diffusions" with $D_T \sim \ln T$ and other slowly growing functions, see, e.g., [7]. In the present work, to define an anomalous diffusion we propose to consider a nonstationary Ornstein–Uhlenbeck process X_t, $t \geqslant 0$, with time-varying coefficients (in Eq. (1) the constants $(-\kappa_1)$, σ are replaced with piecewise continuous functions of time a_t, σ_t). Precise requirements on the coefficients will be shown below; so far we emphasize that these functions can be both unbounded for $t \to \infty$ and can tend to zero. In special cases, modeling of anomalous diffusions with this kind of a process was done in [6, 8] for power law type functions.

The purpose of this work is to classify diffusions into types (subdiffusion, normal diffusion, superdiffusion) based on known parameters of the underlying process X_t and solutions to the inverse problem of finding coefficients X_t for modeling a given mean-squared displacement D_T. We note that by the definition of displacement $Y_T = \int_0^T X_t\,dt$, in the study of anomalous diffusions an important question is to examine Y_T, the integrated Ornstein–Uhlenbeck process ("integrated" here is a term from probability and statistics used to denote integrals of Riemann type for random processes). Analysis of an integrated process also is of independent importance since it is widely used in practical applications (modeling cumulative volatility [9] and asset return [10], logarithm of population size [11], power consumption [12], geology [13] and others). Often it is Y_t which is available for observation instead of X_t [12, 14], and estimation of the parameters of X_t by known data on Y_t is usually done under stationarity assumptions [12, 14] for process X_t, which obviously does not hold in many situations (see, e.g., [15–17]). One possible way to construct Y_T with $D_T = \mathrm{E}Y_T^2 \not\sim T$ is dynamic scaling [13] of the displacement corresponding to the standard process (1), i.e., an external transformation that does not account for possible non-autonomous dynamics of the equations on X_t. In this work we will show that in order to model a known mean-squared displacement function D_T it suffices that the process parameters obey a relation based on Riccati equations known from filtering theory. The paper is organized as follows. In Section 2 we formulate the basic assumptions on the coefficients of dynamics equations, give a formal definition of anomalous diffusion, show a statement that lets one determine the diffusion type based on the parameters of process X_t, and give examples. Section 3 is devoted to solving the problem of finding parameters X_t in order to construct a given mean-squared displacement function and examples of applying the resulting relations.

2. THE NOTION OF ANOMALOUS DIFFUSION AND FINDING ITS TYPE BASED ON THE COEFFICIENTS OF THE GENERATING PROCESS

Suppose that on a complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$ we have a scalar random process X_t, $t \geqslant 0$, which is a Ornstein–Uhlenbeck process with time-varying coefficients defined by a non-autonomous linear stochastic differential equation

$$dX_t = a_t X_t dt + \sigma_t dw_t, \qquad X_0 = x, \tag{2}$$

where the initial state x is non-random; w_t, $t \geqslant 0$, is the standard Wiener process; a_t, σ_t, $t \geqslant 0$, are piecewise continuous functions of time. Note that for $\sigma_t \to 0$, $t \to \infty$, we have the case of the so-called attenuating disturbances (see [18, 19] for applications in control theory); if, on the other hand, $\sigma_t \to \infty$, then disturbances grow with time (see, e.g., the cognitive model in [16] or a description of the population dynamics in [20]), which certainly also influences the dynamics and statistical characteristics of the process (2).

A solution of (2) is a process of the form

$$X_t = \Phi(t,0)x + \int\limits_0^t \Phi(t,s)\sigma_s dw_s, \tag{3}$$

where $\Phi(t,s) = \exp\left\{\int_s^t a_v\, dv\right\}$.

Let us write down the main characteristics of the process (3); see, e.g., [21]:

expectation

$$\mathrm{E}X_t = \Phi(t,0)x,$$

second moment function

$$\mathrm{E}X_t^2 = \Phi^2(t,0)x^2 + \int\limits_0^t \Phi^2(t,s)\sigma_s^2\, ds, \tag{4}$$

covariance function

$$\mathrm{E}(X_t X_s) = \Phi(\chi,\tau)\mathrm{E}X_\tau^2, \quad \text{where} \quad \chi = \max(t,s), \quad \tau = \min(t,s). \tag{5}$$

We assume that function a_t, $t \geqslant 0$, guarantees stability with rate δ_t for solutions of the corresponding deterministic equation. Namely, there exists a function $\delta_t > 0$, $t \geqslant 0$, such that

(i) $$\limsup_{t\to\infty} \left(|a_t|/\delta_t\right) < \infty;$$

(ii) $$\Phi(t,s) \leqslant \kappa \exp\left\{-\int\limits_s^t \delta_v\, dv\right\}, \qquad s \leqslant t,$$

for some constant $\kappa > 0$;

(iii) $$\int\limits_0^t \delta_v\, dv \to \infty, \quad t \to \infty.$$

Note that function δ_t can be naturally called the stability rate since according to (ii) this function determines the rate of decrease for the upper bound for $\Phi(t,s)$. Next we comment on the introduced conditions. Condition (i) means that the chosen stability rate δ_t cannot be improved for the class of functions of the form $\delta_t^{(1)} = \lambda\delta_t$ ($\lambda > 0$ is a constant). For example, for an unbounded (for $t \to \infty$) a_t it is impossible to have optimal stability rate which is equal to a constant. Relation (ii) together with (iii) yields a definition of asymptotic stability for the solutions of (2) when $\sigma_t \equiv 0$, see [22].

Note that in case $\delta_t \equiv \kappa_1 > 0$ we get an exponential stability type; for $\delta_t \to 0$, $t \to \infty$, the resulting stability is weaker, subexponential, and for $\delta_t \to \infty$, on the contrary, we see superexponential stability (the terminology arose in the analysis of nonlinear equations, see, e.g., [23]).

Assuming that the initial position is zero, we define the displacement process

$$Y_T = \int_0^T X_t\, dt, \quad T \geqslant 0,$$

and find the mean-squared displacement

$$D_T = \mathrm{E}\left(\int_0^T X_t\, dt\right)^2. \tag{6}$$

We transform (6), using a formula for the variance of Riemann type integrals of random processes (see, e.g., [24])

$$D_T = \int_0^T \int_0^T \mathrm{E}(X_t X_s)\, ds\, dt;$$

substituting (5), we get

$$D_T = \int_0^T \int_0^t \Phi(t,s)\mathrm{E}X_s^2\, ds\, dt + \int_0^T \int_t^T \Phi(s,t)\mathrm{E}X_t^2\, ds\, dt. \tag{7}$$

It is interesting to study the behavior of D_T for $T \to \infty$, so if for some nondecreasing function $\tilde{D}_T > 0$ it holds that $\tilde{D}_T \sim D_T$ (see Definition 1), this asymptotically equivalent function will also characterize the order of change in the mean-squared displacement D_T. Note that due to (ii) and inequality $\Phi(t,s) \geqslant \kappa_0 \exp\{-\int_s^t \bar{\kappa}\delta_v\, dv\}$ (κ_0, $\bar{\kappa}$ are some positive constants) mean-squared displacement D_T can be estimated as

$$\kappa_0^3 D_T^{(1)} \leqslant D_T \leqslant \kappa^3 D_T^{(2)}, \tag{8}$$

where $D_T^{(1)}$ and $D_T^{(2)}$ are the mean-squared displacements defined based on (2) for the cases $a_t = -\bar{\kappa}\delta_t$ and $a_t = -\delta_t$ respectively. For these situations, we have also introduced separate notations for second moment functions (see (4)):

$$m_t^{(1)} = \exp\left\{-2\int_0^t \bar{\kappa}\delta_v\, dv\right\} x^2 + \int_0^t \exp\left\{-2\int_s^t \bar{\kappa}\delta_v\, dv\right\} \sigma_s^2\, ds, \tag{9}$$

$$m_t^{(2)} = \exp\left\{-2\int_0^t \delta_v\, dv\right\} x^2 + \int_0^t \exp\left\{-2\int_s^t \delta_v\, dv\right\} \sigma_s^2\, ds. \tag{10}$$

Integrating by parts, we transform representation (7) to the form

$$D_T = 2\int_0^T \int_0^t \Phi(t,s)\mathrm{E}X_s^2\, ds\, dt. \tag{11}$$

Next we formulate the definition of an anomalous diffusion.

Definition 2. Let $d_1 = \liminf_{t\to\infty} (D_T/T)$ and $d_2 = \limsup_{t\to\infty} (D_T/T)$. If $0 < d_1 \leqslant d_2 < \infty$ then the diffusion is called normal, otherwise it is anomalous: for $d_2 = 0$, a subdiffusion; for $d_1 = \infty$, a superdiffusion.

In the following statement we show conditions that let us classify diffusion (2) depending on the characteristics of processes that define mean-squared displacements $D_T^{(1)}$ and $D_T^{(2)}$.

Theorem 1. *Let* $d^{(1)} = \liminf_{t\to\infty} (m_t^{(1)}/\delta_t)$ *and* $d^{(2)} = \limsup_{t\to\infty} (m_t^{(2)}/\delta_t)$, *where* $m_t^{(1)}$ *and* $m_t^{(2)}$ *have been defined in* (9) *and* (10). *Then the following diffusion types are possible:*

1) *for* $0 < d^{(1)} \leqslant d^{(2)} < \infty$*—normal diffusion;*
2) *for* $d^{(2)} = 0$*—subdiffusion;*
3) *for* $d^{(1)} = \infty$*—superdiffusion.*

Proofs of Theorem 1 and the Theorem 2 shown below are given in the Appendix. Next we show sample applications of Theorem 1 for classification of diffusions under various coefficients a_t and σ_t.

Example 1. Let us consider Eq. (2) for an important special case [8] of power functions $a_t = -\delta_t$, $\sigma_t = \delta_t^{3/2}$, where $\delta_t = (1+t)^{\alpha-1}$ with power $\alpha > 0$, and classify the resulting diffusions. We use Theorem 1 for this purpose. First, $\mathrm{E}X_t^2 = m_t^{(1)} = m_t^{(2)} = \exp\left\{-2\int_0^t \delta_v\,dv\right\} x^2 + \int_0^t \exp\left\{-2\int_s^t \delta_v\,dv\right\} \delta_s^3\,ds$. Next we note that $\mathrm{E}X_t^2 \sim \delta_t^2$ with $\delta_t^2 = (1+t)^{2\alpha-2}$ for $\alpha \neq 0$ and $\mathrm{E}X_t^2 \sim \ln(t+1)/(t+1)^2$ for $\alpha = 0$. We find $d^{(1)} = d^{(2)} = \lim_{t\to\infty} (\mathrm{E}X_t^2/\delta_t) = \lim_{t\to\infty} \delta_t$ if $\alpha \neq 0$ and $d^{(1)} = d^{(2)} = 0$ in the case $\alpha = 0$. Then by Theorem 1 we have that: the value $0 \leqslant \alpha < 1$ corresponds to a subdiffusion; $\alpha = 1$, to normal diffusion; $\alpha > 1$, to superdiffusion. Indeed, it is easy to see (see also [8]) that the mean-squared displacement $D_T \sim T^\alpha$ $(\alpha > 0)$ or $D_T \sim \ln^3 T$ $(\alpha = 0)$.

In many cases, the statement of Theorem 1 readily enables us to classify the diffusion based on analyzing the relations between parameters δ_t and σ_t.

Corollary. *Consider the constants* $s_1 = \liminf_{t\to\infty}(\sigma_t^2/\delta_t)$, $s_2 = \limsup_{t\to\infty}(\sigma_t^2/\delta_t)$. *Then:*

(a) *for exponential stability type with* $\delta_t \equiv \kappa_1$: $0 < s_1 \leqslant s_2 < \infty$ *is normal diffusion;* $s_2 = 0$*—subdiffusion;* $s_1 = \infty$*—superdiffusion;*

(b) *for subexponential stability type with* $\delta_t \to 0$, $t \to \infty$: $s_1 > 0$*—superdiffusion;*

(c) *for superexponential stability type with* $\delta_t \to \infty$, $t \to \infty$: $s_2 < \infty$*—subdiffusion.*

Example 2. Let $\sigma_t^2 \sim \delta_t$ and suppose that $\sigma_t^2 = k^2\delta_t$, where $k > 0$ is a constant. In this case an increment of process X_t defined by Eq. (2) is proportional (with a time-varying coefficient δ_t) to the differential of the deviation of the displacement process Y_t from the scaled Brownian motion $Y_t^{(W)}$, see [6], which is also used as the displacement model in case of an anomalous diffusion in [25]. Namely, $dX_t = \delta_t dZ_t$, where $Z_t = Y_t^{(W)} - Y_t$, for $Y_t = \int_0^t X_s\,ds$, and $Y_t^{(W)} = \int_0^t (k/\sqrt{\delta_t})dw_s$. Note that the faster δ_t increases, i.e., the larger is the stability rate in Eq. (2), the more stable (in terms of the distribution) will the process $Y_t^{(W)}$ be and the smaller will be its contribution to the dynamics of Z_t. Indeed, the quadratic variation is $\langle Y_t^{(W)}\rangle = \mathrm{E}[Y_t^{(W)}]^2 = \int_0^t (k^2/\delta_t)\,dt$ and for the limiting value $\langle Y_\infty^{(W)}\rangle$ it will hold that $Y_t^{(W)} \to Y_\infty^{(W)}$ with probability one for $t \to \infty$, see [26], where $Y_\infty^{(W)}$ is a Gaussian random variable. Under the assumptions of the corollary, we have constants $s_1 = s_2 > 0$. Then it becomes obvious that for subexponential stability we have the resulting superdiffusion, exponential stability type leads to normal diffusion; superexponential, to subdiffusion.

3. DIFFUSION PARAMETER DETERMINATION FOR A GIVEN FUNCTION OF MEAN-SQUARED DISPLACEMENT

Suppose that we know the function D_T, $T \geqslant 0$, and the problem is to find $a_t = -\delta_t$ and σ_t for Eqs. (2) for which the corresponding displacement process would be asymptotically equivalent to D_T in mean squares. It turns out that this problem always has a solution under the natural condition that the mean-squared displacement function increases monotonely.

Theorem 2. *Let D_t be a three times differentiable function, and $D'_t > 0$, $t \geqslant 0$. Then there exists a pair of functions (δ_t, σ_t^2), where $\delta_t > 0$ defines the stability rate and $\sigma_t^2 > 0$, related by*

$$\dot{\delta}_t + \frac{3D''_t}{D'_t}\delta_t + 2\delta_t^2 + \frac{D'''_t}{D'_t} = \frac{2\sigma_t^2}{D'_t} \quad \delta_0 = \bar{\delta}, \tag{12}$$

($\bar{\delta} > 0$ is an arbitrary initial condition), such that for the displacement process

$$\tilde{D}_T = 2\int_0^T \exp\left\{-\int_0^t \delta_v\,dv\right\}\int_0^t \exp\left\{\int_0^s \delta_v\,dv\right\} m_s^{(2)}\,ds\,dt, \tag{13}$$

it holds that $\tilde{D}_T \sim D_T$. Here the function $m_t^{(2)}$ in (13) is defined by (10) with $x = 0$.

Remark 1. A known comparison theorem for solutions of Riccati equations (see [27, Theorem 4.1]) implies that δ_t monotonically increases with respect to σ_t^2. Let $\delta_t^{(1)}$ and δ_t^2 be the solutions of Eqs. (12) with the same initial condition $\bar{\delta}$ for different $\sigma_t^{(1)}$ and $\sigma_t^{(2)}$. If $\left(\sigma_t^{(1)}\right)^2 \geqslant \left(\sigma_t^{(2)}\right)^2$, $t \geqslant 0$, then $\delta_t^{(1)} \geqslant \delta_t^{(2)}$, $t \geqslant 0$. Consider (12) for $2\sigma_t^2 - D'''_t \equiv 0$, then (12) becomes a Bernoulli equation

$$\dot{\delta}_t + \frac{3D''_t}{D'_t}\delta_t + 2\delta_t^2 = 0, \quad \delta_0 = \bar{\delta}$$

whose solution can be written down in a closed form:

$$\tilde{\delta}_t = \frac{1}{(D'_t)^3\left[1/\bar{\delta} + \int_0^t 1/(D'_s)^3\,ds\right]}. \tag{14}$$

By the comparison theorem [27, Theorem 4.1] it follows that $\delta_t \geqslant \tilde{\delta}_t$, where δ_t is the solution of (12) for the case when $2\sigma_t^2 - D'''_t \geqslant 0$. For example, for $D_T = T$ we have the bound $\delta_t \geqslant 1/(\bar{\delta}^{-1} + t)$. The resulting boundary (14) will correspond to the case of diffusion with coefficient $\tilde{\sigma}_t^2 = D'''_t/2$ if $D'''_t > 0$, $t \geqslant 0$.

Remark 2. When finding δ_t and σ_t^2 in case of a normal diffusion ($D_T = T$) Eq. (12) takes the form $2\sigma_t^2 = 2\delta_t^2 + \dot{\delta}_t$. Suppose that coefficients δ_t and σ_t^2 are known, and we want to find out which diffusion type they generate. Let $2\tilde{\sigma}_t^2 = 2\delta_t^2 + \dot{\delta}_t > 0$. Then the condition from Theorem 1 can be rewritten as $\lim_{t\to\infty}[\mathrm{E}X_t^2/\delta_t] = \lim_{t\to\infty}[\sigma_t^2/(2\delta_t^2 + \dot{\delta}_t)] = \lim_{t\to\infty}[\sigma_t^2/\tilde{\sigma}_t^2]$, i.e., we compare the actual diffusion coefficient σ_t and theoretical one $\tilde{\sigma}_t$ corresponding to regular diffusion for a fixed stability rate δ_t.

Example 3. As an example, consider determination of the stability rate δ_t for a constant coefficient $\sigma_t \equiv \sigma > 0$ for three types of diffusions with the following mean-squared displacement functions: (a) $D_T = T$; (b) $D_T = (T+1)^2$; (c) $D_T = \ln(T+1)$. We use Theorem 2 and in all cases will assume that the initial value of stability rate $\delta_0 = \sigma$ is the same. Then the following situations hold:

(a) for $D_T = T$ Eq. (12) becomes a Riccati equation with constant coefficients

$$\dot{\delta}_t + 2\delta_t^2 = 2\sigma^2$$

and for initial condition $\delta_0 = \sigma$ it has a solution $\delta_t \equiv \sigma$, $t \geqslant 0$, i.e., it defines the exponential stability type;

(b) for $D_T = (T+1)^2$ Eq. (12) takes the form

$$\dot{\delta}_t + \frac{3}{t+1}\delta_t + 2\delta_t^2 = \frac{\sigma^2}{t+1}, \quad \delta_0 = \sigma,$$

and has a solution $\delta_t > 0$, $t \geqslant 0$, for $\sigma > 0$; based on our analysis of the dual problem for a linear controller (see the proof of Theorem 2) we find that $\delta_t \sim 1/\sqrt{t+1} = \delta_t^{(0)}$, i.e., stability will be subexponential, and the mean-squared displacement corresponding to $\delta_t^{(0)}$ is $D_T^{(0)} \sim (T+1)^2$;

(c) if $D_T = \ln(T+1)$ then (12) becomes the following equation:

$$\dot{\delta}_t - \frac{3}{t+1}\delta_t + 2\delta_t^2 = 2\sigma^2(t+1) - \frac{2}{t+1}, \quad \delta_0 = \sigma,$$

with a positive solution $\delta_t > 0$, $t \geqslant 0$, for $\sigma > 1$. Considering the dual problem for a linear controller, we conclude that $\delta_t \sim \sqrt{t+1} = \delta_t^{(0)}$ is the superexponential stability type. It is easy to see that in this case $D_T^{(0)} \sim \ln(T+1)$.

In conclusion we note that δ_t found in Example 3 and the corresponding constant diffusion coefficients σ in the modeling of D_T for cases (a)–(c) are in accordance with the conclusions made in Corollary:

(a) $\sigma^2/\delta_t = \sigma^2/\sigma$ is constant, and $\delta_t = \sigma$ does indeed generate normal diffusion ($D_T = T$);

(b) $\delta_t \sim 1/\sqrt{t+1}$ and $\sigma^2/\delta_t \to \infty$, $t \to \infty$, and we get superdiffusion ($D_T = (T+1)^2$);

(c) $\delta_t \sim \sqrt{t+1}$, $\sigma^2/\delta_t \to 0$, $t \to \infty$, then subdiffusion is obvious, and it does hold with $D_T = \ln(T+1)$.

4. CONCLUSION

In this work we have considered the problem of modeling anomalous diffusions with the Ornstein–Uhlenbeck process with time-varying coefficients. The statement of Theorem 1 on the classification of diffusions lets us conclude that anomalous diffusions may result from a disproportional change in the time variance of the process compared to the stability rate. The result formulated as a Riccati equation (12), where two parameters δ_t and σ_t turn out to be connected by a single relation, matches the remark made in [12] that it is impossible to independently determine the stability rate and diffusion coefficient when observing the displacement process (the work [12] analyzed a standard Ornstein–Uhlenbeck process (1)). At the same time, the relation shown in Theorem 2 between stability rate and the diffusion coefficient corresponding to mean-squared displacement D_T can also be used to solve the problem when one of the parameters (2) is already known. More precisely, if we fix δ_t or σ_t^2 then the second parameter is defined by (12) provided that the resulting function has given properties: $\sigma_t^2 > 0$ (for known δ_t) or $\delta_t > 0$, $\int_0^t \delta_v\, dv \to \infty$, $t \to \infty$ (for known σ_t^2).

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APPENDIX

Proof of Theorem 1. In case 1 function $c_1\delta_t \leqslant m_t^{(1)} \leqslant c_2\delta_t$ and $c_1\delta_t \leqslant m_t^{(2)} \leqslant c_2\delta_t$, for some constants $c_1, c_2 > 0$ at time $t > t_0$ $(t_0 \geqslant 0)$. Then these relations and representation (11) yield the bound

$$I_T + c_1\int\limits_0^T\left[1-\exp\left\{-\int\limits_{t_0}^t \delta_v dv\right\}\right]dt \leqslant D_t^{(2)}$$
$$\leqslant I_T + c_2\int\limits_0^T\left[1-\exp\left\{-\int\limits_{t_0}^t \delta_v dv\right\}\right]dt, \quad T > t_0,$$

where integral I_T is defined as

$$I_T = \int\limits_0^{t_0}\exp\left\{\int\limits_0^s \delta_v\, dv\right\} m_s^{(2)}\, ds \int\limits_0^T \exp\left\{-\int\limits_0^t \delta_v\, dv\right\} dt.$$

Convergence $\int_0^T \exp\left\{-\int_0^t \delta_v\, dv\right\} dt/T \to 0$, $T\to\infty$ follows from condition (iii), so $0 < \liminf\limits_{T\to\infty}(D_T^{(2)}/T) \leqslant \limsup\limits_{T\to\infty}(D_T^{(2)}/T) < \infty$. We can perform similar considerations for $D_T^{(1)}$ as well, showing that $0 < \liminf\limits_{T\to\infty}(D_T^{(1)}/T) \leqslant \limsup\limits_{T\to\infty}(D_T^{(1)}/T) < \infty$. Due to inequality (8), we conclude that D_T corresponds to normal diffusion. In case 2 the function $m_t^{(2)} < \epsilon\delta_t$ for an arbitrarily small number $\epsilon > 0$ and $t > t_0(\epsilon)$, so $D_T^{(2)} < I_T + \epsilon T$, which means that $(D_T^{(2)}/T) \to 0$, $T\to\infty$, and we get subdiffusion. In case 3, on the contrary, we have the bound $m_t^{(1)} \geqslant c\bar\kappa\delta_t$ for an arbitrarily large $c > 0$ for $t > t_0(c)$. Then $D_T^{(1)} \geqslant \bar I_T + cT - c\int_0^T \exp\left\{-\int_{t_0}^t \bar\kappa\delta_v dv\right\} dt$, where $\bar I_T = \int_0^{t_0}\exp\left\{\int_0^s \bar\kappa\delta_v dv\right\} m_s^{(2)} ds \int_0^T \exp\left\{-\int_0^t \bar\kappa\delta_v dv\right\} dt$, which leads to $(D_T^{(1)}/T) \to \infty$, $T\to\infty$ and characterizes superdiffusion. This concludes the proof of Theorem 1.

Proof of Theorem 2. Equation (12) is a Riccati equation arising in filtering theory [28]:

$$\dot\delta_t = 2g_t\delta_t - b_t^2\delta_t^2 + q_t, \quad \delta_0 = \bar\delta, \tag{A.1}$$
$$\text{where } g_t = -\frac{3D_t''}{2D_t'}, \quad b_t^2 = 2, \quad q_t = \frac{2\sigma_t^2 - D_t'''}{D_t'}.$$

It is known that Eq. (A.1) has a nonnegative solution δ_t given that $\bar\delta \geqslant 0$, $q_t \geqslant 0$, $t \geqslant 0$, and the solution can be represented as (see [29])

$$\delta_t = \delta_0\exp\left\{2\int\limits_0^t (g_v - b_v^2\delta_v/2)\, dv\right\} + \int\limits_0^t \exp\left\{2\int\limits_s^t (g_v - b_v^2\delta_v/2)\, dv\right\} q_s\, ds.$$

Since $D_t' > 0$, $\delta_0 > 0$, and the sign of D_t''' is not known in the general case, then, letting the coefficient $\sigma_t^2 > 0$ to be such that $2\sigma_t^2 - D_t''' > 0$, $t \geqslant 0$, we get a function $\delta_t > 0$, $t \geqslant 0$. Further we need to show that σ_t^2 can be found in such a way that the δ_t derived from (12) defines the stability rate, i.e., it holds that $\int_0^t \delta_v\, dv \to \infty$, $t\to\infty$. To achieve this, we define a lower bound δ_t by estimating function $p_t = 1/\delta_t$ from above. It is easy to see that the equation on p_t will also be a Riccati equation:

$$\dot p_t = -2g_t p_t + b_t^2 - q_t p_t^2, \quad p_0 = 1/\delta_0 > 0.$$

We construct a deterministic control problem (linear controller, see [28]) by passing to reverse time. Let $\tilde{p}_t = p_{-t}$. Then function $\tilde{p}_t$ for $t \leqslant 0$ is a solution of a Riccati equation with boundary condition at time moment $T = 0$:

$$\dot{\tilde{p}}_t - 2g_{-t}\tilde{p}_t + b^2_{-t} - q_{-t}\tilde{p}^2_t = 0, \quad \tilde{p}_T = p_0. \tag{A.2}$$

The corresponding equation of control system state dynamics is

$$dx_t = (-g_{-t}x_t + u_t)dt, \quad x_\tau = 1,$$

where u_t is the control function and $\tau < T$ is an arbitrary fixed moment of time. The objective functional is

$$J_{T,\tau}(u) = \int\limits_\tau^T \left(b^2_{-t}x^2_t + \frac{u^2_t}{q_{-t}} \right) dt + p_0 x^2_T, \quad \tau < T.$$

It is well known (see [3, 28]) that $\min_u J_{T,\tau}(u) = J_{T,\tau}(u^*) = \tilde{p}_\tau$, where the control law is $u^*_t = -q_{-t}\tilde{p}_t x^*_t$, and the function $\tilde{p}_t = p_{-t}$ is a solution of (A.2). If we take a control $u^{(0)}_t$ competing with strategy u^*_t, then for a suitably defined q_{-t} in the objective functional we can find an upper bound on $\tilde{p}_\tau$ that ensures (iii) for stability rate δ_t. For example, the exponentially stable law $u^{(0)}_t = (g_{-t} - 1)x^{(0)}_t$ defines the trajectory $x^{(0)}_t = \exp\{-(t-\tau)\}$. The objective functional value here is $J_{T,\tau}(u) = \int_\tau^T \exp\{-2(t-\tau)\}\left[b^2_{-t} + (g_{-t}-1)^2/q_{-t}\right] dt + p_0 \exp\{-2(T-\tau)\}$. Since $b^2_{-t} = 2$ then q_{-t} can be chosen to be such that the coefficient $(g_{-t}-1)^2/q_{-t}$ is bounded, e.g., $q_{-t} = 1 + (g_{-t}-1)^2$, $t \leqslant 0$. Then $\tilde{p}_\tau = J_{T,\tau}(u^*) \leqslant J_{T,\tau}(u^0) \leqslant c$ for some constant $c > 0$ and any $\tau < T$. Due to relation $\tilde{p}_\tau = p_{-\tau}$ function $p_t \leqslant c$, $t > 0$. Consequently, $\delta_t \geqslant 1/c$ and $\int_0^t \delta_v\, dv \to \infty$, $t \to \infty$, for $\sigma^2_t > 0$ and $(2\sigma^2_t - D'''_t)/D'_t > (1-g_t)^2$, where $g_t = -3D''_t/2D'_t$.

Thus, we have shown that there exist δ_t and σ^2_t with the required properties. To prove that $\tilde{D}_T \sim D_T$ with mean-squared displacement $\tilde{D}_T$ defined with (11) and parameters δ_t and σ^2_t related by (12), we rewrite (12) as

$$2\sigma^2_t \exp\left\{2\int\limits_0^t \delta_v\, dv\right\} = \left[2\delta_t\left(D''_t + D'_t\delta_t\right) + \left(D'''_t + D''_t\delta_t + D'_t\dot{\delta}_t\right)\right] \exp\left\{2\int\limits_0^t \delta_v\, dv\right\}.$$

Integrating the above relation twice with suitably chosen multipliers, we get that

$$2\int\limits_0^T \exp\left\{-\int\limits_0^t \delta_v\, dv\right\} \int\limits_0^t \exp\left\{\int\limits_0^s \delta_v\, dv\right\} m^{(2)}_s\, ds\, dt = D_T - D_0 - I_T,$$

where $m^{(2)}_t$ has been defined in (10) for $x = 0$,

$$I_T = \int\limits_0^T \exp\left\{-\int\limits_0^t \delta_v\, dv\right\} \left[c_1 \int\limits_0^t \exp\left\{-\int\limits_0^s \delta_v\, dv\right\} ds + c_2\right] dt,$$

and constants $c_1 = \delta_0 D'_0 + D''_0$, $c_2 = D'_0$. Due to (13) we get a representation $\tilde{D}_T = D_T - D_0 - I_T$. It is easy to see that now $0 \leqslant \lim\limits_{T\to\infty} (I_T/\tilde{D}_T) < \infty$, which implies that $\tilde{D}_T \sim D_T$. This concludes the proof of Theorem 2.

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